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Wronskian and Grammian Solutions for Generalized (n + 1)-Dimensional KP Equation with Variable Coefficients

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ABSTRACT

The generalized (n + 1)-dimensional KP equation with variable coefficients is investigated in this paper. The bilinear form of the equation has been obtained by the Hirota direct method. In addition, with the help of Wronskian technique and the Pfaffian properties, Wronskian and Grammian solutions have been generated.

Keywords: Generalized Variable Coefficient (n + 1)-Dimensional KP Equation; Hirota Bilinear Method; Wronskian Solution; Grammian Solution

1. Introduction

Recently, there has been a growing interest in studying variable-coefficient nonlinear evolution equations (NLEEs). Quite a few researchers studied the variable-coefficient KP equations [1-3], which provides us with more realistic models in such physical situations as the canonical and cylindrical cases, propagation of surface waves in large channels of varying width and depth with nonvanishing vorticity and so on. In this paper, we consider the generalized variable-coefficient (n + 1)-dimensional KP equation

$$\left(u_t + s(t)uu_x + m(t)u_{xxx}\right)_x + \sum_{k=2}^{n} h_k(t)u_{x_k x_k} = 0, \quad (1)$$

where $m(t)$, $h_k(t)(k \geq 2)$ are arbitrary functions with respect to *t*. Equation (1) can be reduced to the (3 + 1)-dimensional KP equation

$$\left(u_t + 6uu_x + u_{xxx}\right)_x + 3u_{yy} + 3u_{zz} = 0, \quad (2)$$

by setting

$$s(t) = 6, m(t) = 1, h_2(t) = h_3(t) = 3, h_k(t) = 0\ (k \geq 4),$$
$$x_2 = y, x_3 = z.$$

Equation (2) describes the dynamics of solitons and nonlinear waves in plasmas physics and fluid dynamics. Obviously, (1) is the generalization of (2).

It is well known that the bilinear method first proposed by Hirota provides us with a comprehensive approach to construct exact solutions [4-6]. Once a NLEE is written in bilinear form, we are able to derive systematically particular solutions including the multi-soliton solutions. Soliton solutions can also be written in Wronskian form, which was first introduced by Satsuma in 1979 [7]. Freeman and Nimmo developed the Wronskian technique, which admits direct verifications of solutions in Wronskian form to the bilinear equations [8]. It is noted that Grammian is another type of solution representation for soliton equations, which can be rewritten as a Pfaffian and the proof can easily be completed by virtue of Pfaffian properties [9,10].

The organization of the paper is as follows. In Section 2, based on the Hirota bilinear method, we obtain the bilinear forms of (1). Then the Wronskian and Grammian solutions of (1) are derived in Sections 3 and 4, respectively. Finally, the conclusions and discussions will be given in Section 5.

2. Bilinear Form of (1)

By the dependent variable transformation

$$u = 12m(t)s^{-1}(t)(\ln f)_{xx}, \quad (3)$$

Equation (1) can be transformed into the following bilinear form:

$$\left[D_x D_t + m(t)D_x^4 + \sum_{k=2}^{n} h_k(t)D_{x_k}^2\right] f \cdot f = 0, \quad (4)$$

where the Hirota bilinear operators D_x, D_t and $D_{x_k}(k \geq 2)$ are defined by

$$D_x^m D_t^n a\cdot b=\left(\partial_x-\partial_{x'}\right)^m\left(\partial_t-\partial_{t'}\right)^n a\left(x,t\right)b\left(x',t'\right)\Big|_{x'=x,t'=t}. \tag{5}$$

Equation (4) can be rewritten as

$$\begin{aligned}&\left(f_{tx}f-f_t f_x\right)+m\left(t\right)\left(ff_{xxxx}-4f_x f_{xxx}+3f_{xx}^2\right)\\&+\sum_{k=2}^{n}h_k\left(t\right)\left(f_{x_k x_k}f-f_{x_k}^2\right)=0.\end{aligned} \tag{6}$$

3. Wronskian Solution of (1)

In this section, the N-soliton solutions of (1) in Wronskian form have been generated.

Theorem 1. Equation (4) has the solution in terms of the Wronskian determinant

$$f_N\left(x,x_k,t\right)=W\left(\phi_1,\phi_2,\cdots,\phi_N\right)=\begin{vmatrix}\phi_1^{(0)}&\phi_1^{(1)}&\cdots&\phi_1^{(N-1)}\\\phi_2^{(0)}&\phi_2^{(1)}&\cdots&\phi_2^{(N-1)}\\\vdots&\vdots&\ddots&\vdots\\\varphi_N^{(0)}&\phi_N^{(1)}&\cdots&\varphi_N^{(N-1)}\end{vmatrix}, \tag{7}$$

where $\phi_j^i=\partial^i\phi_j/\partial x^i$ and $\phi_j=\phi_j\left(x,x_k,t\right)$ satisfy the set of linear partial differential equations

$$\begin{aligned}&\phi_{j,t}=-4m\left(t\right)\phi_{j,xxx},\ \phi_{j,x_k}=c_k\left(t\right)\phi_{j,xx},\\&\sum_{k=2}^{n}c_k^2\left(t\right)h_k\left(t\right)=3m\left(t\right).\end{aligned} \tag{8}$$

Proof. To conveniently write (7), we adopt the compact notation

$$W\left(\phi_1,\phi_2,\cdots,\phi_N\right)=\left|0,1,2,\cdots,N-1\right|=\left|\overline{N-1}\right|. \tag{9}$$

Under the properties of the Wronskian determinant and the conditions (8), we obtain

$$\begin{aligned}f_{N,x}&=\left|\overline{N-2},N\right|,\\f_{N,xx}&=\left|\overline{N-3},N-1,N\right|+\left|\overline{N-2},N+1\right|,\\f_{N,xxx}&=\left|\overline{N-4},N-2,N-1,N\right|+2\left|\overline{N-3},N-1,N+1\right|+\left|\overline{N-2},N+2\right|,\\f_{N,xxxx}&=\left|\overline{N-5},N-3,N-2,N-1,N\right|+2\left|\overline{N-3},N,N+1\right|+3\left|\overline{N-4},N-2,N-1,N+1\right|\\&\quad+\left|\overline{N-2},N+3\right|+3\left|\overline{N-3},N-1,N+2\right|,\\f_{N,t}&=-4m\left(t\right)\left(\left|\overline{N-4},N-2,N-1,N\right|+\left|\overline{N-2},N+2\right|-\left|\overline{N-3},N-1,N+1\right|\right),\\f_{N,xt}&=-4m\left(t\right)\left(\left|\overline{N-5},N-3,N-2,N-1,N\right|-\left|\overline{N-3},N,N+1\right|+\left|\overline{N-2},N+3\right|\right),\\f_{N,x_k}&=c_k\left(t\right)\left(\left|\overline{N-2},N+1\right|-\left|\overline{N-3},N-1.N\right|\right),\\f_{N,x_k x_k}&=c_k^2\left(t\right)\left(\left|\overline{N-5},N-3,N-2,N-1,N\right|\right)-\left|\overline{N-4},N-2,N-1,N+1\right|\\&\quad+2\left|\overline{N-3},N,N+1\right|-\left|\overline{N-3},N-1,N+2\right|+\left|\overline{N-2},N+3\right|\Big).\end{aligned} \tag{10}$$

Substituting these derivatives into (6), the left side becomes

$$\begin{aligned}&\left(f_{tx}f-f_t f_x\right)+m\left(t\right)\left(ff_{xxxx}-4f_x f_{xxx}+3f_{xx}^2\right)+\sum_{k=2}^{n}h_k\left(t\right)\left(f_{x_k x_k}f-f_{x_k}^2\right)\\&=12m\left(t\right)\left(\left|\overline{N-1}\right|\left|\overline{N-3},N,N+1\right|-\left|\overline{N-3},N-1,N+1\right|\left|\overline{N-2},N\right|+\left|\overline{N-2},N+1\right|\left|\overline{N-3},N-1,N\right|\right)\\&=12m\left(t\right)\left(\left|\overline{N-3},N-2,N-1\right|\left|\overline{N-3},N,N+1\right|-\left|\overline{N-3},N-2,N\right|\left|\overline{N-3},N-1,N+1\right|+\left|\overline{N-3},N-2,N+1\right|\left|\overline{N-3},N-1,N\right|\right)\\&=0.\end{aligned} \tag{11}$$

Thus, we have the N-soliton solutions of (1) in Wronskian form

$$u=12m\left(t\right)s^{-1}\left(t\right)\times\frac{\left|\overline{N-1}\right|\left(\left|\overline{N-3},N-1,N\right|+\left|\overline{N-2},N+1\right|\right)-\left|\overline{N-2},N\right|^2}{\left|\overline{N-1}\right|^2}, \tag{12}$$

where $\varphi_j = \varphi_j(x, x_k, t)$ satisfies the conditions (8).

4. Grammian Solution of (1)

In what follows, we focus on the Grammian type solution and construct a broad set of sufficient conditions which make the Grammian determinant a solution of the bilinear Equation (4).

Theorem 2. Equation (4) has the Grammian solution as follows:

$$f_N = \det\left|a_{ij}\right|_{1\le i\le j\le N},\ a_{ij} = \delta_{ij} + \int^x \phi_i\psi_j \mathrm{d}x, 1\le i, j\le N, \quad (13)$$

where the functions $\phi_i = \phi_i(x, x_k, t)$ and $\psi_j = \psi_j(x, x_k, t)$ satisfy the two sets of conditions

$$\phi_{i,x_k} = c_k(t)\phi_{i,xx},\ \phi_{i,t} = -4m(t)\phi_{i,xxx}, \quad (14)$$

$$\psi_{j,x_k} = -c_k(t)\phi_{j,xx}, \psi_{j,t} = -4m(t)\phi_{j,xxx}. \quad (15)$$

Proof. A differential of the determinant f_N expressed by means of a Pfaffian is

$$\begin{aligned} &f_N = \left(1,2,\cdots,N,N^*,\cdots,2^*,1^*\right), \\ &a_{ij} = \left(i, j^*\right) = \delta_{ij} + \int^x \phi_i\psi_j \mathrm{d}x, \left(i, j\right) = \left(i^*, j^*\right) = 0. \end{aligned} \quad (16)$$

Next we introduce the Pfaffians $(m, n = 0,1,2,\cdots,N)$ defined by

$$\left(d_n, j^*\right) = \frac{\partial^n}{\partial x^n}\psi_j, \left(d_m, d_n^*\right) = 0, \quad (17)$$

$$\left(d_n^*, i\right) = \frac{\partial^n}{\partial x^n}\phi_i, \left(d_n, i\right) = \left(d_m^*, i^*\right) = 0. \quad (18)$$

Based on the Pfaffians defined above, differentials of the elements $a_{ij}\left(i = 1,2,\cdots,N; j = 1,2,\cdots,N\right)$ can be obtained as follows:

$$\begin{aligned} \frac{\partial}{\partial x}a_{ij} &= \phi_i\psi_j = \left(d_0, d_0^*, i, j^*\right), \\ \frac{\partial}{\partial t}a_{ij} &= \int^x \left(\phi_{i,t}\psi_j + \phi_i\psi_{j,t}\right)\mathrm{d}x = 4m(t)\left(\phi_{i,x}\psi_{j,x} - \phi_{i,xx}\psi_j - \phi_i\psi_{j,xx}\right) \\ &= 4m(t)\left[\left(d_1, d_1^*, i, j^*\right) - \left(d_0, d_2^*, i, j^*\right) - \left(d_2, d_0^*, i, j^*\right)\right], \\ \frac{\partial}{\partial x_k}a_{ij} &= \int^x \left(\phi_{i,x_k}\psi_j + \phi_i\psi_{j,x_k}\right)\mathrm{d}x = \int^x \left(c_k(t)\phi_{i,xx}\psi_j - c_k(t)\phi_i\psi_{j,xx}\right)\mathrm{d}x \\ &= c_k(t)\left(\phi_{i,x}\psi_j - \phi_i\psi_{j,x}\right) = c_k(t)\left[\left(d_0, d_1^*, i, j^*\right) - \left(d_1, d_0^*, i, j^*\right)\right]. \end{aligned} \quad (19)$$

We denote $f_N = \left(1,2,\cdots,N,N^*,\cdots,2^*,1^*\right) = (\bullet)$, then

$$\begin{aligned} f_{N,x} &= \left(d_0, d_0^*, \bullet\right), \\ f_{N,xx} &= \left(d_1, d_0^*, \bullet\right) + \left(d_0, d_1^*, \bullet\right), \\ f_{N,xxx} &= \left(d_2, d_0^*, \bullet\right) + 2\left(d_1, d_1^*, \bullet\right) + \left(d_0, d_2^*, \bullet\right), \\ f_{N,xxxx} &= \left(d_3, d_0^*, \bullet\right) + 3\left(d_2, d_1^*, \bullet\right) + 3\left(d_1, d_2^*, \bullet\right) + 2\left(d_0, d_0^*, d_1, d_1^*, \bullet\right) + \left(d_0, d_3^*, \bullet\right), \\ f_{N,t} &= 4m(t)\left[\left(d_1, d_1^*, \bullet\right) - \left(d_0, d_2^*, \bullet\right) - \left(d_2, d_0^*, \bullet\right)\right], \\ f_{N,xt} &= 4m(t)\left[\left(d_0, d_0^*, d_1, d_1^*, \bullet\right) - \left(d_0, d_3^*, \bullet\right) - \left(d_3, d_0^*, \bullet\right)\right], \\ f_{N,x_k} &= c_k(t)\left[\left(d_0, d_1^*, \bullet\right) - \left(d_1, d_0^*, \bullet\right)\right], \\ f_{N,x_kx_k} &= c_k^2(t)\left[\left(d_3, d_0^*, \bullet\right) - \left(d_1, d_2^*, \bullet\right) + 2\left(d_0, d_0^*, d_1, d_1^*, \bullet\right) + \left(d_0, d_3^*, \bullet\right) - \left(d_2, d_1^*, \bullet\right)\right]. \end{aligned} \quad (20)$$

Substituting the above Pfaffians into (4), after some calculations, we have

$$\begin{aligned} &\left[D_xD_t + m(t)D_x^4 + \sum_{k=2}^n h_k(t)D_{x_k}^2\right] f\cdot f = \left(f_{tx}f - f_tf_x\right) + m(t)\left(ff_{xxxx} - 4f_xf_{xxx} + 3f_{xx}^2\right) + \sum_{k=2}^n h_k(t)\left(f_{x_kx_k}f - f_{x_k}^2\right) \\ &= 12m(t)\left[\left(d_0, d_0^*, d_1, d_1^*, \bullet\right)(\bullet) - \left(d_0, d_0^*, \bullet\right)\left(d_1, d_1^*, \bullet\right) + \left(d_1, d_0^*, \bullet\right)\left(d_0, d_1^*, \bullet\right)\right] = 0. \end{aligned} \quad (21)$$

This shows that the Grammian determinant f_N with the conditions of (14) and (15) solves (4).

5. Conclusions and Discussions

In summary we have extended the Wronskian method and Pfaffian properties to the generalized variable-coefficient (n + 1)-dimensional KP Equation (1). As a result, the Wronskian solutions and the Grammian solutions of (1) have been derived. It is known that if one gets the solutions of the conditions (8) or that of (14) and (15), then one can obtain the corresponding solutions of (1), which need to be further studied.

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