Applied Mathematics, **2012, 3, 52-55**
http://dx.doi.org/10.4236/am.2012.31009 Published Online January 2012 (http://www.SciRP.org/journal/am)

Mapping Properties of Generalized Robertson Functions under Certain Integral Operators

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Received July 24, 2011; revised November 24, 2011; accepted December 2, 2011

ABSTRACT

In the present article, certain classes of generalized p-valent Robertson functions are considered. Mapping properties of these classes are investigated under certain p-valent integral operators introduced by Frasin recently.

Keywords: p-Valent Analytic Functions; Bounded Boundary Rotations; Bounded Radius Rotations; Integral Operators

1. Introduction

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{j=p+1}^{\infty}a_j z^j \quad \left(p\in\mathbb{N}=\{1,2,\cdots\}\right),$$

which are analytic in the open unit disc $E=\{z:|z|<1\}$. We write $A(1)=A$. A function $f(z)\in A$ is said to be spiral-like if there exists a real number λ $\left(|\lambda|<\frac{\pi}{2}\right)$ such that

$$\operatorname{Re} e^{i\lambda}\frac{zf'(z)}{f(z)}>0 \quad (z\in E).$$

The class of all spiral-like functions was introduced by L. Spacek [1] in 1933 and we denote it by S_λ^*. Later in 1969, Robertson [2] considered the class C_λ of analytic functions in E for which $zf'(z)\in S_\lambda^*$.

Let $P_k^\lambda(p,\rho)$ be the class of functions $p(z)$ analytic in E with $p(0)=1$ and

$$\int_0^{2\pi}\left|\frac{\operatorname{Re} e^{i\lambda}p(z)-\rho\cos\lambda}{p-\rho}\right|\mathrm{d}\theta\le k\pi\cos\lambda, \quad \left(z=re^{i\theta}\right),$$

where $k\ge 2$, $0\le\rho<1$ and λ is real with $|\lambda|<\frac{\pi}{2}$. For $\lambda=0$, $p=1$, this class was introduced in [3] and for $\rho=0$, see [4]. For $k=2$, $\lambda=0$ and $\rho=0$, the class $P_k^\lambda(p,\rho)$ reduces to the class P of functions $p(z)$ analytic in E with $p(0)=1$ and whose real part is positive.

We define the following classes

$$R_k^\lambda(p,\rho)=\left\{f(z):f(z)\in A(p) \text{ and } \frac{zf'(z)}{f(z)}\in P_k^\lambda(p,\rho),\ 0\le\rho<1\right\},$$

$$V_k^\lambda(p,\rho)=\left\{f(z):f(z)\in A(p) \text{ and } \frac{(zf'(z))'}{f'(z)}\in P_k^\lambda(p,\rho),\ 0\le\rho<1\right\}.$$

For $\lambda=0$, $\rho=0$ and $p=1$, we obtain the well known classes R_k and V_k of analytic functions with bounded radius and bounded boundary rotations studied by Tammi [5] and Paatero [6] respectively. For details see [7-12]. Also it can easily be seen that $R_2^\lambda(0)=S_\lambda^*$ and $V_2^\lambda(0)=C_\lambda$.

Let us consider the integral operators

$$F_p(z)=\int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1}\cdots\left(\frac{f_n(t)}{t^p}\right)^{\alpha_n}\mathrm{d}t \quad (1.1)$$

and

$$G_p(z)=\int_0^z pt^{p-1}\left[\frac{f_1'(t)}{pt^{p-1}}\right]^{\alpha_1}\cdots\left[\frac{f_n'(t)}{pt^{p-1}}\right]^{\alpha_n}\mathrm{d}t, \quad (1.2)$$

where $f_i(z)\in A(p)$ and $\alpha_i>0$ for all $i\in\{1,2,\cdots,n\}$.

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These operators, given by (1.1) and (1.2), are defined by Frasin [13]. If we take $p = 1$, we obtain the integral operators $F_1(z) = F_n(z)$ and $G_1(z) = F_{\alpha_1 \cdots \alpha_n}(z)$ introduced and studied by Breaz and Breaz [14] and Breaz *et al.* [15], for details see also [16-20]. Also for $p = n = 1$, $\alpha_1 = \alpha \in [0,1]$ in (1.1), we obtain the integral operator studied in [21] given as

$$\int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} \mathrm{d}t,$$

and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| \le \frac{1}{4}$ in (1.2), we obtain the integral operator

$$\int_0^z \left(f'(t) \right)^{\delta} \mathrm{d}t,$$

discussed in [22,23].

In this paper, we investigate some propeties of the above integral operators $F_p(z)$ and $G_p(z)$ for the classes $V_k^{\lambda}(p,\rho)$ and $R_k^{\lambda}(p,\rho)$ respectively.

2. Main Result

Theorem 2.1. Let $f_i(z) \in R_k^{\lambda}(p,\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let λ is real with $|\lambda| < \frac{\pi}{2}$, $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - p) \sum_{i=1}^{n} \alpha_i + p < 1,$$

then $F_p(z) \in V_k^{\lambda}(p, \lambda_1)$ with

$$\lambda_1 = (\rho - p) \sum_{i=1}^{n} \alpha_i + p. \tag{2.1}$$

Proof. From (1.1), we have

$$\frac{zF_p''(z)}{F_p'(z)} = (p-1) + \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - p \right), \tag{2.2}$$

or, equivalently

$$\mathrm{e}^{i\lambda} \left(1 + \frac{zF_p''(z)}{F_p'(z)} \right) = \mathrm{e}^{i\lambda} p \left(1 - \sum_{i=1}^{n} \alpha_i \right) + \mathrm{e}^{i\lambda} \sum_{i=1}^{n} \alpha_i \frac{zf_i'(z)}{f_i(z)}. \tag{2.3}$$

Subtracting and adding $\rho \cos\lambda \sum_{i=1}^{n} \alpha_i$ on the right hand side of (2.3), we have

$$\begin{aligned} \mathrm{e}^{i\lambda} \left(1 + \frac{zF_p''(z)}{F_p'(z)} \right) &= p\mathrm{e}^{i\lambda} + \left(\rho\cos\lambda - p\mathrm{e}^{i\lambda} \right) \sum_{i=1}^{n} \alpha_i \\ &\quad + \sum_{i=1}^{n} \alpha_i \left[\mathrm{e}^{i\lambda} \frac{zf_i'(z)}{f_i(z)} - \rho\cos\lambda \right], \end{aligned} \tag{2.4}$$

Taking real part of (2.4) and then simple computation gives

$$\begin{aligned} &\int_0^{2\pi} \left| \mathrm{Re}\left[\mathrm{e}^{i\lambda} \left(1 + \frac{zF_p''(z)}{F_p'(z)} \right) - \lambda_1 \cos\lambda \right] \right| \mathrm{d}\theta \\ &\le \sum_{i=1}^{n} \alpha_i \int_0^{2\pi} \left| \mathrm{Re}\left[\mathrm{e}^{i\lambda} \frac{zf_i'(z)}{f_i(z)} - \rho\cos\lambda \right] \right| \mathrm{d}\theta, \end{aligned} \tag{2.5}$$

where λ_1 is given by (2.1). Since $f_i(z) \in R_k^{\lambda}(p,\rho)$ for $1 \le i \le n$, we have

$$\int_0^{2\pi} \left| \mathrm{Re}\left[\mathrm{e}^{i\lambda} \frac{zf_i'(z)}{f_i(z)} - \rho\cos\lambda \right] \right| \mathrm{d}\theta \le (p-\rho)\cos\lambda k\pi. \tag{2.6}$$

Using (2.6) and (2.1) in (2.5), we obtain

$$\int_0^{2\pi} \left| \mathrm{Re}\left[\mathrm{e}^{i\lambda} \left(1 + \frac{zF_p''(z)}{F_p'(z)} \right) - \lambda_1 \cos\lambda \right] \right| \mathrm{d}\theta \le (p-\lambda_1)\cos\lambda k\pi.$$

Hence $F_n(z) \in V_k^{\lambda}(p, \lambda_1)$ with λ_1 is given by (2.1).

By setting $p = 1$ and $\lambda = 0$ in Theorem 2.1, we obtain the following result proved in [9].

Corollory 2.2. Let $f_i(z) \in R_k(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1,$$

then $F_n(z) \in V_k(\lambda_1)$ and λ_1 is given by (2.1).

Now if we take $k = 2$ and $\lambda = 0$ in Theorem 2.1, we obtain the following result.

Corollory 2.3. Let $f_i(z) \in S_p^*(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - p) \sum_{i=1}^{n} \alpha_i + p < 1,$$

then $F_p(z) \in C_p(\lambda_1)$ and λ_1 is given by (2.1).

Letting $p = n = 1$, $\lambda = 0$, $\alpha_1 = \alpha$ and $f_1(z) = f(z)$ in Theorem 2.1, we have.

Corollory 2.4. Let $f(z) \in R_k(\rho)$ with $0 \le \rho < 1$. Also let $\alpha > 0$. If

$$0 \le (\rho - 1)\alpha + 1 < 1,$$

then

$$\int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} \mathrm{d}t \in V_k(\lambda_1)$$

with $\lambda_1 = (\rho - 1)\alpha + 1$.

Theorem 2.5. Let $f_i(z) \in V_k^{\lambda}(p,\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let λ is real is real with $|\lambda| < \frac{\pi}{2}$, $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - p)\sum_{i=1}^{n} \alpha_i + p < 1,$$

then $G_p(z) \in V_k^\lambda(p, \lambda_1)$ and λ_1 is given by (2.1).

Proof. From (1.2), we have

$$1 + \frac{zG_p''(z)}{G_p'(z)} = p + \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)} + 1 \right) - p\sum_{i=1}^{n} \alpha_i,$$

or, equivalently

$$e^{i\lambda}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) = pe^{i\lambda}\left(1 - \sum_{i=1}^{n}\alpha_i\right) + \sum_{i=1}^{n}\alpha_i e^{i\lambda}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right).$$

This relation is equivalent to

$$e^{i\lambda}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) = pe^{i\lambda} + \left(\rho\cos\lambda - pe^{i\lambda}\right)\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\alpha_i\left[e^{i\lambda}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) - \rho\cos\lambda\right]. \quad (2.7)$$

Taking real part of (2.7) and then simple computation gives us

$$\int_0^{2\pi}\left|\mathrm{Re}\left[e^{i\lambda}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) - \lambda_1\cos\lambda\right]\right| d\theta \le \sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\left|\mathrm{Re}\left[e^{i\lambda}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) - \rho\cos\lambda\right]\right| d\theta, \quad (2.8)$$

where λ_1 is given by (2.1). Since $f_i(z) \in V_k^\lambda(p, \rho)$ for $1 \le i \le n$, we have

$$\int_0^{2\pi}\left|Re\left[e^{i\lambda}\left(1 + \frac{zf_i''(z)}{f_i'(z)}\right) - \rho\cos\lambda\right]\right| d\theta \le (p - \rho)\cos\lambda k\pi. \quad (2.9)$$

Using (2.9) in (2.8), we obtain

$$\int_0^{2\pi}\left|Re\left[e^{i\lambda}\left(1 + \frac{zG_p''(z)}{G_p'(z)}\right) - \lambda_1\cos\lambda\right]\right| d\theta \le (p - \lambda_1)\cos\lambda k\pi.$$

Hence $G_p(z) \in V_k^\lambda(p, \lambda_1)$ with λ_1 is given by (2.1).

By setting $k = 2$ and $\lambda = 0$ in Theorem 2.5, we obtain the following result.

Corollory 2.6. Let $f_i(z) \in C_p(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - p)\sum_{i=1}^{n}\alpha_i + p < 1,$$

then $G_p(z) \in C_p(\lambda_1)$ with λ_1 is given by (2.1).

Letting $p = n = 1$, $\lambda = 0$, $\alpha_1 = \delta$ and $f_1(z) = f(z)$ in Theorem 2.5, we have.

Corollory 2.7. Let $f(z) \in V_k(\rho)$ with $0 \le \rho < 1$. Also let $\delta > 0$. If $0 \le (\rho - 1)\delta + 1 < 1$, then

$$\int_0^z \left(f'(t)\right)^\delta dt \in V_k(\lambda_1)$$

with $\lambda_1 = (\rho - 1)\delta + 1$.

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