Applied Mathematics, 2012, 3, 45-51
http://dx.doi.org/10.4236/am.2012.31008 Published Online January 2012 (http://www.SciRP.org/journal/am)

L_p-Estimations of Vector Fields in Unbounded Domains

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Received November 15, 2011; revised December 15, 2011; accepted December 23, 2011

ABSTRACT

Some new estimations of scalar products of vector fields in unbounded domains are investigated. L_p-estimations for the vector fields were proved in special weighted functional spaces. The paper generalizes our earlier results for bounded domains. Estimations for scalar products make it possible to investigate wide classes of mathematical physics problems in physically inhomogeneous domains. Such estimations allow studying issues of correctness for problems with non-smooth coefficients. The paper analyses solvability of stationary set of Maxwell equations in inhomogeneous unbounded domains based on the proved L_p-estimations.

Keywords: Estimations; Scalar Product; Vector Field; Functional Spaces; Maxwell Equations; Solvability; Inhomogeneous Domains

1. Introduction

The estimations of scalar products of vector fields and their norms play a significant role in proving the solvability of mathematical physics problems. Many researches are devoted to the study of estimates of the norms of vector functions in different functional spaces [1-4]. But in the most cases such estimations require the homogeneous areas when their parameters don't depend on space coordinates [5,6].

For inhomogeneous areas we suggest using estimations of scalar products of vector fields for the mathematical physics problems. In the publications [7-10] some L_p-estimations of scalar product of vector fields in the limited areas were obtained and was investigated the possibility of their application to study the solvability of different problems of electromagnetic theory.

It is natural to study problem formulations in non-homogeneous unbounded domains for most problems of mathematical physics. In the publications [11,12] we proved L_2-estimations of scalar products of vector fields in unlimited areas.

The paper is dedicated to solvability of a stationary set of Maxwell equations in the whole $\mathbb{R}^3$ space, based on the proved L_p-estimations of scalar product in the weighted functional spaces.

2. Main Functional Spaces

Let $\Omega \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$ space (particularly $\Omega \equiv \mathbb{R}^3$).

Let $L_p(\Omega)$ be a Banach space of functions $u:\Omega \to \mathbb{R}$, summable with power p, where a norm is

$$\|u\|_{L_p(\Omega)} = \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right).$$

Let $\{L_p(\Omega)\}^{31/p}$ be a Banach space of vector-functions $\boldsymbol{u}:\Omega \to \mathbb{R}^3$,

$$\boldsymbol{u}(x) = (u_1(x), u_2(x), u_3(x))$$

where $u_i \in L_p(\Omega)$, ($i = 1,2,3$), with a norm

$$\|\boldsymbol{u}\|_{\{L_p(\Omega)\}^3} = \left(\sum_{i=1}^{3} \|u_i\|_{L_p(\Omega)}^p\right)^{1/p}.$$

Let $H_p(\mathrm{div};\Omega)$ and $H_p(\mathrm{rot};\Omega)$ be Banach spaces

$$H_p(\mathrm{div};\Omega) = \left\{\boldsymbol{u} \in \{L_p(\Omega)\}^3 : \mathrm{div}\,\boldsymbol{u} \in L_p(\Omega)\right\},$$

$$H_p(\mathrm{rot};\Omega) = \left\{\boldsymbol{u} \in \{L_p(\Omega)\}^3 : \mathrm{rot}\,\boldsymbol{u} \in \{L_p(\Omega)\}^3\right\}$$

with norms

$$\|\boldsymbol{u}\|_{H_p(\mathrm{div};\Omega)} = \left\{\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}x + \int_\Omega |\mathrm{div}\,\boldsymbol{u}|^p \,\mathrm{d}x\right\}^{1/p},$$

$$\|\boldsymbol{u}\|_{H_p(\mathrm{rot};\Omega)} = \left\{\int_\Omega |\boldsymbol{u}|^p \,\mathrm{d}x + \int_\Omega |\mathrm{rot}\,\boldsymbol{u}|^p \,\mathrm{d}x\right\}^{1/p},$$

respectively.

We denote by $H_p^0(\mathrm{rot};\Omega)$ and $H_p^0(\mathrm{div};\Omega)$ the closures of the set of test vector-functions in $H_p(\mathrm{rot};\Omega)$

and $H_p(\mathrm{div};\Omega)$, respectively.

The following estimates for scalar products of vector fields in the bounded star-shaped domain $\Omega \subset \mathbb{R}^3$ with the regular boundary were obtaned in [8,9,11].

Lemma 2.1. *Let* $p > 3/2$, $q = p/(p-1)$. *There exists a constant* $C_1(\Omega, p) > 0$, *that for any* $\boldsymbol{u} \in H_q^0(\mathrm{rot};\Omega)$ *and* $\boldsymbol{v} \in H_p(\mathrm{div};\Omega)$

$$\left|\int_\Omega (\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\mathrm{d}x\right| \le C_1(\Omega, p)$$
$$\times\left(\|\,\mathrm{rot}\boldsymbol{u}\,\|_{\{L_q(\Omega)\}^3}\|\boldsymbol{v}\|_{\{L_p(\Omega)\}^3} + \|\boldsymbol{u}\|_{\{L_q(\Omega)\}^3}\|\mathrm{div}\boldsymbol{v}\|_{L_p(\Omega)}\right).$$

Lemma 2.2. *Let* $p > 3/2$, $q = p/(p-1)$. *There exists a constant* $C_2(\Omega, p) > 0$, *that for any* $\boldsymbol{u} \in H_q^0(\mathrm{div};\Omega)$, $\boldsymbol{v} \in H_p(\mathrm{div};\Omega)$

$$\left|\int_\Omega (\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\mathrm{d}x\right|$$
$$\le C_2(\Omega, p)\cdot\left(\|\boldsymbol{u}\|_{\{L_q(\Omega)\}^3}\|\mathrm{rot}\boldsymbol{v}\|_{\{L_p(\Omega)\}^3}\right.$$
$$\left.+\|\mathrm{div}\boldsymbol{u}\|_{L_q(\Omega)}\|\boldsymbol{v}\|_{\{L_p(\Omega)\}^3} + \|\mathrm{div}\boldsymbol{u}\|_{L_q(\Omega)}\|\mathrm{rot}\boldsymbol{v}\|_{\{L_p(\Omega)\}^3}\right).$$

Lemma 2.3. *Let* $p > 3$, $q = p/(p-1)$. *There exists a constant* $C_3(\Omega, p) > 0$, *that for any* $\boldsymbol{u} \in H_q^0(\mathrm{div};\Omega)$ *and* $\boldsymbol{v} \in H_p(\mathrm{rot};\Omega)$

$$\left|\int_\Omega (\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\mathrm{d}x\right| \le C_3(\Omega, p)$$
$$\times\left(\|\boldsymbol{u}\|_{\{L_q(\Omega)\}^3}\|\mathrm{rot}\boldsymbol{v}\|_{\{L_p(\Omega)\}^3} + \|\mathrm{div}\boldsymbol{u}\|_{L_q(\Omega)}\|\boldsymbol{v}\|_{\{L_p(\Omega)\}^3}\right).$$

The main result of this paper is a proof of similar estimates for $\Omega \equiv \mathbb{R}^3$.

Let $\Omega \equiv \mathbb{R}^3$. For each $\alpha \in \mathbb{R}$ and $p \ge 1$ we define Banach spaces of vector-functions:

$$H_p^\alpha(\mathrm{rot};\mathbb{R}^3)$$
$$=\left\{\boldsymbol{u} \in \{L_p(\mathbb{R}^3)\}^3 : (1+|\cdot|^2)^{\alpha/p}\,\mathrm{rot}\boldsymbol{u} \in \{L_p(\mathbb{R}^3)\}^3\right\},$$

$$H_p^\alpha(\mathrm{div};\mathbb{R}^3)$$
$$=\left\{\boldsymbol{u} \in \{L_p(\mathbb{R}^3)\}^3 : (1+|\cdot|^2)^{\alpha/p}\,\mathrm{div}\boldsymbol{u} \in L_p(\mathbb{R}^3)\right\}$$

with the corresponding norms

$$\|\boldsymbol{u}\|_{\alpha,p,\mathrm{rot}} = \left(\|\boldsymbol{u}\|^p_{\{L_p(\Omega)\}^3} + \left\|(1+|\cdot|^2)^{\alpha/p}\,\mathrm{rot}\boldsymbol{u}\right\|^p_{\{L_p(\Omega)\}^3}\right)^{1/p},$$

$$\|\boldsymbol{u}\|_{\alpha,p,\mathrm{div}} = \left(\|\boldsymbol{u}\|^p_{\{L_p(\Omega)\}^3} + \left\|(1+|\cdot|^2)^{\alpha/p}\,\mathrm{div}\boldsymbol{u}\right\|^p_{L_p(\Omega)}\right)^{1/p}.$$

For $p = 2$ [12] these spaces are defined as:

$$H^\alpha(\mathrm{rot};\mathbb{R}^3) = H_2^\alpha(\mathrm{rot};\mathbb{R}^3),$$
$$H^\alpha(\mathrm{div};\mathbb{R}^3) = H_2^\alpha(\mathrm{div};\mathbb{R}^3).$$

3. Estimations of Scalar Products

The main result of the current article is

Theorem 3.1. *Let* $1 < p < \infty$, $p \ne 3/2$, $q = \dfrac{p}{p-1}$, $\alpha > \dfrac{1}{2}\max\{p,q\}$. *Then there exists a positive constant* $C(\alpha, p)$, *which does not depend on vector-functions* $\boldsymbol{u} \in H_q^\alpha(\mathrm{rot};\mathbb{R}^3)$ *and* $\boldsymbol{v} \in H_p^\alpha(\mathrm{div};\mathbb{R}^3)$, *and the inequality*

$$\int_{\mathbb{R}^3}(\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\mathrm{d}x$$
$$\le C(\alpha, p)\cdot\left(\left\|(1+|\cdot|^2)^{\alpha/q}\,\mathrm{rot}\boldsymbol{u}\right\|_{\{L_q(\mathbb{R}^3)\}^3}\|\boldsymbol{v}\|_{\{L_p(\mathbb{R}^3)\}^3}\right. \quad (1)$$
$$\left.+\|\boldsymbol{u}\|_{\{L_q(\mathbb{R}^3)\}^3}\left\|(1+|\cdot|^2)^{\alpha/p}\,\mathrm{div}\boldsymbol{v}\right\|_{L_p(\mathbb{R}^3)}\right)$$

is correct.

In proving Theorem 3.1 the following statement is used.

Lemma 3.2 [7]. *Let* Ω *be an open set in* $\mathbb{R}^3$ *(particularly,* $\Omega \equiv \mathbb{R}^3$*) star-shaped on* $0 \in \Omega$. *Then the following identities are true for all* $x \in \Omega$ *and each function* $\boldsymbol{u} \in \{C^1(\Omega)\}^3$

$$\boldsymbol{u}(x) = \mathrm{grad}_x\left(\int_0^1(\boldsymbol{u}(z)\cdot x)\mathrm{d}\tau\right) + \int_0^1\tau\left[\mathrm{rot}_z\boldsymbol{u}(z)\times x\right]\mathrm{d}\tau, \quad (2)$$

$$\boldsymbol{u}(x) = \mathrm{rot}_x\left(\int_0^1\tau\left[\boldsymbol{u}(z)\times x\right]\mathrm{d}\tau\right) + \int_0^1\tau^2 x\,\mathrm{div}_z\boldsymbol{u}(z)\mathrm{d}\tau, \quad (3)$$

$z = \tau x$, $\tau \in [0,1]$.

Let $r \equiv |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\xi = \tau r$, $s = x/|x|$. Then the identities (2) and (3) are equivalent to

$$\begin{aligned}\boldsymbol{u}(rs) &= \mathrm{grad}_x\left(\int_0^r(\boldsymbol{u}(\xi s)\cdot s)\mathrm{d}\xi\right)\\ &\quad+\frac{1}{r}\int_0^r\xi\left[\mathrm{rot}\boldsymbol{u}(\xi s)\times s\right]\mathrm{d}\xi,\end{aligned} \quad (4)$$

$$\begin{aligned}\boldsymbol{u}(rs) &= \mathrm{rot}_x\left(\int_0^r\xi\left[\boldsymbol{u}(\xi s)\times s\right]\mathrm{d}\xi\right)\\ &\quad+\frac{s}{r^2}\int_0^r\xi^2\,\mathrm{div}\boldsymbol{u}(\xi s)\mathrm{d}\xi.\end{aligned} \quad (5)$$

Proof (Theorem 3.1). Let $p > 3/2$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be smooth vector-functions on $\mathbb{R}^3$

$$\boldsymbol{u} \in \{C^1(\mathbb{R}^3)\}^3 \bigcap H_p^\alpha(\mathrm{rot};\mathbb{R}^3),$$

$$\boldsymbol{v} \in \{C^1(\mathbb{R}^3)\}^3 \bigcap H_q^\alpha(\mathrm{div};\mathbb{R}^3).$$

Let B_R denote a closed solid sphere with radius $R>0$ centered at the origin and with the boundary ∂B_R. Consider the integral

$$\int_{\mathbb{R}^3} \theta_R(|x|)(\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\,\mathrm{d}x, \qquad (6)$$

where $\theta_R(|x|)$ is a function of scalar argument

$$\theta_R(r) = \begin{cases} 1, & r \le R/2, \\ R/r - 1, & R/2 < r \le R, \\ 0, & r > R. \end{cases}$$

We use the representation (3) for the vector-function $\boldsymbol{v}$ in the integral (6)

$$\int_{B_R} \theta_R(|x|)(\boldsymbol{u}(x)\cdot\boldsymbol{v}(x))\,\mathrm{d}x$$
$$= \int_{B_R}\left(\theta_R(|x|)\boldsymbol{u}(x)\cdot\mathrm{rot}\left(\int_0^1 \tau\left[\boldsymbol{v}(\tau x)\times x\right]\mathrm{d}\tau\right)\right)\mathrm{d}x$$
$$+\int_{B_R}\left(\theta_R(|x|)\boldsymbol{u}(x)\cdot\int_0^1 \tau^2 x\,\mathrm{div}\boldsymbol{v}(\tau x)\,\mathrm{d}\tau\right)\mathrm{d}x = I_1 + I_2,$$

For the first of resulting integrals (I_1) we use a vector field relation

$$\mathrm{div}[\boldsymbol{a}\times\boldsymbol{b}] = (\mathrm{rot}\boldsymbol{a}\cdot\boldsymbol{b}) - (\boldsymbol{a}\cdot\mathrm{rot}\boldsymbol{b}),$$

and then we invoke the Gauss-Ostrogradsky theorem and use the fact that $\theta_R(|x|)=0$ when $x\in\partial B_R$. So

$$I_1 = \int_{B_R}\left(\theta_R(|x|)\boldsymbol{u}(x)\cdot\mathrm{rot}\left(\int_0^1\tau\left[\boldsymbol{v}(\tau x)\times x\right]\mathrm{d}\tau\right)\right)\mathrm{d}x$$
$$= \int_{B_R}\left(\mathrm{rot}\left(\theta_R(|x|)\boldsymbol{u}(x)\right)\cdot\int_0^1\tau\left[\boldsymbol{v}(\tau x)\times x\right]\mathrm{d}\tau\right)\mathrm{d}x$$

or passing to spherical coordinates the operator rot

$$I_1 = \int_S \mathrm{d}s\int_0^R r\left(\theta_R(r)\,\mathrm{rot}\boldsymbol{u}(rs)\cdot\int_0^r\xi\left[\boldsymbol{v}(\xi s)\times s\right]\mathrm{d}\xi\right)\mathrm{d}r$$
$$+\int_S\mathrm{d}s\int_0^R r\left(\left[\mathrm{grad}\theta_R(r)\times\boldsymbol{u}(rs)\right]\cdot\int_0^r\xi\left[\boldsymbol{v}(\xi s)\times s\right]\mathrm{d}\xi\right)\mathrm{d}r$$
$$= I_{1,1} + I_{1,2}.$$

We estimate the first integral. Applying Hölder's inequality to $\int_0^r \xi\left|\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi$ we get

$$\int_0^r\xi\left|\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi \le \left(\int_0^r\xi^2\left|\boldsymbol{v}(\xi s)\right|^p\mathrm{d}\xi\right)^{1/p}\cdot\left(\int_0^r\xi^{\left(1-\frac{2}{p}\right)q}\mathrm{d}\xi\right)^{1/q}$$
$$= \frac{r^{\frac{3-q}{q}}}{(3-q)^{1/q}}\cdot\left(\int_0^r\xi^2\left|\boldsymbol{v}(\xi s)\right|^p\mathrm{d}\xi\right)^{1/p}$$

then

$$\left|I_{1,1}\right| \le \int_S\mathrm{d}s\int_0^R r\theta_R(r)\left|\mathrm{rot}\boldsymbol{u}(rs)\right|\left|\int_0^r\xi\left|\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi\right|\mathrm{d}r$$
$$\le \frac{1}{(3-q)^{1/q}}\int_S\mathrm{d}s\int_0^R r^{3/q}\theta_R(r)\left|\mathrm{rot}\boldsymbol{u}(rs)\right|$$
$$\left(\int_0^r\xi^2\left|\boldsymbol{v}(\xi s)\right|^p\mathrm{d}\xi\right)^{1/p}\mathrm{d}r$$
$$\le \frac{1}{(3-q)^{1/q}}\cdot\int_S\mathrm{d}s\left[\left(\int_0^R\xi^2\left|\boldsymbol{v}(\xi s)\right|^p\mathrm{d}\xi\right)^{1/p}\right.$$
$$\left.\times\int_0^R\frac{r^{1/q}\theta_R(r)}{\left(1+r^2\right)^{\alpha/q}}\cdot r^{2/q}\left(1+r^2\right)^{\alpha/q}\left|\mathrm{rot}\boldsymbol{u}(rs)\right|\mathrm{d}r\right].$$

Applying Hölder's inequality to the second inner integral, we have:

$$\left|I_{1,1}\right| \le \frac{1}{(3-q)^{1/q}}\int_S\mathrm{d}s\left(\int_0^R\xi^2\left|\boldsymbol{v}(\xi s)\right|^p\mathrm{d}\xi\right)^{1/p}$$
$$\times\left(\int_0^R\frac{r^{p-1}\theta_R^p(r)}{\left(1+r^2\right)^{\alpha(p-1)}}\mathrm{d}r\right)^{1/p}\left(\int_0^R r^2\left(1+r^2\right)^{\alpha}\left|\mathrm{rot}\boldsymbol{u}(rs)\right|^q\mathrm{d}r\right)^{1/q}.$$

We can write the estimation as

$$\int_0^R\frac{r^{p-1}\theta_R^p(r)}{\left(1+r^2\right)^{\alpha(p-1)}}\mathrm{d}r \le \int_0^1\frac{r^{p-1}\mathrm{d}r}{\left(1+r^2\right)^{\alpha(p-1)}} + \int_1^R\frac{r^{p-1}\mathrm{d}r}{\left(1+r^2\right)^{\alpha(p-1)}}$$
$$\le \int_0^1 r^{p-1}\mathrm{d}r + \int_1^R\frac{r^{p-1}\mathrm{d}r}{r^{2\alpha(p-1)}}$$
$$= \frac{1}{p} + \frac{1}{2\alpha(p-1)-p}\left(1-R^{p-2\alpha(p-1)}\right).$$

It is obvious that if $\alpha > q/2$ an expression $p-2\alpha(p-1)<0$, then

$$\left|I_{1,1}\right|$$
$$\le C_{1,1}(\alpha,p,R)\left\|\left(1+|\cdot|^2\right)^{\alpha/q}\mathrm{rot}\boldsymbol{u}\right\|_{\{L_q(B_R)\}^3}\left\|\boldsymbol{v}\right\|_{\{L_p(B_R)\}^3},$$

$$C_{1,1}(\alpha,p,R)$$
$$= \left(\frac{2p-3}{p-1}\right)^{1-\frac{1}{p}}\left(\frac{1}{p}+\frac{1}{2\alpha(p-1)-p}\left(1-\frac{1}{R^{2\alpha(p-1)-p}}\right)\right)^{1/p}.$$

Let us estimate the integral $I_{1,2}$. It is evident the the $\mathrm{grad}\theta_R(|x|) = \frac{\mathrm{d}\theta_R(r)}{\mathrm{d}r}\cdot s$, where

$$\frac{\mathrm{d}\theta_R(r)}{\mathrm{d}r} = \begin{cases} -R/r^2, & R/2 < r < R, \\ 0, & \text{otherwise.}\end{cases}$$

Applying Hölder's inequality several times, we get

$$\left|I_{1,2}\right| \le \frac{1}{(3-q)^{1/q}}\left(\int_{R/2}^{R} r^{p-1}\left|\frac{\mathrm{d}\theta_R(r)}{\mathrm{d}r}\right|^p \mathrm{d}r\right)^{1/p}$$
$$\times\int_S \mathrm{d}s\left(\int_0^R \xi^2\left|\boldsymbol{v}(\xi s)\right|^p \mathrm{d}\xi\right)^{1/p}\left(\int_{R/2}^{R} r^2\left|\boldsymbol{u}(rs)\right|^q \mathrm{d}r\right)^{1/q}$$
$$\le \frac{1}{(3-q)^{1/q}}\cdot\left(\int_{R/2}^{R} r^{p-1}\left|\frac{\mathrm{d}\theta_R(r)}{\mathrm{d}r}\right|^p \mathrm{d}r\right)^{1/p}\cdot\|\boldsymbol{v}\|_{\{L_p(\mathbb{R}^3)\}^3}$$
$$\times\left(\int_S \mathrm{d}s\int_{R/2}^{R} r^2\left|\boldsymbol{u}(rs)\right|^q \mathrm{d}r\right)^{1/q}.$$

The following estimation is obvious

$$0<\left(\int_{R/2}^{R} r^{p-1}\left|\frac{\mathrm{d}\theta_R(r)}{\mathrm{d}r}\right|^p \mathrm{d}r\right)^{1/p} = \left(\frac{2^p-1}{p}\right)^{1/p}$$

Hense

$$\int_S \mathrm{d}s\int_{R/2}^{R} r^2\left|\boldsymbol{u}(rs)\right|^q \mathrm{d}r \to 0 \text{ when } r\to\infty < \infty.$$

then $\left|I_{1,2}\right| \to 0$ when $R\to\infty$.

Next we construct an estimation for integral I_2. We apply Hölder's inequality to $\int_0^r \xi^2\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi$.

$$\int_0^r \xi^2\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi$$
$$=\int_0^r \frac{\xi^{2/q}}{\left(1+\xi^2\right)^{\alpha/p}}\cdot\xi^{2/p}\left(1+\xi^2\right)^{\alpha/p}\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|\mathrm{d}\xi$$
$$\le\left(\int_0^r \frac{\xi^2}{\left(1+\xi^2\right)^{\alpha(q-1)}}\mathrm{d}\xi\right)^{1/q}$$
$$\times\left(\int_0^r \xi^2\left(1+\xi^2\right)^{\alpha}\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|^p \mathrm{d}\xi\right)^{1/p}.$$

So, we get an estimation for I_2:

$$\left|I_2\right| \le \int_S \mathrm{d}s\left(\int_0^R \xi^2\left(1+\xi^2\right)^{\alpha}\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|^p \mathrm{d}\xi\right)^{1/p}$$
$$\times\int_0^R\left(\int_0^r \frac{\xi^2}{\left(1+\xi^2\right)^{\alpha(q-1)}}\mathrm{d}\xi\right)^{1/q}\theta_R(r)\left|\boldsymbol{u}(rs)\right|\mathrm{d}r.$$

We use Hölder's inequality for the second integral again.

$$\left|I_2\right| \le \int_S \mathrm{d}s\left(\int_0^R \xi^2\left(1+\xi^2\right)^{\alpha}\left|\mathrm{div}\boldsymbol{v}(\xi s)\right|^p \mathrm{d}\xi\right)^{1/p}$$
$$\times\left(\int_0^R \frac{\theta_R^p(r)}{r^{2(p-1)}}\left(\int_0^r \frac{\xi^2}{\left(1+\xi^2\right)^{\alpha(q-1)}}\mathrm{d}\xi\right)^{p-1}\mathrm{d}r\right)^{1/p}$$
$$\times\left(\int_0^R r^2\left|\boldsymbol{u}(rs)\right|\mathrm{d}r\right)^{1/q}.$$

Let's estimate the following integral

$$\int_0^R \frac{\theta_R^p(r)}{r^{2(p-1)}}\left(\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}}\right)^{p-1}\mathrm{d}r$$
$$\le\int_0^1 \frac{1}{r^{2(p-1)}}\left(\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}}\right)^{p-1}\mathrm{d}r$$
$$+\int_1^R \frac{1}{r^{2(p-1)}}\left(\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}}\right)^{p-1}\mathrm{d}r.$$

$$\int_0^1 \frac{1}{r^{2(p-1)}}\left(\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}}\right)^{p-1}\mathrm{d}r$$
$$\le\int_0^1 \frac{1}{r^{2(p-1)}}\left(\int_0^r \xi^2\,\mathrm{d}\xi\right)^{p-1}\mathrm{d}r = \frac{1}{3^{p-1}p}.$$

Denote $\kappa = 2\alpha(q-1)$ and consider

$$J=\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}} = \int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{\kappa/2}}.$$

When $\kappa<3$ (*i.e.* $\alpha<\frac{3}{2}(p-1)$), then

$$J\le\int_0^r \xi^{2-\kappa}\mathrm{d}\xi = \frac{1}{3-\kappa}r^{3-\kappa}$$

and, respectively

$$\int_1^R \frac{1}{r^{2(p-1)}}J^{p-1}\mathrm{d}r \le \frac{1}{(3-\kappa)^{p-1}}\int_1^R \frac{\mathrm{d}r}{r^{(\kappa-1)(p-1)}}$$
$$=\frac{1}{(3-\kappa)^{p-1}(2\alpha-p)}\left(1-\frac{1}{R^{2\alpha-p}}\right) \le \frac{1}{(3-\kappa)^{p-1}(2\alpha-p)}.$$

If $\kappa>3$ we get

$$J\le\int_0^1 \xi^2\,\mathrm{d}\xi + \int_1^R \frac{\mathrm{d}\xi}{\xi^{\kappa-2}} = \frac{1}{3}+\frac{1}{\kappa-3}\left(1-\frac{1}{r^{\kappa-3}}\right) \le \frac{\kappa}{3(\kappa-3)},$$

$$\int_1^R \frac{1}{r^{2(p-1)}}J^{p-1}\mathrm{d}r \le \left(\frac{\kappa}{3(\kappa-3)}\right)^{p-1}\int_1^R \frac{\mathrm{d}r}{r^{2(p-1)}}$$
$$=\left(\frac{\kappa}{3(\kappa-3)}\right)^{p-1}\frac{1}{2p-3}\left(1-\frac{1}{R^{2p-3}}\right)$$
$$\le\frac{1}{2p-3}\left(\frac{\kappa}{3(\kappa-3)}\right)^{p-1}.$$

At last, when $\kappa=3$, then

$$J=\int_0^r \frac{\xi^2\,\mathrm{d}\xi}{\left(1+\xi^2\right)^{3/2}} \le \frac{1}{3}+\int_1^r \frac{\xi^{2+\frac{3-q}{2}}\mathrm{d}\xi}{\left(1+\xi^2\right)^{3/2}}$$
$$=\frac{1}{3}+\frac{2}{3-q}\left(r^{\frac{3-q}{2}}-1\right) \le \frac{2}{3-q}\cdot r^{\frac{3-q}{2}},$$

$$\int_1^R \frac{1}{r^{2(p-1)}} J^{p-1} \mathrm{d}r \le \left(\frac{2}{3-q}\right)^{p-1} \int_1^R \frac{\mathrm{d}r}{r^{(2p-1)/2}}$$

$$= \left(\frac{2}{3-q}\right)^{p} \left(1 - \frac{1}{R^{p-3/2}}\right) \le \left(\frac{2}{3-q}\right)^{p}.$$

Thus, we obtain

$$\int_0^R \frac{\theta_R^p(r)}{r^{2(p-1)}} \left(\int_0^r \frac{\xi^2 \mathrm{d}\xi}{\left(1+\xi^2\right)^{\alpha(q-1)}} \right)^{p-1} \mathrm{d}r \le C_2(\alpha, p),$$

and therefore

$$|I_2| \le \left(C_2(\alpha,p)\right)^{1/p} \int_S \mathrm{d}s \Bigg[\left(\int_0^R r^2 |\boldsymbol{u}(rs)| \mathrm{d}r \right)^{1/q}$$

$$\times \left(\int_0^R \xi^2 \left(1+\xi^2\right)^{\alpha} |\mathrm{div}\boldsymbol{v}(\xi s)|^p \mathrm{d}\xi \right)^{1/p} \Bigg]$$

$$\le \left(C_2(\alpha,p)\right)^{1/p} \|\boldsymbol{u}\|_{\{L_q(B_R)\}^3} \left\| \left(1+|\cdot|^2\right)^{\alpha/p} \mathrm{div}\boldsymbol{v} \right\|_{L_p(B_R)}.$$

Bringing together the constructed estimates, we derive the following inequality for integral (6)

$$\int_{B_R} \theta_R(|x|) \left(\boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\right) \mathrm{d}x$$

$$\le C(\alpha,p) \left(\left\| \left(1+|\cdot|^2\right)^{\alpha/q} \mathrm{rot}\boldsymbol{u} \right\|_{\{L_q(\mathbb{R}^3)\}^3} \|\boldsymbol{v}\|_{\{L_p(\mathbb{R}^3)\}^3} \right.$$

$$\left. + \left\| \left(1+|\cdot|^2\right)^{\alpha/p} \mathrm{div}\boldsymbol{v} \right\|_{L_p(\mathbb{R}^3)} \|\boldsymbol{u}\|_{\{L_q(\mathbb{R}^3)\}^3} \right),$$

where

$$C(\alpha,p) = \max\left\{ \lim_{R\to\infty} C_{1,1}(\alpha,p,R), C_2(\alpha,p) \right\}.$$

Going to the limit for $R \to \infty$ in the last inequality, we will obtain estimation (1).

Note, that for $1 < p < 3/2$ the theorem may be proved similarly using the Equivalence (2).

4. Discussion of the Stationary Problem of Electromagnetic Theory

As an example of using the estimations proved in Section 3, we will consider a problem of determining the magnetic field stretch $\boldsymbol{H}(x)$ in the whole $\mathbb{R}^3$ space with a bounded conducting subdomain.

Stationary electromagnetic field is described by the set of stationary Maxwell's equations

$$\mathrm{rot}\boldsymbol{H}(x) = \frac{4\pi}{c} \sigma(x) \left(\boldsymbol{E}(x) + \boldsymbol{E}^{\mathrm{ext}}(x)\right), \tag{7}$$

$$\mathrm{rot}\boldsymbol{E}(x) = 0, \tag{8}$$

$$\mathrm{div}\left(\mu(x)\boldsymbol{H}(x)\right) = 0, \tag{9}$$

$$\mathrm{div}\left(\varepsilon(x)\boldsymbol{E}(x)\right) = 4\pi\rho(x). \tag{10}$$

Here $x \in \mathbb{R}^3$. The conductivity of the atmosphere is denoted as $\sigma \in L_\infty\left(\mathbb{R}^3\right)$. Let Ω_σ denotes a bounded open star-shaped subset of $\mathbb{R}^3$ defined by conditions

$$\sigma(x) \ge \sigma_* > 0, \text{ for almost all } x \in \Omega_\sigma, \tag{11}$$

$$\sigma(x) = 0, \text{ for almost all } x \in \mathbb{R}^3 \setminus \Omega_\sigma. \tag{12}$$

Functions $\mu, \varepsilon \in L_\infty\left(\mathbb{R}^3\right)$ are permeability and permittivity. They satisfy the following conditions

$$0 < \mu_* \le \mu(x) \le \mu^*, \ x \in \mathbb{R}^3;$$

$$0 < \varepsilon_* \le \varepsilon(x) \le \varepsilon^* \text{ for } x \in \Omega_\sigma; \ \varepsilon(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega_\sigma.$$

The $\boldsymbol{E}^{\mathrm{ext}} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3$ is a vector-function of the external electromotive force, which is asumed given and satisfying the condition

$$\boldsymbol{E}^{\mathrm{ext}}(x) = 0 \text{ for almost all } x \in \mathbb{R}^3 \setminus \Omega_\sigma.$$

Function $\rho \in L_2\left(\mathbb{R}^3\right)$ equals zero for almost all $x \in \mathbb{R}^3 \setminus \Omega_\sigma$.

We introduce the necessary functional spaces

$$\mathrm{Ker}\left(\mathrm{div};\mathbb{R}^3\right) = \left\{ \boldsymbol{u} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3 : \mathrm{div}\boldsymbol{u} = 0 \right\},$$

$$\mathrm{Ker}\left(\mathrm{rot};\mathbb{R}^3\right) = \left\{ \boldsymbol{u} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3 : \mathrm{rot}\boldsymbol{u} = 0 \right\},$$

$$\mathrm{Ker}\left(\mathrm{div}\mu;\mathbb{R}^3\right) = \left\{ \boldsymbol{u} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3 : \mu\boldsymbol{u} \in \mathrm{Ker}\left(\mathrm{div};\mathbb{R}^3\right) \right\},$$

$$V\left(\mathrm{rot};\Omega_\sigma\right) = \left\{ \boldsymbol{u} \in \left\{W_2^1\left(\mathbb{R}^3\right)\right\}^3 : \mathrm{rot}\boldsymbol{u} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3, \right.$$

$$\left. \mathrm{rot}\boldsymbol{u}(x) = 0 \text{ for almost all } x \in \mathbb{R}^3 \setminus \Omega_\sigma \right\}.$$

Denote $U\left(\Omega_\sigma\right) \equiv \mathrm{Ker}\left(\mathrm{div}\mu;\mathbb{R}^3\right) \cap V\left(\mathrm{rot};\Omega_\sigma\right)$. It is readily proved that this functional space will be Hilbert space relatively to scalar product

$$\left(\boldsymbol{u}\cdot\boldsymbol{v}\right)_{U(\Omega_\sigma)} = \left(\boldsymbol{u}\cdot\boldsymbol{v}\right)_{\{L_2(\mathbb{R}^3)\}^3} + \left(\mathrm{rot}\boldsymbol{u}\cdot\mathrm{rot}\boldsymbol{v}\right)_{\{L_2(\mathbb{R}^3)\}^3}.$$

We name the solution of the Problem (7)-(10) the functions $\boldsymbol{H} \in U\left(\Omega_\sigma\right)$, $\boldsymbol{E} \in \mathrm{Ker}\left(\mathrm{rot};\mathbb{R}^3\right)$ and $\rho \in H^{-1}\left(\mathbb{R}^3\right)$ satisfying condition $\rho(x) = 0$ for almost all $x \in \mathbb{R}^3 \setminus \Omega_\sigma$.

The validity of (10) implies the distibution ρ for all $\varphi \in \mathcal{D}\left(\Omega_\sigma\right)$ defined by the formula

$$\langle \rho, \varphi \rangle = -\frac{1}{4\pi} \int_{\Omega_\sigma} \varepsilon(x) \left(\boldsymbol{E}(x) \cdot \mathrm{grad}\varphi(x)\right) \mathrm{d}x. \tag{13}$$

Equation (7) in conducting subdomain will be

$$\frac{c}{4\pi} \sigma^{-1} \cdot \mathrm{rot}\boldsymbol{H} = \boldsymbol{E} + \boldsymbol{E}^{\mathrm{ext}},$$

and in nonconducting subdomain ($\mathbb{R}^3 \setminus \Omega_\sigma$) it becomes an identity.

Multiplying the last equation by $\mathrm{rot}\boldsymbol{\psi}$, $\boldsymbol{\psi} \in U\left(\Omega_\sigma\right)$, integrating along Ω_σ, and using $\boldsymbol{E} \in \mathrm{Ker}\left(\mathrm{rot};\mathbb{R}^3\right)$ or $\int_{\Omega_\sigma}\left(\boldsymbol{E}(x)\cdot \mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x = \int_{\mathbb{R}^3}\left(\boldsymbol{E}(x)\cdot \mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x = 0$.

It becomes obvious that the problem of determining the stationary magnetic field can be formulated as follows:

Determine vector-function $\boldsymbol{H} \in U\left(\Omega_\sigma\right)$ satisfying the integral identity

$$\begin{aligned}&\frac{c}{4\pi}\int_{\Omega_\sigma}\left(\sigma^{-1}(x)\mathrm{rot}\boldsymbol{H}(x)\cdot\mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x\\&=\int_{\Omega_\sigma}\left(\boldsymbol{E}^{\mathrm{ext}}(x)\cdot\mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x\end{aligned}\tag{14}$$

for all functions $\boldsymbol{\psi} \in U\left(\Omega_\sigma\right)$.

We need the following statement to prove the theorem of solvability (Theorem 4.2) for the Problem (14).

Lemma 4.1 (Lax-Milgram [13]). *Let V be a Hilbert space over the field of real numbers. Let $a(\cdot,\cdot): V\times V \to \mathbb{R}$ be a symmetric bilinear bounded coercive form, $f(\cdot): V\to\mathbb{R}$ —linear bounded functional. Then there exists a unique element $u\in V$ satisfying the equality*

$$a(u,v) = f(v)$$

for all $v \in V$.

Theorem 4.2 (Solvability of the Problem (14)). *Let $\sigma \in L_\infty\left(\mathbb{R}^3\right)$ satisfy* (11), (12), *$\boldsymbol{E}^{\mathrm{ext}} \in \left\{L_2\left(\mathbb{R}^3\right)\right\}^3$ and $\boldsymbol{E}^{\mathrm{ext}}(x) = 0$ for almost all $x \in \mathbb{R}^3 \setminus \Omega_\sigma$. Then the solution $\boldsymbol{H} \in U\left(\Omega_\sigma\right)$ of the generalized Problem* (14) *exists and is unique.*

Proof. Let's verify the conditions of the Lax-Milgram lemma.

Let us denote

$$a(\boldsymbol{H},\boldsymbol{\psi}) = \frac{c}{4\pi}\int_{\Omega_\sigma}\left(\sigma^{-1}(x)\mathrm{rot}\boldsymbol{H}(x)\cdot\mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x,$$

$$f(\boldsymbol{\psi}) = \int_{\Omega_\sigma}\left(\boldsymbol{E}^{\mathrm{ext}}(x)\cdot\mathrm{rot}\boldsymbol{\psi}(x)\right)\mathrm{d}x.$$

Obviously, $a(\cdot,\cdot)$ is a bilinear and symmetric form. The finiteness is easily proved by condition (11):

$$\begin{aligned}\left|a(\boldsymbol{u},\boldsymbol{v})\right| &= \frac{c}{4\pi}\left|\int_{\Omega_\sigma}\left(\sigma^{-1}(x)\mathrm{rot}\boldsymbol{u}(x)\cdot\mathrm{rot}\boldsymbol{v}(x)\right)\mathrm{d}x\right|\\&\le \frac{c}{4\pi\sigma_*}\left|\int_{\Omega_\sigma}\left(\mathrm{rot}\boldsymbol{u}(x)\cdot\mathrm{rot}\boldsymbol{v}(x)\right)\mathrm{d}x\right|.\end{aligned}$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}&\left|a(\boldsymbol{u},\boldsymbol{v})\right|\\&\le \frac{c}{4\pi\sigma_*}\left(\int_{\Omega_\sigma}\left|\mathrm{rot}\boldsymbol{u}(x)\right|^2\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_\sigma}\left|\mathrm{rot}\boldsymbol{v}(x)\right|^2\mathrm{d}x\right)^{1/2}\\&\le \frac{c}{4\pi\sigma_*}\left\|\boldsymbol{u}\right\|_{U(\Omega_\sigma)}\cdot\left\|\boldsymbol{v}\right\|_{U(\Omega_\sigma)}.\end{aligned}$$

Let's show coercivity of the form $a(\cdot,\cdot)$.

Whereas $\boldsymbol{H} \in V\left(\mathrm{rot};\mathbb{R}^3\right)$ thus $\boldsymbol{H} \in H^\alpha\left(\mathrm{rot};\mathbb{R}^3\right)$ for each $\alpha > 1$, *i.e.* the vector-function $\boldsymbol{H}(x)$ satisfies estimation

$$\left\|\boldsymbol{H}\right\|^2 \le \frac{1}{\mu_*}\left(\boldsymbol{H}\cdot\mu\boldsymbol{H}\right)_{\left\{L_2\left(\mathbb{R}^3\right)\right\}^3}.$$

The following notation is used $\|\cdot\| = \|\cdot\|_{\left\{L_2\left(\mathbb{R}^3\right)\right\}^3}$. Let's use Estimation (1)

$$\begin{aligned}\left\|\boldsymbol{H}\right\|^2 \le \frac{C(\alpha,2)}{\mu_*}\cdot&\left(\left\|\left(1+|\cdot|^2\right)^{\alpha/2}\mathrm{rot}\boldsymbol{H}\right\|\cdot\left\|\mu\boldsymbol{H}\right\|\right.\\&\left.+\left\|\boldsymbol{H}\right\|\cdot\left\|\left(1+|\cdot|^2\right)^{\alpha/2}\mathrm{div}\,\mu\boldsymbol{H}\right\|_{L_2\left(\mathbb{R}^3\right)}\right).\end{aligned}$$

Since $\boldsymbol{H} \in W\left(\Omega_\sigma\right)$, the last summand is zero. Then using the Hölder's inequality, we obtain

$$\left\|\boldsymbol{H}\right\|^2 \le \frac{C(\alpha,2)}{\mu_*}\mathrm{esssup}\,\mu\cdot\left\|\left(1+|\cdot|^2\right)^{\alpha/2}\mathrm{rot}\boldsymbol{H}\right\|\cdot\left\|\boldsymbol{H}\right\|.$$

Hense

$$\left\|\boldsymbol{H}\right\|_{\left\{L_2\left(\mathbb{R}^3\right)\right\}^3} \le \frac{C(\alpha,2)}{\mu_*}\mathrm{esssup}\,\mu\cdot\left(1+R^2\right)^{\frac{\alpha}{2}}\left\|\mathrm{rot}\boldsymbol{H}\right\|_{\left\{L_2\left(\Omega_\sigma\right)\right\}^3},$$

where $R = \mathrm{diam}\Omega_\sigma$. The estimates show the coercivity of the bilinear form, because

$$\begin{aligned}\left\|\boldsymbol{H}\right\|^2_{V\left(\mathrm{rot};\mathbb{R}^3\right)} &\le \left(\left(\frac{C(\alpha,2)}{\mu_*}\mathrm{esssup}\,\mu\right)^2\cdot\left(1+R^2\right)^\alpha+1\right)\\&\quad\times\int_{\Omega_\sigma}\left(\mathrm{rot}\boldsymbol{H}(x)\cdot\mathrm{rot}\boldsymbol{H}(x)\right)\mathrm{d}x\\&\le\frac{4\pi}{c}\left(\left(\frac{C(\alpha,2)}{\mu_*}\mathrm{esssup}\,\mu\right)^2\left(1+R^2\right)^\alpha+1\right)\\&\quad\times\mathrm{esssup}\,\sigma\cdot a(\boldsymbol{H},\boldsymbol{H}).\end{aligned}$$

Now we verify the conditions for functional $f(\cdot)$. The linearity is obvious. Let's show the finiteness using the Cauchy-Bunyakovsky-Schwarz inequality

$$\begin{aligned}\left|f(\boldsymbol{\psi})\right| &\le \left(\int_{\Omega_\sigma}\left|\boldsymbol{E}^{\mathrm{ext}}(x)\right|^2\mathrm{d}x\right)^{1/2}\cdot\left(\int_{\Omega_\sigma}\left|\mathrm{rot}\boldsymbol{\psi}(x)\right|^2\mathrm{d}x\right)^{1/2}\\&\le\left\|\boldsymbol{E}^{\mathrm{ext}}\right\|_{\left\{L_2\left(\mathbb{R}^3\right)\right\}^3}\cdot\left\|\mathrm{rot}\boldsymbol{\psi}\right\|_{U(\Omega_\sigma)}.\end{aligned}$$

Thus, all the constraints of the Lax-Milgram lemma are satisfied, and the solution of the Problem (14) exists and is unique.

Remark. *The solvability of the studied problem is also true when σ is a positive-definite tensor. The scheme of the proof is similar to Theorem* 4.2.

Let $\boldsymbol{H} \in U\left(\Omega_\sigma\right)$ satisfies relation (14) for all $\boldsymbol{\psi} \in U\left(\Omega_\sigma\right)$. Let's show that other indefinite functions will be defined in Ω_σ from Equaitons (7)-(10) as values depending on $\boldsymbol{H}$.

We determine function $\boldsymbol{E}$ in the conductivity area by equality

$$\boldsymbol{E} = \frac{c}{4\pi\sigma}\operatorname{rot}\boldsymbol{H} - \boldsymbol{E}^{\text{ext}}.$$

Let $\boldsymbol{v} \in \{\mathcal{D}(\Omega_\sigma)\}$. Let's extend $\boldsymbol{v}$ by zero in $\mathbb{R}^3$. According to the Lax-Milgram lemma there is the unique function $\boldsymbol{g} \in \operatorname{Ker}(\operatorname{rot};\mathbb{R}^3)$, which for each $\boldsymbol{\eta} \in \operatorname{Ker}(\operatorname{rot};\mathbb{R}^3)$ satisfies the equality

$$\int_{\mathbb{R}^3}(\mu\boldsymbol{g}\cdot\boldsymbol{\eta})\,\mathrm{d}x = \int_{\Omega_\sigma}(\mu\boldsymbol{v}\cdot\boldsymbol{\eta})\,\mathrm{d}x.$$

Then $\boldsymbol{\psi} = \boldsymbol{v} - \boldsymbol{g} \in \operatorname{Ker}(\operatorname{rot};\mathbb{R}^3)$ and as $\operatorname{rot}\boldsymbol{\psi} = \operatorname{rot}\boldsymbol{v}$, then $\boldsymbol{\psi} \in U(\Omega_\sigma)$. Therefore we obtain that

$$\int_{\Omega_\sigma}(\boldsymbol{E}\cdot\operatorname{rot}\boldsymbol{v})\,\mathrm{d}x = \int_{\Omega_\sigma}(\boldsymbol{E}\cdot\operatorname{rot}\boldsymbol{\psi})\,\mathrm{d}x$$

$$= \frac{c}{4\pi}\int_{\Omega_\sigma}\left(\sigma^{-1}\operatorname{rot}\boldsymbol{H}\cdot\operatorname{rot}\boldsymbol{\psi}\right)\mathrm{d}x - \int_{\Omega_\sigma}\left(\boldsymbol{E}^{\text{ext}}\cdot\operatorname{rot}\boldsymbol{\psi}\right)\mathrm{d}x = 0.$$

This shows that $\boldsymbol{E} \in \operatorname{Ker}(\operatorname{rot};\Omega_\sigma)$.

The function $\rho \in H^{-1}(\Omega_\sigma)$ is defined by relation (13) as shown above.

5. Conclusion

The paper was devoted to the proof of L_p-estimation of vector fields in weighted functional spaces. Also we discussed a solvability of the problem of determinig the magnetic field stretch in the whole $\mathbb{R}^3$ space. The proof of solvability is based on the proved estimation.

6. Acknowledgements

This work was supported by Analytical Departmental Program "Highschool scientific potential growth" (2009-2011) Russian Ministry of Education and Science (reg. no. 2.1.1/3927), Federal Target Program "Scientific and Scientific-Pedagogical Personnel of Innovative Russia" (2009-2013) (project NK-13P-13) and RFBR Grant (project 09-01-97019-r_povolzhie_a).

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