*Applied Mathematics*, 2011, 2, 1479-1485
doi:10.4236/am.2011.212210 Published Online December 2011 (http://www.SciRP.org/journal/am)

# Numerical Solution of Nonlinear Klein-Gordon Equation Using Lattice Boltzmann Method

**Qiaojie Li[1], Zong Ji[2], Zhoushun Zheng[1], Hongjuan Liu[1]**
[1]*School of Mathematical Science and Computing Technology, Central South University, Changsha, China*
[2]*Modern Service and Trade College, Yunnan University of Finance and Economics, Kunming, China*
*E-mail: qiaojie_li@foxmail.com*
*Received October* 24, 2011; *revised November* 24, 2011; *accepted December* 1, 2011

## Abstract

In this paper, in order to extend the lattice Boltzmann method to deal with more nonlinear equations, a one-dimensional (1D) lattice Boltzmann scheme with an amending function for the nonlinear Klein-Gordon equation is proposed. With the Taylor and Chapman-Enskog expansion, the nonlinear Klein-Gordon equation is recovered correctly from the lattice Boltzmann equation. The method is applied on some test examples, and the numerical results have been compared with the analytical solutions or the numerical solutions reported in previous studies. The $L_2$, $L_\infty$ and Root-Mean-Square (RMS) errors in the solutions show the efficiency of the method computationally.

**Keywords:** Lattice Boltzmann, Chapman-Enskog Expansion, Nonlinear Klein-Gordon Equation

## 1. Introduction

Nonlinear phenomena modeled by partial differential equation appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. The nonlinear Klein-Gordon equation has attracted much attention in studying solutions and condensed mater physics, in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [1,2]. In the last few decades, many powerful methods, such as the inverse scattering method, Baklund transformation, the auxiliary equation method [3,4], the Wadati trace method, Hirota bilinear forms, the tanh-sech method, the sine-cosine method, Jacobi elliptic functions, and the Riccati equation expansion method were used to investigate these types of equations (see [5] and references therein). A variety of finite difference scheme have been presented (see [6] and references therein) and the alternative approaches using spectral and pseudo-spectral methods have recently been presented [7,8]. To avoid the mesh generation, meshless techniques have attracted the attention of researchers in recent years. The radial basis function (RBF) as a truly meshless method was used to solve nonlinear Klein-Gordon equation in [9].

Recently, unlike convectional numerical methods which search for the macroscopic equation, the lattice Boltzmann method (LBM) has achieved much success in studying nonlinear equations and the evolution of complex systems [10,11]. By choosing appropriate collision or equilibrium distribution, the lattice Boltzmann model is able to recover the PDE of interest. This method is a new technique based on a mesoscopic kinetic equation for the particle distribution functions. Compared with the conventional numerical methods, the LBM provides many of the advantages, including geometrical flexibility, clear physical pictures, ease in incorporating complex boundary conditions, simplicity of programming and numerical efficiency. Recently, it has been developed to simulate linear and nonlinear PDE such as Laplace equation [12], Poisson equation [13,14], the shallow water equation [15], Burgers equation [16], Korteweg-de Vires equation [17], Wave equation [18,19], reaction-diffusion equation [20,21], convection-diffusion equation [22-24].

In this paper, the initial-value problem of the one-dimensional nonlinear Klein-Gordon equations is given by the following equation,

$$u_{tt} + \alpha u_{xx} + g(u) = f(x,t) \tag{1}$$

where $u = u(x,t)$ represents the wave displacement at position $x$ and time $t$, $\alpha$ is a known constant and $g(u)$ is the nonlinear force.

The present work is motivated by the desire to extend

Copyright © 2011 SciRes. 

the lattice Boltzmann method to deal with evolution models characterized by nonlinear wave dispersion. By using Taylor expansion and the Chapman-Enskog expansion, the second-order nonlinear Klein-Gordon equation can be recovered from the present model correctly. The local equilibrium distribution function and the amending function are obtained. To make a comparison between numerical solutions and analytical ones, four Klein-Gordon equations with quadratic or cubic nonlinearity are considered. From the simulations, we find that the numerical results are in excellent agreement with the analytical solutions. This indicates that the present method is an efficient and flexible approach for practical application.

The organization of the paper as follows. In Section 2, the lattice Boltzmann model is described. Numerical examples are simulated in Section 3. Summary and conclusion are presented in Section 4.

## 2. The Lattice Boltzmann Model

The lattice Boltzmann model used on this study is the three-velocity lattice Bhatnagar-Gross-Krook (LBGK) model. The directions of the discrete velocity are defined as $c_i$ $(i=0,1,2)$

$$[c_0, c_1, c_2] = [0, -c, c].$$

where $c$ is a constant. The lattice Boltzmann equation with an amending function is given as follow

$$\begin{aligned} & f_i\left(x+c_i\Delta t, t+\Delta t\right) - f_i(x,t) \\ & = -\frac{1}{\tau}\left[f_i(x,t) - f_i^{eq}(x,t)\right] + \Delta t F_i(x,t) \end{aligned} \tag{2}$$

where $f_i(x,t)$ and $f_i^{eq}(x,t)$ are defined as the distribution and equilibrium distribution function, respectively. $F_i(x,t)$ is an amending function and $\tau$ is the dimensionless relaxation time. $\Delta x = c_i \Delta t$ and $\Delta t$ are the lattice spacing and time step, respectively.

Unlike for the normal LBM, the first derivative of the macroscopic variable $u(x,t)$ meets the following conservation laws

$$\sum_i f_i(x,t) = \sum_i f_i^{eq}(x,t) = \frac{\partial u(x,t)}{\partial t} \tag{3}$$

Then, through choosing appropriate local equilibrium distributions, we can retrieve the corresponding macroscopic equation correctly.

Indeed, applying the Taylor expansion to left-hand side of Equation (2) and retaining terms up to $\mathcal{O}(\Delta t^3)$, we get

$$\begin{aligned} & \Delta t\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)f_i + \frac{\Delta t^2}{2}\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)^2 f_i \\ & + \mathcal{O}(\Delta t^3) = -\frac{1}{\tau}\left(f_i - f_i^{eq}\right) + \Delta t F_i \end{aligned} \tag{4}$$

The macroscopic equation can be recovered in the multi-scale analysis using a small expansion parameter $\varepsilon$ which is proportional to the ration of the lattice spacing to the characteristic macroscopic length. To do this, the Chapman-Enskog expansion in time and space is applied:

$$\begin{aligned} & f_i = f_i^{eq} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)},\ F_i = \varepsilon^2 F_i^{(2)} \\ & \frac{\partial}{\partial t} = \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2},\ \frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial x_1} \end{aligned} \tag{5}$$

where $f_i^{(k)}$ and $F_i^{(2)}$ are the non-equilibrium distribution functions and non-equilibrium amending function, which satisfy the solvability conditions

$$\begin{aligned} & \sum_i f_i^{(k)} = 0\ (k \ge 1) \\ & \sum_i F_i^{(2)} = F^{(2)} \end{aligned} \tag{6}$$

Substituting Equation (5) into (4), we have

$$\begin{aligned} & \left(\varepsilon c_i\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}\right)\left(f_i^{eq} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)}\right) \\ & + \frac{\Delta t}{2}\left(\varepsilon c_i\frac{\partial}{\partial x_1} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2}\right)^2\left(f_i^{eq} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)}\right) \\ & = -\frac{1}{\tau}\left(\varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)}\right) + \varepsilon^2 F_i^{(2)} \end{aligned} \tag{7}$$

Comparing the two sides of Equation (7) and treating terms in order of $\varepsilon$ and $\varepsilon^2$ gives

$$\mathcal{O}(\varepsilon):\ \left(c_i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)f_i^{eq} = -\frac{1}{\tau\Delta t}f_i^{(1)} \tag{8}$$

$$\begin{aligned} \mathcal{O}(\varepsilon^2):\ & \frac{\partial}{\partial t_2}f_i^{eq} + \left(c_i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)f_i^{(1)} \\ & + \frac{\Delta t}{2}\left(c_i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)^2 f_i^{eq} = -\frac{1}{\tau\Delta t}f_i^{(2)} + F_i^{(2)} \end{aligned} \tag{9}$$

Applying Equation (8) to the left side of Equation (9), we can rewrite Equation (9) as

$$\begin{aligned} & \frac{\partial}{\partial t_2}f_i^{eq} + \left(1 - \frac{1}{2\tau}\right)\left(c_i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right)f_i^{(1)} \\ & = -\frac{1}{\tau\Delta t}f_i^{(2)} + F_i^{(2)} \end{aligned} \tag{10}$$

In order to recover Equation (1), we must give appropriate local equilibrium distribution function. We choose $f_i^{eq}$ such that,

$$\begin{aligned} & \sum_i f_i^{eq} = \frac{\partial u(x,t)}{\partial t} \\ & \sum_i c_i f_i^{eq} = 0, \sum_i c_i c_i f_i^{eq} = c_s^2 u(x,t) \end{aligned} \tag{11}$$

Copyright © 2011 SciRes. 

where $c_s^2 = c^2/3$ is called the lattice Boltzmann sound speed. Equation (11) leads to three linear equations for $f_i^{eq}(x,t)$. Solving these equations determines the equilibrium distribution functions

$$f_0^{eq}(x,t) = \frac{\partial u(x,t)}{\partial t} - \frac{u(x,t)}{3}$$
$$f_1^{eq}(x,t) = \frac{u(x,t)}{6} \tag{12}$$
$$f_2^{eq}(x,t) = \frac{u(x,t)}{6}$$

Meanwhile, the amending function $F_i(x,t)$ is taken as

$$F_i(x,t) = \omega_i F(x,t) = \omega_i\left(f(x,t) - g(u)\right) \tag{13}$$

such that $\sum_i F_i(x,t) = F(x,t)$. For simplicity, only one case is given here

$$F_0(x,t) = \frac{2}{3}\left(f(x,t) - g(u)\right)$$
$$F_1(x,t) = \frac{1}{6}\left(f(x,t) - g(u)\right) \tag{14}$$
$$F_2(x,t) = \frac{1}{6}\left(f(x,t) - g(u)\right)$$

Summing Equation (8) and Equation (10) over *i*, and using Equation (6) and (11), we obtain

$$\frac{\partial}{\partial t_1}\left(\frac{\partial u}{\partial t}\right) = 0 \tag{15}$$

$$\frac{\partial}{\partial t_2}\left(\frac{\partial u}{\partial t}\right) + \left(1 - \frac{1}{2\tau}\right)\frac{\partial}{\partial x_1}\left(\sum_i c_i f_i^{(1)}\right) = F^{(2)} \tag{16}$$

Using Equation (8) and (11), we get

$$\begin{aligned}\sum_i c_i f_i^{(1)} &= -\tau\Delta t \sum_i c_i\left(c_i \frac{\partial}{\partial x_1} + \frac{\partial}{\partial t_1}\right) f_i^{eq} \\ &= -\tau\Delta t \sum_i\left[\frac{\partial}{\partial x_1}(c_i c_i f_i^{eq}) + \frac{\partial}{\partial t_1}(c_i f_i^{eq})\right] \\ &= -\tau\Delta t \frac{\partial}{\partial x_1}(c_s^2 u)\end{aligned} \tag{17}$$

Then substituting Equation (17) into Equation (16), we have

$$\frac{\partial}{\partial t_2}\left(\frac{\partial u}{\partial t}\right) + c_s^2\Delta t\left(\frac{1}{2} - \tau\right)\frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) = F^{(2)} \tag{18}$$

When Equation $(15)\times\varepsilon + (18)\times\varepsilon^2$ is applied, the final nonlinear Klein-Gordon equation is recovered as

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial^2 u}{\partial x^2} = F(x,t) = f(x,t) - g(u) \tag{19}$$

where $\alpha = c_s^2\Delta t\left(\frac{1}{2} - \tau\right)$

In the computational process, in order to obtain *u*(*x*,*t*), we can apply backward difference to the item $\frac{\partial u(x,t)}{\partial t}$

$$\frac{\partial u(x,t)}{\partial t} = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} \tag{20}$$

## 3. Numerical Simulation Results

In this section, we present the result of our LBM numerical experiments for the relevant equations. In comparison with the analytical solutions and results derived by existing literature, the efficiency of proposed model is validated. The distribution function $f_i(x,t)$ is initialized with $f_i^{eq}(x,t)$ for all nodes at $t = 0$. The macroscopic variable $u(x,t)$ is initialized by the initial condition and the non-equilibrium extrapolation scheme proposed by Guo [25] is used for boundary treatment. The following error norms are used to measure the accuracy

1) $L_2$-error

$$L_2 - error = \left(\sum_{i=1}^{n} e_i^2\right)^{\frac{1}{2}}$$

2) $L_\infty$-error

$$L_\infty - error = \text{Max } e_i, \quad 1 \le i \le n$$

3) The root mean square (RMS) error

$$RMS - error = \left(\sum_{i=1}^{n}\frac{e_i^2}{n}\right)^{\frac{1}{2}}$$

where $e_i = \left|u(x_i,t) - u^*(x_i,t)\right|$, $u(x_i,t)$ and $u^*(x_i,t)$ are the numerical solution and analytical one.

Example 1. The Klein-Gordon equation with quadratic nonlinearity in the interval $-1 \le x \le 1$

$$u_{tt} - u_{xx} = -x\cos t + x^2\cos^2 t - u^2$$

The initial conditions are given by

$$u(x,0) = x,\ u_t(x,0) = 0$$

The exact solution is given in [9]

$$u(x,t) = x\cos t$$

We extract the boundary condition from the exact solution. In **Table 1**, the $L_\infty$, $L_2$ and RMS errors are obtained for $t = 1,3,5,7,10$. The graph of analytical and LBM solution for $t = 1$ and $t = 10$ are given in **Figure 1** and the space-time graph of the LBM solution is given in **Figure 2**.

Example 2. Consider the nonlinear Klein-Gordon equation with quadratic nonlinearity in interval $0 \le x \le 1$.

Copyright © 2011 SciRes. 

**Table 1. $L_\infty$, $L_2$ and RMS errors with dx = 0.02 and dt = 2 × $10^{-5}$.**

| t | errors | | |
|---|---|---|---|
| | $L_\infty$-error | $L_2$-error | RMS |
| 1 | 1.9558e–03 | 1.1135e–03 | 1.1294e–04 |
| 3 | 1.3664e–03 | 7.6676e–03 | 7.6295e–04 |
| 5 | 1.5260e–03 | 8.5602e–03 | 8.5178e–04 |
| 7 | 1.6201e–03 | 9.5926e–03 | 9.5450e–04 |
| 10 | 1.0465e–03 | 6.9848e–03 | 6.9501e–04 |

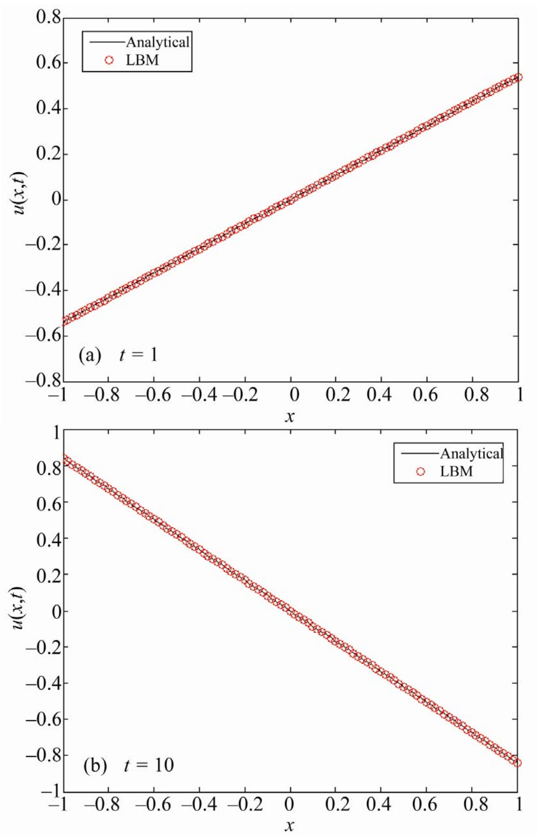


**Figure 1. Analytical and LBM solutions with dx = 0.02 and dt = 2 × $10^{-5}$ for different time.**

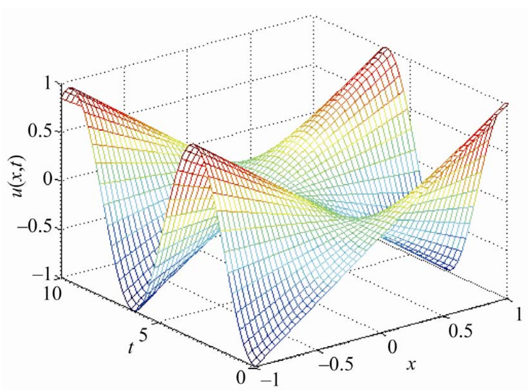


**Figure 2. Space-time graph of the LBM solutions up to *t* = 10 with dx = 0.02 and dt = 2 × $10^{-5}$.**

$$u_{tt} - u_{xx} = 6xt\left(x^2 - t^2\right) + x^6t^6 - u^2$$

The initial conditions are given by

$$u(x,0) = 0,\ u_t(x,0) = 0$$

The exact solution is given in [9]

$$u(x,t) = x^3t^3$$

The Boundary condition is determined by the analytical solution. In **Table 2**, the $L_\infty$, $L_2$ and RMS errors are obtained for $t = 1,2,3,4,5$. The graph of analytical and LBM solution for $t = 5$ and the space-time graph of the LBM solution are given in **Figure 3** and **Figure 4**, respectively.

Example 3. The nonlinear Klein-Gordon equation with cubic nonlinearity in interval $-1 \le x \le 1$.

$$u_{tt} + \alpha u_{xx} = -\beta u - \gamma u^3$$

We take $\alpha = -2.5, \beta = 1, \gamma = 1.5$ as the same in [9]. The initial conditions are given by

$$u(x,0) = B\tan(Kx),\ u_t(x,0) = BcK\sec^2(Kx)$$

The exact solution is

$$u(x,t) = B\tan\left(K\left(x + ct\right)\right)$$

where $B = \sqrt{\beta/\gamma}$ and $K = \sqrt{-\beta/2\left(\alpha + c^2\right)}$. In **Table 3**, the $L_\infty$, $L_2$ and RMS errors are obtained for two values of $c$ ($c = 0.5$ and $c = 0.05$) for $t = 1,2,3,4$. The graph of analytical and LBM solution for $t = 4$ and the space-time graph of the LBM solution for each value of $c$ are given in **Figure 5** and **Figure 6**, respectively.

Example 4. We consider the nonlinear Klein-Gordon equation with the form [9].

$$u_{tt} - u_{xx} = -u - u^3;\quad x \in [0, 1.28]$$

with initial data

$$u(x,0) = A\left[1 + \cos\left(\frac{2\pi x}{1.28}\right)\right],\ u_t(x,0) = 0$$

The boundary conditions are given by

$$u_x(0,t) = 0, u_x(1.28,t) = 0$$

**Table 2. $L_\infty$, $L_2$ and RMS errors with dx = 0.01 and dt = 5 × $10^{-5}$.**

| t | errors | | |
|---|---|---|---|
| | $L_\infty$-error | $L_2$-error | RMS |
| 1 | 5.8742e–04 | 1.9270e–03 | 1.9174e–04 |
| 2 | 4.6618e–03 | 2.1643e–02 | 2.1535e–03 |
| 3 | 1.5139e–02 | 4.9465e–02 | 4.9219e–03 |
| 4 | 3.4225e–02 | 8.5102e–02 | 8.4679e–03 |
| 5 | 6.3219e–02 | 9.3035e–02 | 1.2970e–02 |

Copyright © 2011 SciRes. 

**Table 3. $L_\infty$, $L_2$ and RMS errors with d$x$ = 0.01 and d$t$ = 5 × $10^{-5}$.**

| $t$ | errors | | |
|---|---|---|---|
| | $L_\infty$-error | $L_2$-error | RMS |
| $c$ = 0.5 | | | |
| 1 | 1.4189e–04 | 6.6508e–04 | 6.6171e–05 |
| 3 | 4.6601e–04 | 1.5438e–03 | 1.5362e–04 |
| 3 | 1.9445e–03 | 4.9588e–03 | 4.9342e–04 |
| 4 | 2.8219e–02 | 7.1870e–02 | 7.1513e–03 |
| $c$ = 0.05 | | | |
| 1 | 5.6970e–05 | 2.9718e–04 | 2.9570e–05 |
| 2 | 7.4878e–05 | 3.8699e–04 | 3.8507e–05 |
| 3 | 1.1972e–04 | 5.2203e–04 | 5.1944e–05 |
| 4 | 1.4008e–04 | 4.4143e–04 | 4.3924e–05 |

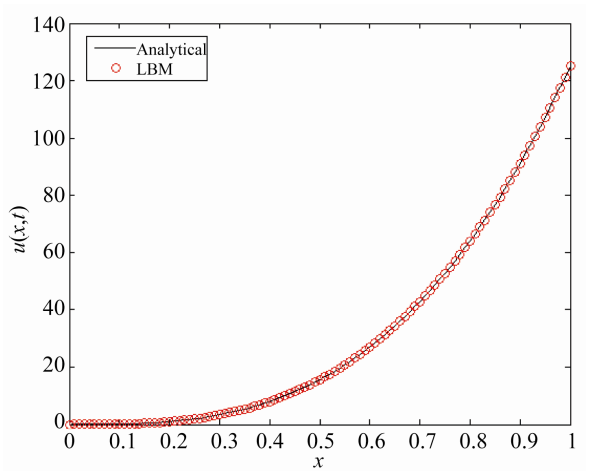


**Figure 3. Analytical and LBM solutions at $t$ = 5 with d$x$ = 0.01 and d$t$ = 5 × $10^{-5}$.**

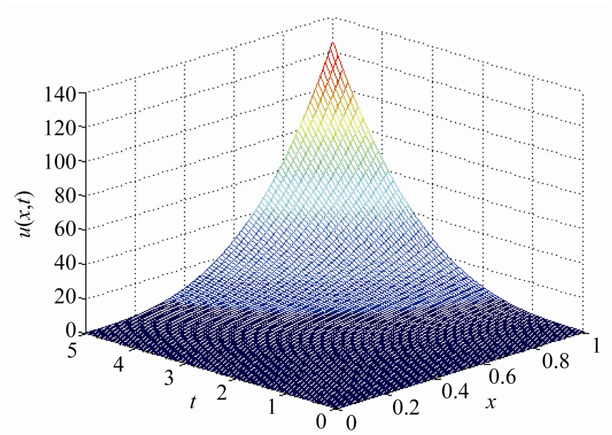


**Figure 4. Space-time graph of the LBM solutions up to $t$ = 5 with d$x$ = 0.01 and d$t$ = 5 × $10^{-5}$.**

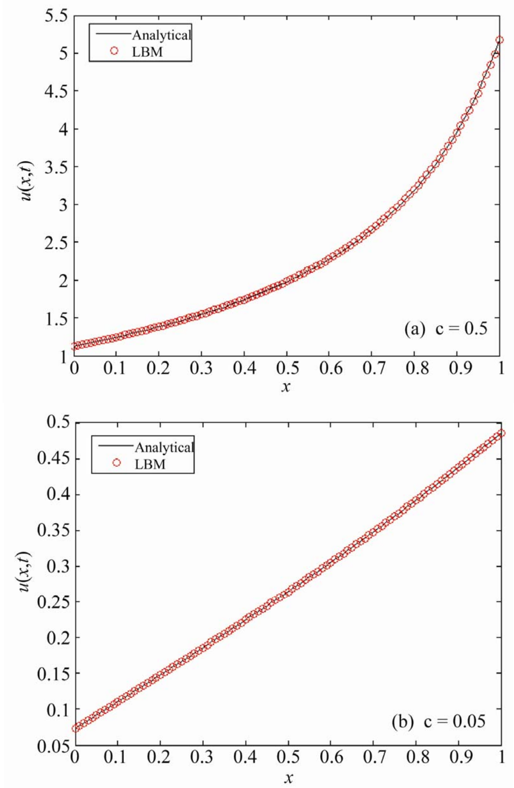


**Figure 5. Analytical and LBM solutions at $t$ = 4 with d$x$ = 0.01 and d$t$ = 5 × $10^{-5}$ for different $c$.**

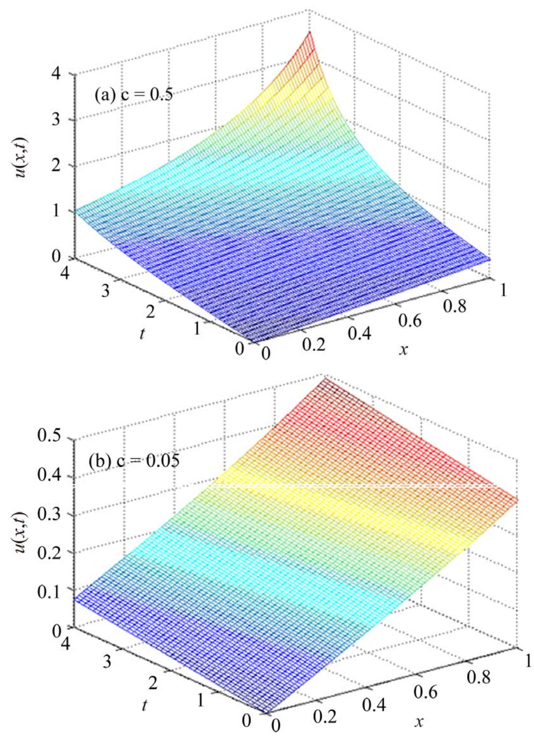


**Figure 6. Space-time graph of the LBM solutions up to $t$ = 4 with d$x$ = 0.01 and d$t$ = 5 × $10^{-5}$ for different $c$.**

Copyright © 2011 SciRes. 

For the above problem due to the periodic boundary conditions, the continuous solutions remain always symmetric with respect to the center of the spatial interval. Authors of [26] also studied this problem and found undesirable characteristics in some of the numerical schemes, in particular a loss of spatial symmetry and the onset of instability for larger values of the parameter A (amplitude) in the initial condition of the equation. We solved the above problem using lattice Boltzmann method for several values of $A$. In **Figure 7**, we show the approximate solutions for $A = 1$ with $\Delta x = 0.0128$ and $\Delta t = 1.8286 \times 10^{-5}$. **Figure 8** presents the approximate solutions for $A = 100$ with $\Delta x = 0.0128$ and $\Delta t = 1.8286 \times 10^{-5}$. From **Figure 7** and **Figure 8**, we can find that the spatial symmetry is kept for different amplitude $A$. It indicates that the present lattice Boltzmann method is comparable with other numerical schemes.

## 4. Conclusions

In the current study, a new lattice Boltzmann model is proposed to solve 1D nonlinear Klein-Gordon equation. The efficiency and accuracy of the proposed model are validated through detail numerical simulation with quadratic and cubic nonlinearity. It can be found that the LBGK results are in excellent agreement with the analytical solution. It should be point out that in order to attain better accuracy the lattice Boltzmann model requires a relatively small time step $\Delta t$ and the proper range is form $10^{-4}$ to $10^{-6}$. Detailed stability analysis of present model is needed in further study.

## 5. Acknowledgements

This work was supported by the National Natural Science Foundation of China (50874123, 51174236) and National Basic Research Program of China (2011CB606306).

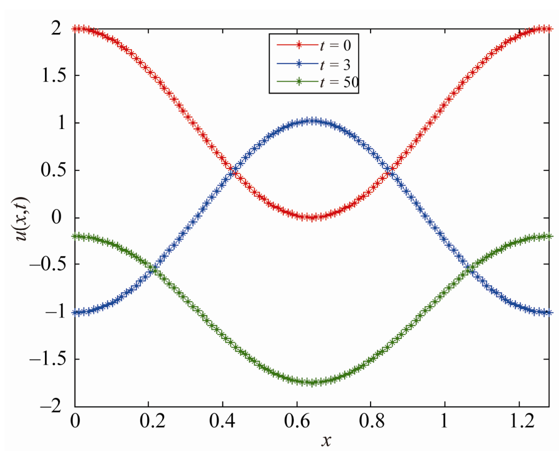


**Figure 7. LBM solution at *t* = 0, 3, 50 with *A* = 1, dx = 0.0128 and d*t* = 1.8286 × 10$^{-5}$.**

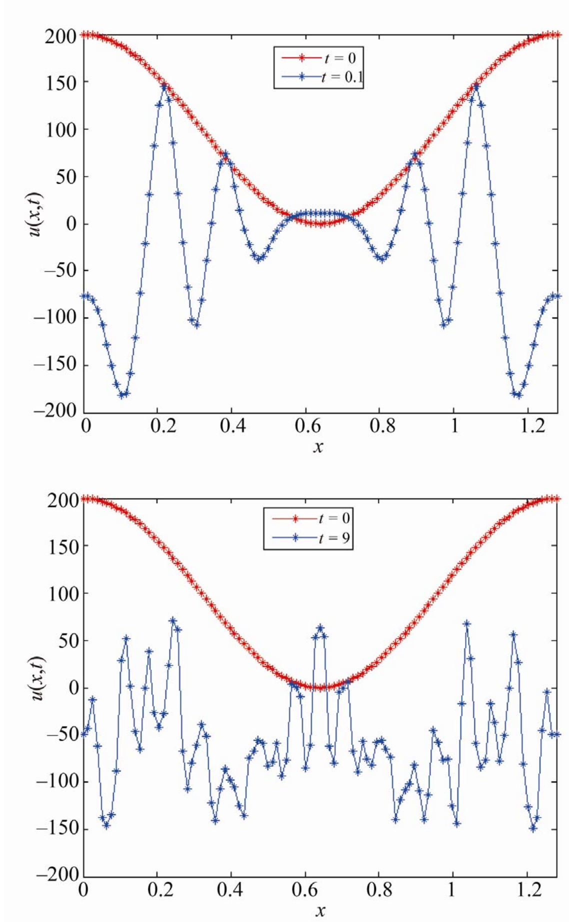


**Figure 8. LBM solution with *A* = 100 at different time *t*.**

## 6. References

[1] P. J. Caudrey, I. C. Eilbeck and J. D. Gibbon, "The Sine-Gordon Equation as a Model Classical Field Theory," *Nuovo Cimento*, Vol. 25, No. 2, 1975, pp. 497-511.

[2] R. K. Dodd, I. C. Eilbeck, J. D. Gibbon and H. C. Morris, "Solitons and Nonlinear Wave Equations," Academic, London, 1982.

[3] Sirendaoreji, "Auxiliary Equation Method and New Solutions of Klein-Gordon Equations," *Chaos*, *Solitons & Fractals*, Vol. 31, No. 4, 2007, pp. 943-950. doi:10.1016/j.chaos.2005.10.048

[4] Sirendaoreji, "A New Auxiliary Equation and Exact Travelling Wave Solutions of Nonlinear Equation," *Physics Letters A*, Vol. 356, No. 2, 2006, pp. 124-130. doi:10.1016/j.physleta.2006.03.034

[5] A. M. Wazwaz, "New Travelling Wave Solutions to the Boussinesq and Klein-Gordon Equations," *Communications in Nonlinear Science and Numerical Simulation*,

Copyright © 2011 SciRes. 

Vol. 13, No. 5, 2008, pp. 889-901.
doi:10.1016/j.cnsns.2006.08.005

[6] M. A. Lynch, "Large Amplitude Instability in Finite Difference Approximates to the Klein-Gordon Equation," *Applied Numerical Mathematics*, Vol. 31, No. 2, 1999, pp. 173-182. doi:10.1016/S0168-9274(98)00128-7

[7] X. Li, B. Y. Guo and L. Vazquez, "A Legendre Spectral Method for Solving the Nonlinear Klein-Gordon Equation," *Mathematics Applied and Computation*, Vol. 15, No. 1, 1996, pp. 19-36.

[8] X. Li and B. Y. Guo, "A Legendre Spectral Method for Solving Nonlinear Klein-Gordon Equation," *Journal Computation of Mathematics*, Vol. 15, No. 2, 1997, pp. 105-126.

[9] M. Deghan and A. Shokri, "Numerical Solution of the Nonlinear Klein-Gordon Equation Using Radial Basis Functions," *Journal of Computational and Applied Mathematics*, Vol. 230, No. 2, 2009, pp. 400-410.
doi:10.1016/j.cam.2008.12.011

[10] R. Benzi, S. Succi and M. Vergassola, "The Lattice Boltzmann Equation: Theory and Application," *Physics Reports*, Vol. 222, No. 3, 1992, pp. 145-197.
doi:10.1016/0370-1573(92)90090-M

[11] S. Y. Chen and G. D. Doolen, "Lattice Boltzmann Method for Fluid Flows," *Annual Review of Fluid Mechanics*, Vol. 30, No. 1, 1997, pp. 329-364.
doi:10.1146/annurev.fluid.30.1.329

[12] J. Y. Zhang, G. W. Yan and Y. F. Dong, "A New Lattice Boltzmann Model for the Laplace Equation," *Applied Mathematics and Computation*, Vol. 215, No. 2, 2009, pp. 539-547.doi:10.1016/j.amc.2009.05.047

[13] Z. H. Chai and B. C. Shi, "A Novel Lattice Boltzmann Model for the Poisson Equation," *Applied Mathematical Modelling*, Vol. 32, No. 10, 2008, pp. 2050-2058.
doi:10.1016/j.apm.2007.06.033

[14] M. Hirabayashi, Y. Chen and H. Ohashi, "The Lattice BGK Model for the Poisson Equation," *JSME International Journal Series B*, Vol. 44, No. 1, 2001, pp. 45-52.
doi:10.1299/jsmeb.44.45

[15] J. G. Zhou, "Lattice Boltzmann Method for Shallow Water Flows," Springer Verlag, New York, 2004.

[16] Z. Shen, G. Yuan and L. Shen, "Lattice Boltzmann Method for Burgers Equation," *Chinese Journal of Computational Physics*, Vol. 175, No. 1, 2000, pp. 172-177.

[17] J. Y. Zhang and G. W. Yan, "A Lattice Boltzmann Model for the Korteweg-de Vries Equation with Two Conservation Laws," *Computer Physics Communications*, Vol. 180, No. 7, 2009, pp. 1054-1062.
doi:10.1016/j.cpc.2008.12.027

[18] G. W. Yan, "A Lattice Boltzmann Equation for Waves," *Journal of Computational Physics*, Vol. 161, No. 1, 2000, pp. 61-69. doi:10.1006/jcph.2000.6486

[19] J. Y. Zhang, G. W. Yan and X. Shi, "Lattice Boltzmann Model for Wave Propagation," *Physics Review E*, Vol. 80, 2009, Article ID 026706.
doi:10.1103/PhysRevE.80.026706

[20] S. P. Dawson, S. Chen and G. D. Doolen, "Lattice Boltzmann Computations for Reaction-Diffusion Equation," *Journal of Chemical Physics*, Vol. 98, No. 2, 1993, pp. 1514-1523. doi:10.1063/1.464316

[21] X. Yu and B. C. Shi, "A Lattice Boltzmann Model for Reaction Dynamical Systems with Time Delay," *Applied Mathematics and Computation*, Vol. 181, No. 2, 2006, pp. 958-965. doi:10.1016/j.amc.2006.02.020

[22] S. R. Vander and M. Ernst, "Convection-Diffusion Lattice Boltzmann Scheme for Irregular Lattice," *Journal of Computational Physics*, Vol. 160, No. 2, 2000, pp. 766-782. doi:10.1006/jcph.2000.6491

[23] Z. L. Guo, B. C. Shi and N. C. Wang, "Fully Lagrangian and Lattice Boltzmann Method for the Advection-Diffusion Equation," *Journal of Scientific Computing*, Vol. 14, No. 3, 1999, pp. 291-300.
doi:10.1023/A:1023273603637

[24] B. C. Shi and Z. L. Guo, "Lattice Boltzmann Model for Nonlinear Convection-Diffusion Equations," *Physics Review E*, Vol. 79, 2009, Article ID 016701.
doi:10.1103/PhysRevE.79.016701

[25] Z. L. Guo, C. G. Zheng and B. C. Shi, "Non-Equilibrium Extrapolation Method for Velocity and Pressure Boundary Conditions in the Lattice Boltzmann Method," *Chinese Physics*, Vo. 11, No. 4, 2002, pp. 366-374.
doi:10.1088/1009-1963/11/4/310

[26] S. Jiminez and L. Vazquez, "Analysis of Four Numerical Scheme for a Nonlinear Klein-Gordon Equation," *Applied Mathematics and Computation*, Vol. 35, No. 1, 1990, pp. 61-94. doi:10.1016/0096-3003(90)90091-G

Copyright © 2011 SciRes. 