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Strong Convergence of an Iterative Method for Generalized Mixed Equilibrium Problems and Fixed Point Problems

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Abstract

In this paper, we introduce a hybrid iterative method for finding a common element of the set of common solutions of generalized mixed equilibrium problems and the set of common fixed points of an finite family of nonexpansive mappings. Furthermore, we show a strong convergence theorem under some mild conditions.

Keywords: Generalized Mixed Equilibrium Problem, Hybrid Iterative Scheme, Fixed Point, Nonexpansive Mapping, Strong Convergence

1. Introduction

Equilibrium problems theory provides us with a natural, novel and unified framework for studying a wide class of problems arising in economics, finance, transportation, network and structural analysis, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative.

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and $T: C \to 2^H$ a multivalued mapping. Let $\phi: C\times C \to R$ be a real-valued function and $\Theta: H\times C\times C \to R$ be an equilibrium-like function, *i.e.*, $\Theta(w,u,v)+\Theta(w,v,u)=0$ for each $(w,u,v)\in H\times C\times C$. The generalized mixed equilibrium problem (for short, GMEP) is to find $u\in C$ and $w\in T(u)$ such that

$$GMEP: \Theta(w,u,v)+\phi(v,u)-\phi(u,u)\geq 0, \forall v\in C. \quad (1.1)$$

in particular, if T is single-valued mapping, this problem is equivalent to finding $u\in C$ such that

$$\Theta(T(u),u,v)+\phi(v,u)-\phi(u,u)\geq 0, \forall v\in C. \quad (1.2)$$

Denote the set of solutions of GMEP by Ω.

Now, we recall the following definitions.

A mapping $f: C\to C$ is said to be contractive if there exists a constant $\alpha\in(0,1)$ such that $\|f(x)-f(y)\|\leq\alpha\|x-y\|$ for any $x,y\in C$. A mapping $g: C\to C$ is said to be firmly nonexpansive if $\|g(x)-g(y)\|^2\leq\langle g(x)-g(y), x-y\rangle$. A mapping $T: C\to C$ is said to be nonexpansive if $\|Tx-Ty\|\leq\|x-y\|$ for any $x,y\in C$. The set of fixed points of T is denoted by $F(T)$.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H and $\bigcap_{i=1}^N F(T_i)\neq\emptyset$. Define the mappings

$$\begin{cases} U_{n,1}=\lambda_{n,1}T_1+(1-\lambda_{n,1})I \\ U_{n,2}=\lambda_{n,2}T_2U_{n,1}+(1-\lambda_{n,2})I \\ \vdots \\ U_{n,N-1}=\lambda_{n,N-1}T_{N-1}U_{n,N-2}+(1-\lambda_{n,N-1})I \\ W_n=U_{n,N}=\lambda_{n,N}T_NU_{n,N-1}+(1-\lambda_{n,N})I \end{cases} \quad (1.3)$$

where $\{\lambda_{n,i}\}_{i=1}^N\subset(0,1]$ for all $n\geq 1$. Such a mapping W_n is called $W-mapping$ generated by $T_1,\cdots,T_N$ and $\{\lambda_{n,i}\}_{i=1}^N$.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Then, for any $x\in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x-P_C(x)\|\leq\|x-y\|$$

for all $y\in C$. Such a P_C is called the metric projection of H into C. We know that P_C is nonexpansive. What's more,

$$x^* = P_C(x) \Leftrightarrow \langle x - x^*, x^* - y \rangle \geq 0, \quad \forall y \in C.$$

Let C be a convex subset of a real Hilbert space H, $\eta : C \times C \to H$ and $k : C \to R$ a Frechet differential function. Then k is said to be η-strongly convex if there exists a constant $\mu > 0$ such that

$$k(y) - k(x) - \langle k'(x), \eta(y,x) \rangle \geq \frac{\mu}{2} \|x - y\|^2, \quad \forall x, y \in C.$$

If $\mu = 0$, then k is said to be η-convex. In particular, if $\eta(y,x) = y - x$ for all $y, x \in C$, then k is said to be strongly convex.

Let C be a nonempty subset of a real Hilbert space H. A bifunction $\phi(\cdot,\cdot) : C \times C \to R$ is said to be skew-symmetric if

$$\phi(u,v) + \phi(v,u) - \phi(u,u) - \phi(v,v) \leq 0, \quad \forall u, v \in C.$$

It is easy to see that if the skew-symmetric bifunction $\phi(\cdot,\cdot)$ is linear in both arguments, then

$$\phi(u,v) \geq 0, \forall u \in C.$$

We denote $\rightharpoonup$ for weak convergence and $\to$ for strong convergence. A bifunction $\phi : C \times C \to R$ is called weakly sequentially continuous at $(x_0, y_0) \in C \times C$ if $\phi(x_n, y_n) \to \phi(x_0, y_0)$ as $n \to \infty$ for each sequence $\{(x_n, y_n)\}$ in $C \times C$ converging weakly to (x_0, y_0). The function $\phi(\cdot,\cdot)$ is called weakly sequentially continuous on $C \times C$ if it is weakly sequentially continuous at each point of $C \times C$.

Let $CB(X)$ denote the set of nonempty closed bounded subset of X. For $A, B \in CB(X)$, define the Hausdorff metric $\hbar$ as follows:

$$\hbar(A,B) = \max\{sup_{a \in A} inf_{b \in B} d(a,b), \; sup_{b \in A} inf_{a \in B} d(b,a)\}.$$

In order to solve the generalized mixed equilibrium problems for an equilibrium-like bifunction $\Theta : H \times C \times C \to R$, we assume that Θ satisfies the following conditions with respect to the multivalued mapping $T : C \to 2^H$:

(Θ_1) for each fixed $v \in C$, $(w,u) \mapsto \Theta(w,u,v)$ is an upper semicontinuous function from $H \times C$ to R, that is, $w_n \to w$ and $u_n \to u$ imply $\limsup_{n\to\infty} \Theta(w_n, u_n, v) \leq \Theta(w,u,v)$;

(Θ_2) for each fixed $(w,v) \in H \times C$, $u \mapsto \Theta(w,u,v)$ is a concave function;

(Θ_3) for each fixed $(w,u) \in H \times C$, $v \mapsto \Theta(w,u,v)$ is a convex function;

(Θ_4) $\Theta(w_1, T_r(x), T_s(y)) + \Theta(w_2, T_s(y), T_r(x)) \leq -\gamma \|T_r(x) - T_s(y)\|^2$

for all $x, y \in C$ and $r, s \in (0,\infty)$, where $\gamma > 0$, $w_1 \in T(x)$ and $w_2 \in T(y)$.

Let $k : C \to R$ be a differential function with Frechet derivative $k'(x)$ at x satisfying the following:

(k_1) k' is continuous from the weak topology to the strong topology;

(k_2) k' is Lipschitz continuous with constant $\nu > 0$.

Let $\eta : C \times C \to H$ be a function satisfying the following:

(η_1) $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in C$;

(η_2) $\eta(\cdot,\cdot)$ is affine in the first coordinate variable;

(η_3) for each fixed $y \in C$, $y \in Cx \mapsto \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

Let C be a nonempty closed convex subset of a real Hilbert space and $T : C \to 2^H$ a multivalued mapping. For $x \in C$, let $w \in T(x)$. Let $\phi : C \to R$ be a real-valued function satisfying the following:

(ϕ_1) $\phi(\cdot,\cdot)$ is skew symmetric;

(ϕ_2) for each fixed $y \in C$, $\phi(\cdot, y)$ is convex and upper semicontinuous;

(ϕ_3) $\phi(\cdot,\cdot)$ is weakly continuous on $C \times C$.

Recently Wei-You Zeng, Nan-Jing Huang and Chang-Wen Zhao [1] introduce and consider a new class of equilibrium problems, which is known as the generalized mixed equilibrium problems. Furthermore, they introduce an iterative scheme (1.4) by the viscosity approximation method for finding a common element of the set of common solutions for generalized mixed equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings in Hilbert space.

$$\begin{cases} \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \hbar(T(x_n), T(x_{n+1})); \\ \Theta(w_n, u_n, v) + \phi(v, u_n) - \phi(u_n, u_n) + \frac{1}{r}\langle k'(u_n) - k'(x_n), \eta(v, u_n) \rangle \geq 0, \forall v \in C \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n u_n \end{cases} \tag{2.1}$$

Motivated and inspired by the research going on in this important field, we introduce the following hybrid iterative scheme (1.5) for finding a common element of the set of common solutions for generalized mixed equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings. We show that the approximation solution converges strongly to a unique solution of a class of variational inequalities under some mild conditions. Results obtained in this paper can be viewed as an improvement and refinement of the recent results in this direction.

Algorithm 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H, $T: C \to CB(H)$ be a multivalued mapping, f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let $W_n: C \to C$ be defined by (1.3), and $r > 0$. For given $x_1 \in C$ and $w_1 \in T(x_1)$, there exists sequences $\{x_n\}$, $\{u_n\}$ in C and $\{w_n : w_n \in T(x_n)\}$ in H such that for all $n = 1, 2, \cdots$,

$$\begin{cases} \|w_n - w_{n+1}\| \le \left(1+\frac{1}{n}\right)\hbar\left(T(x_n), T(x_{n+1})\right); \\ \Theta(w_n, u_n, v) + \phi(v, u_n) - \phi(u_n, u_n) + \frac{1}{r}\left\langle k'(u_n) - k'(x_n), \eta(v, u_n)\right\rangle \ge 0, \forall v \in C \\ x_{n+1} = a_n f(W_n x_n) + b_n x_n + c_n W_n u_n \end{cases} \tag{2.2}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences in (0, 1) such that $a_n + b_n + c_n = 1$.

It is easy to see that the iterative scheme (1.5) may be well defined.

Let r be a positive number. For a given point $x \in C$ and $w_x \in T(x)$, consider the following auxiliary problem for GMEP: find $u \in C$ such that

$$\Theta(w_x, u, v) + \phi(v, u) - \phi(u, u) + \frac{1}{r}\left\langle k'(u) - k'(x), \eta(v, u)\right\rangle \ge 0, \forall v \in C, \tag{2.3}$$

It is easy to see that if $u = x$, then u is a solution of GMEP.

We need the following important results.

Lemma 1.1. [2] Let C be a nonempty closed convex bounded subset of a real Hilbert space H and let $\phi: C \times C \to R$ be a real-valued function satisfying (ϕ_1)-(ϕ_3). Let $T: C \to 2^H$ be a multivalued mapping and $\Theta: H \times C \times C \to R$ be an equilibrium-like bifunction satisfying the conditions (Θ_1)-(Θ_4). Assume that $\eta: C \times C \to H$ is a Lipschitz function with lipschitz constant $\lambda > 0$ which satisfies the conditions (η_1)-(η_3). Let $k: C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies the conditions (k_1) and (k_2). For each $x \in C$, let $w_x \in T(x)$. For $r > 0$, define a mapping $T_r: C \to C$ by

$$T_r(x) = \left\{u \in C : \Theta(w_x, u, v) + \phi(v, u) - \phi(u, u) + \frac{1}{r}\left\langle k'(u) - k'(x), \eta(v, u)\right\rangle \ge 0, \forall v \in C\right\} \tag{2.4}$$

Then there hold the following:

1) the auxiliary problem (1.6) has a unique solution;

2) T_r is single-valued;

3) if $\lambda v/\mu \le 1$, it follows that T_r is firmly nonexpansive;

4) $F(T_r) = \Omega$;

5) Ω is closed and convex.

Lemma 1.2. [3] Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into H and $\bigcap_{i=1}^N F(T_i) \ne \varnothing$, and let $\{\lambda_{n,i}\}_{i=1}^N$ be a sequence in $(0, b]$ for some $b \in (0,1)$. Then, $F(W_n) = \bigcap_{i=1}^N F(T_i)$.

Lemma 1.3. [4] If the sequences $\{u_n\}$ and $\{x_n\}$ are bounded and W_n is defined by (1.3), then the following estimates hold:

$$\|W_{n+1}u_{n+1} - W_n u_n\| \le \|u_{n+1} - u_n\| + 2M\sum_{i=1}^N \left|\lambda_{n+1,i} - \lambda_{n,i}\right|, \ \forall n \ge 0$$

and

$$\|W_{n+1}x_{n+1} - W_n x_n\| \le \|x_{n+1} - x_n\| + 2M\sum_{i=1}^N \left|\lambda_{n+1,i} - \lambda_{n,i}\right|, \ \forall n \ge 0$$

for some constant $M > 0$.

Lemma 1.4. [4] In a real Hilbert space *H*, $\forall x, y, z \in H$ and $t_1, t_2, t_3 \in [0,1]$ with $t_1 + t_2 + t_3 = 1$, there holds the following equality:

$$\|t_1 x + t_2 y + t_3 z\|^2 \le t_1 \|x\|^2 + t_2 \|y\|^2 + t_3 \|z\|^2.$$

Lemma 1.5. [6] Let $\{x_n\}$ and $\{u_n\}$ be bounded sequences in a Banach space *X* and let $\{b_n\}$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$. Suppose

$$x_{n+1} = (1 - b_n) z_n + b_n x_n$$

for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} \left(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \right) \le 0.$$

Then,

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Lemma 1.6. [5] Let $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1 - \delta_n) a_n + b_n, \forall n = 1, 2, \cdots$$

where $\{\delta_n\}$ is a sequence in (0,1), $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n / \delta_n \le 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.7. [2] Let $\{x_n\}$ be a sequence in a normed space $(X, \|\cdot\|)$ such that

$$\|x_{n+1} - x_{n+2}\| \le \theta \|x_n - x_{n+1}\| s_n + r_n, \forall n = 1, 2, \cdots$$

where $\theta \in (0,1)$, and $\{s_n\}$ and $\{r_n\}$ are sequences satisfying the following conditions:

1) $s_n \ge 1$ and $\sum_{n=1}^{\infty} (s_n - 1) < \infty$;

2) $r_n \ge 0$, and $\sum_{n=1}^{\infty} r_n < \infty$.

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.8. [7] Let $A, B \in CB(X)$ and $a \in A$. Then for $\rho > 1$, there must exist a point $b \in B$ such that $d(a,b) \le \rho \hbar (A, B)$.

Lemma 1.9. [5] In a real Hilbert space *H*, there holds the following equality:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle, \forall x, y \in H.$$

3. Main Results

Theorem 2.1. Let *C* be a nonempty closed convex bounded subset of a real Hilbert space *H* and $r > 0$, $T : C \to CB(H)$ be a multivalued $\hbar$-Lipschitz continuous mapping with constant $L > 0$, and let $\phi : C \times C \to R$ be a real-valued function satisfying (ϕ_1) - (ϕ_3). and $\Theta : H \times C \times C \to R$ be an equilibrium-like function satisfying the conditions (Θ_1) - (Θ_4). Assume that $\eta : C \times C \to H$ is a Lipschitz function with lipschitz constant $\lambda > 0$ which satisfies the conditions (η_1) - (η_3). Let $k : C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies the conditions (k_1) and (k_2). with $\lambda\nu/\mu \le 1$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on *H* such that $\bigcap_{i=1}^N F(T_i) \cap \Omega \ne \varnothing$. Let *f* be a contraction of *C* into itself with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ be sequences generated by (1.5), where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences in (0,1) with $a_n + b_n + c_n = 1$ satisfying the following conditions:

1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

2) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$ and $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$;

3) $\sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}| < \infty$;

4) $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \cap_{i=1}^N F(T_i) \cap \Omega$, and $\{w_n\}$ converges strongly to $w^* \in T(x^*)$, where $x^* = P_{\cap_{i=1}^N F(T_i) \cap \Omega} f(x^*)$.

To proof Theorem 2.1, we first establish the following lemma.

Lemma 2.1. Let *C* be a nonempty closed convex bounded subset of a real Hilbert space *H* and $r > 0$, $T : C \to CB(H)$ be a multivalued $\hbar$-Lipschitz continuous mapping with constant $L > 0$, and let $\phi : C \times C \to R$ be a real-valued function satisfying (ϕ_1) - (ϕ_3). and $\Theta : H \times C \times C \to R$ be an equilibrium-like function satisfying the conditions (Θ_1) - (Θ_4). Assume that $\eta : C \times C \to H$ is a Lipschitz function with lipschitz constant $\lambda > 0$ which satisfies the conditions (η_1) - (η_3). Let $k : C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies the conditions (k_1) and (k_2) with $\lambda\nu/\mu \le 1$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on *H* such that $\bigcap_{i=1}^N F(T_i) \cap \Omega \ne \varnothing$. Let *f* be a contraction of *C* into itself with coefficient $\alpha \in (0,1)$. Let $\{x_n\}$, $\{u_n\}$, $\{w_n\}$ be sequences generated by (1.5), where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three sequences in (0,1) with $a_n + b_n + c_n = 1$, satisfying the following conditions:

1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

2) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$;

3) $\lim_{n\to\infty} |\lambda_{n,i+1} - \lambda_{n,i}| = 0$;

4) $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

then

1) $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

2) $\lim_{n\to\infty} \|x_n - W_n u_n\| = 0$, $\lim_{n\to\infty} \|x_n - u_n\| = 0$.

Proof. 1) From the nonexpansity of T_r, we have

$$\|u_{n+1}-u_n\| = \|T_r x_{n+1} - T_r x_n\| \le \|x_{n+1}-x_n\| \tag{3.1}$$

and set $z_n = \dfrac{x_{n+1}-b_n x_n}{1-b_n}$, we obtain

$$\begin{aligned} z_{n+1}-z_n &= \frac{x_{n+2}-b_{n+1}x_{n+1}}{1-b_{n+1}} - \frac{x_{n+1}-b_n x_n}{1-b_n} = \frac{a_{n+1} f(W_{n+1}x_{n+1}) + c_{n+1}W_{n+1}u_{n+1}}{1-b_{n+1}} - \frac{a_n f(W_n x_n) + c_n W_n u_n}{1-b_n} \\ &= \frac{a_{n+1}}{1-b_{n+1}}\left[f(W_{n+1}x_{n+1}) - f(W_n x_n)\right] + \left(\frac{a_{n+1}}{1-b_{n+1}} - \frac{a_n}{1-b_n}\right) f(W_n x_n) \\ &\quad + \frac{c_{n+1}}{1-b_{n+1}}(W_{n+1}u_{n+1} - W_n u_n) + \left(\frac{c_{n+1}}{1-b_{n+1}} - \frac{c_n}{1-b_n}\right) W_n u_n \end{aligned}$$

By Lemma 1.3, we arrive at

$$\begin{aligned} \|z_{n+1}-z_n\| &\le \frac{a_{n+1}}{1-b_{n+1}}\| f(W_{n+1}x_{n+1}) - f(W_n x_n)\| + \left|\frac{a_{n+1}}{1-b_{n+1}} - \frac{a_n}{1-b_n}\right| \left(\| f(W_n x_n)\| + \|W_n u_n\|\right) + \frac{c_{n+1}}{1-b_{n+1}}\|W_{n+1}u_{n+1} - W_n u_n\| \\ &\le \frac{\alpha a_{n+1}}{1-b_{n+1}}\left[\|x_{n+1}-x_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right] + \left|\frac{a_{n+1}}{1-b_{n+1}} - \frac{a_n}{1-b_n}\right| \left(\| f(W_n x_n)\| + \|W_n u_n\|\right) \\ &\quad + \frac{c_{n+1}}{1-b_{n+1}}\left[\|u_{n+1}-u_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right] \end{aligned} \tag{3.2}$$

Hence, it follows from (2.1) that

$$\begin{aligned} \|z_{n+1}-z_n\| &\le \frac{\alpha a_{n+1}}{1-b_{n+1}}\left[\|x_{n+1}-x_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right] + \left|\frac{a_{n+1}}{1-b_{n+1}} - \frac{a_n}{1-b_n}\right| \left(\| f(W_n x_n)\| + \|W_n u_n\|\right) \\ &\quad + \frac{c_{n+1}}{1-b_{n+1}}\left[\|x_{n+1}-x_n\| + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right] \\ &\le \|x_{n+1}-x_n\| + \left|\frac{a_{n+1}}{1-b_{n+1}} - \frac{a_n}{1-b_n}\right| \left(\| f(W_n x_n)\| + \|W_n u_n\|\right) + 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}| \end{aligned} \tag{3.3}$$

It follows from conditions (a) and (c), we have

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\right) \le 0.$$

Hence by Lemma 1.5, we can see that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0$$

Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-b_n)\|z_n - x_n\| = 0 \tag{3.4}$$

From (2.1), we get

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0 \tag{3.5}$$

2) In view of (1.5), we conclude that

$$\begin{aligned} \|x_n - W_n u_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n u_n\| \\ &\le \|x_n - x_{n+1}\| + a_n\| f(W_n x_n) - W_n u_n\| \\ &\quad + b_n\|x_n - W_n u_n\|, \end{aligned}$$

that is

$$\begin{aligned} \|x_n - W_n u_n\| &\le \frac{1}{1-b_n}\|x_n - x_{n+1}\| \\ &\quad + \frac{a_n}{1-b_n}\| f(W_n x_n) - W_n u_n\|, \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|x_n - W_n u_n\| = 0 \tag{3.6}$$

For $p \in \Gamma = \cap_{i=1}^{N} F(T_i) \cap \Omega$, note that T_r is firmly nonexpansive, we can see that

$$\begin{aligned} \|u_n - p\|^2 &= \|T_r x_n - T_r p\|^2 \le \langle T_r x_n - T_r p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2\right) \end{aligned}$$

and so

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|u_n - x_n\|^2 \le \|x_n - p\|^2 \tag{3.7}$$

In view of Lemma 1.4, (2.6) and (2.7), we compute

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|a_n f(W_n x_n)+b_n x_n+c_n W_n u_n-p\|^2\\ &\le a_n\|f(W_n x_n)-p\|^2+b_n\|x_n-p\|^2+c_n\|W_n u_n-p\|^2\\ &\le a_n\|f(W_n x_n)-p\|^2+b_n\|x_n-p\|^2+c_n\|u_n-p\|^2\\ &\le a_n\|f(W_n x_n)-p\|^2+b_n\|x_n-p\|^2+c_n\left(\|x_n-p\|^2-\|x_n-u_n\|^2\right)\\ &\le a_n\|f(W_n x_n)-p\|^2+\|x_n-p\|^2-c_n\|x_n-u_n\|^2\end{aligned}$$

which follows that

$$c_n\|x_n-u_n\|^2 \le \left(\|x_n-p\|+\|x_{n+1}-p\|\right)\left(\|x_{n+1}-x_n\|\right)+a_n\|f(W_n x_n)-p\|^2$$

and hence

$$\lim_{n\to\infty}\|x_n-u_n\|=0$$

This completes the proof.

Proof of Theorem 2.1. We divide our proof into 3 steps.

Step 1. We prove that there exists $x^*\in C$, such that $x_n\to x^*$, $u_n\to x^*$ and $w_n\to w$ as $n\to\infty$, where $w\in T(x^*)$. From (1.5), (2.1) and Lemma 1.3, we compute

$$\begin{aligned}\|x_{n+1}-x_n\| &= \left\|\left[a_n f(W_n x_n)+b_n x_n+c_n W_n u_n\right]-\left[a_{n-1} f(W_{n-1}x_{n-1})+b_{n-1}x_{n-1}+c_{n-1}W_{n-1}u_{n-1}\right]\right\|\\ &\le |a_n-a_{n-1}|\cdot\|f(W_n x_n)\|+a_{n-1}\|f(W_n x_n)-f(W_{n-1}x_{n-1})\|+|b_n-b_{n-1}|\cdot\|x_n\|\\ &\quad +b_{n-1}\|x_n-x_{n-1}\|+|c_n-c_{n-1}|\cdot\|W_n u_n\|+c_{n-1}\|W_n u_n-W_{n-1}u_{n-1}\|\\ &\le |a_n-a_{n-1}|\cdot\|f(W_n x_n)\|+a_{n-1}\alpha\left[\|x_n-x_{n-1}\|+2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right]+|b_n-b_{n-1}|\cdot\|x_n\|\\ &\quad +b_{n-1}\|x_n-x_{n-1}\|+|c_n-c_{n-1}|\cdot\|W_n u_n\|+c_{n-1}\left[\|u_n-u_{n-1}\|+2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right]\\ &\le \left[1-(1-\alpha)a_{n-1}\right]\|x_n-x_{n-1}\|+r_n\end{aligned} \tag{3.8}$$

where $\theta_n=1-(1-\alpha)a_{n-1}\le\frac{1}{2}$, $s_n=1$ and

$$r_n=|a_n-a_{n-1}|\cdot\|f(W_n x_n)\|+|b_n-b_{n-1}|\cdot\|x_n\|+|c_n-c_{n-1}|\cdot\|W_n u_n\|+2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.$$

By Lemma 1.7 and conditions (a)-(d), we conclude that $\{x_n\}$ is a Cauchy sequence in C such that $\lim_{n\to\infty}u_n=x^*$, there exists an element $x^*\in C$. On the other hand, $\lim_{n\to\infty}\|x_n-u_n\|=0$ implies that $\lim_{n\to\infty}u_n=x^*$. From (1.5), we have

$$\begin{aligned}\|w_n-w_{n+1}\| &\le \left(1+\frac{1}{n}\right)\hbar\left(T(x_n),T(x_{n+1})\right)\\ &\le 2\hbar\left(T(x_n),T(x_{n+1})\right)\le 2L\|x_n-x_{n+1}\|\end{aligned} \tag{3.9}$$

and for $m>n\ge 1$,

$$\|w_m-w_n\|\le\sum_{i=n}^{m-1}\|w_i-w_{i+1}\|\le 2L\sum_{i=n}^{m-1}\|x_i-x_{i+1}\| \tag{3.10}$$

$$\begin{aligned}\sum_{i=n}^{m-1}a_{i+1} &= \sum_{i=n}^{m-1}\|x_i-x_{i+1}\|\le\sum_{i=n}^{m-1}(\theta a_i+r_i)\\ &= \theta\sum_{i=n}^{m-1}a_i+\sum_{i=n}^{m-1}r_i\le\theta\sum_{i=n}^{m-1}a_{i+1}+\theta(a_n-a_m)+\sum_{i=n}^{m-1}r_i\\ &\le \theta\sum_{i=n}^{m-1}a_{i+1}+\theta a_n+\sum_{i=n}^{m-1}r_i\end{aligned}$$

Hence

$$\sum_{i=n}^{m-1}\|x_i-x_{i+1}\|\le\frac{\theta}{1-\theta}\|x_n-x_{n-1}\|+\frac{\sum_{i=n}^{m-1}r_i}{1-\theta}$$

In view of (2.4) and (2.8), we obtain

$$\lim_{m,n\to\infty}\|w_m-w_n\|=0 \tag{3.11}$$

which implies that $\{w_n\}$ is a Cauchy sequence in H and therefore there exists an element w in H such that $\lim_{n\to\infty}w_n=w$. Next we can see that

$$\begin{aligned}d\left(w,T(x^*)\right) &= \inf_{b\in T(x^*)}d(w,b)\le\|w-w_n\|+d\left(w,T(x^*)\right)\\ &\le \|w-w_n\|+\hbar\left(T(x_n),T(x^*)\right)\\ &\le \|w-w_n\|+L\|x_n-x^*\|\end{aligned} \tag{3.12}$$

Hence, we derive that $d\left(w,T(x^*)\right)=0$, that is

$w \in T(x^*)$ as $T(x^*) \in CB(H)$.

Step 2. Let $Q = P_{\cap_{i=1}^N F(T_i)\cap\Omega} f$. Then Q is a contraction of C into itself. In fact, for all $x, y \in C$

$$\|Q(x) - Q(y)\| \le \|f(x) - f(y)\| \le \alpha\|x - y\|$$

Therefore there exists a unique element $q \in C$ such that $q = Q(q)$. Noting that $q \in C$ and $Q(q) \in \cap_{i=1}^N F(T_i) \cap \Omega$, we get that $q \in \cap_{i=1}^N F(T_i) \cap \Omega$. Then

$$\langle f(q) - q, p - q \rangle \le 0, \forall p \in \cap_{i=1}^N F(T_i) \cap \Omega. \quad (3.13)$$

Next, we show that $x^* \in \cap_{i=1}^N F(T_i) \cap \Omega$. Since $x_n \to x^*$ and $u_n \to x^*$, we know that $k'(u_n) - k'(x_n) \to 0$, From (1.5) and (Θ_1), we have

$$\Theta(w, x^*, v) + \phi(v, x^*) - \phi(x^*, x^*) \ge 0$$

that is $x^* \in \Omega$. We shall show $x^* \in F(W_n)$. Assume $x^* \notin F(W_n)$, that is $x^* \ne W_n x^*$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges weakly to x^*. By Lemma 2.1, we conclude that $\|W_n u_n - u_n\| \to 0$. From Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty} \|u_{n_j} - x^*\| &< \liminf_{j\to\infty} \|u_{n_j} - W_n x^*\| \\ &\le \liminf_{j\to\infty} \left(\|u_{n_j} - W_n u_{n_j}\| + \|W_n u_{n_j} - W_n x^*\|\right) \\ &\le \liminf_{j\to\infty} \|u_{n_j} - x^*\|\end{aligned}$$

This is a contradiction. So, we get $x^* \in F(W_n) = \cap_{i=1}^N F(T_i)$. Therefore $x^* \in \cap_{i=1}^N F(T_i) \cap \Omega$.

Step 3. From (2.13) and $x_n \to x^*$, we obtain

$$\lim_{n\to\infty} \langle f(q) - q, x_n - q \rangle = \langle f(q) - q, x^* - q \rangle \le 0 \quad (3.14)$$

By Lemma 1.9, (1.5) and (2.7), we compute

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|a_n(f(W_n x_n) - q) + b_n(x_n - q) + c_n(W_n u_n - q)\|^2 \\ &\le \|b_n(x_n - q) + c_n(W_n u_n - q)\|^2 + 2a_n\langle f(W_n x_n) - q, x_{n+1} - q\rangle \\ &\le [b_n\|x_n - q\| + c_n\|u_n - q\|]^2 + 2a_n\langle f(W_n x_n) - f(q), x_{n+1} - q\rangle + 2a_n\langle f(q) - q, x_{n+1} - q\rangle \\ &\le [b_n\|x_n - q\| + c_n\|x_n - q\|]^2 + 2\alpha a_n\|x_n - q\|\|x_{n+1} - q\| + 2a_n\langle f(q) - q, x_{n+1} - q\rangle\end{aligned}$$

Hence

$$\begin{aligned}\|x_{n+1} - q\|^2 &\le \frac{(1 - a_n)^2 + a_n\alpha}{1 - a_n\alpha}\|x_n - q\|^2 + \frac{2a_n}{1 - a_n\alpha}\langle f(q) - q, x_{n+1} - q\rangle \\ &= \left[\frac{1 - 2a_n + \alpha a_n}{1 - a_n\alpha}\right]\|x_n - q\|^2 + \frac{a_n^2}{1 - a_n\alpha}\|x_n - q\|^2 + \frac{2a_n}{1 - a_n\alpha}\langle f(q) - q, x_{n+1} - q\rangle \\ &\le \left[1 - \frac{2(1-\alpha)a_n}{1 - a_n\alpha}\right]\|x_n - q\|^2 + \frac{2(1-\alpha)a_n}{1 - a_n\alpha} \times \left\{\frac{a_n M_1}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle\right\} \\ &= (1 - \delta_n)\|x_n - q\|^2 + \delta_n\sigma_n,\end{aligned}$$

where $M_1 = \sup\{\|x_n - q\|^2 : n \ge 1\}$, $\delta_n = \frac{2(1-\alpha)a_n}{1 - a_n\alpha}$ and $\sigma_n = \frac{a_n M_1}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(q) - q, x_{n+1} - q\rangle$. It is easy to see that $\delta_n \to 0$, $\sum_{n=1}^\infty \delta_n = \infty$, and $\limsup_{n\to\infty} \sigma_n \le 0$. Hence, by Lemma 1.6, the sequence $\{x_n\}$ converges strongly to q. Consequently, we can obtain that $\{u_n\}$ also converges strongly to q, and so $x^* = q$. This completes the proof.

Putting $T_i x = x$ for all $i \ge 1$ in Theorem 2.1, we obtain.

Corollary 2.1. Let C be a nonempty closed convex bounded subset of a real Hilbert space H, $T: C \to CB(H)$ be a multivalued $\hbar$-Lipschitz continuous mapping with constant $L > 0$, and let $\phi: C \times C \to R$ be a real-valued function satisfying (ϕ_1)-(ϕ_3). and $\Theta: H \times C \times C \to R$ be an equilibrium-like function satisfying the conditions (Θ_1)-(Θ_4) and $\Omega \ne \varnothing$. Assume that $\eta: C \times C \to H$ is a Lipschitz function with lipschitz constant $\lambda > 0$ which satisfies the conditions (η_1)-(η_3). Let $k: C \to R$ be an η-strongly convex function with constant $\mu > 0$ which satisfies the conditions (k_1) and (k_2) with $\lambda\nu/\mu \le 1$. Let F be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Then the sequences $\{x_n\}$, $\{u_n\}$, and $\{w_n\}$ generated iteratively by

$$\begin{cases} \|w_n - w_{n+1}\| \le \left(1+\frac{1}{n}\right)\hbar\left(T(x_n), T(x_{n+1})\right); \\ \Theta(w_n, u_n, v) + \phi(v, u_n) - \phi(u_n, u_n) + \frac{1}{r}\left\langle k'(u_n) - k'(x_n), \eta(v, u_n)\right\rangle \ge 0, \forall v \in C \\ x_{n+1} = a_n f(x_n) + b_n x_n + c_n u_n \end{cases} \quad (3.15)$$

converge strongly to $x^* \in \Omega$, and $\{w_n\}$ converges strongly to $w^* \in T(x^*)$, where $x^* = P_\Omega f(x^*)$, and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in (0,1) with $a_n + b_n + c_n = 1$, and $r > 0$ satisfying the following conditions:

1) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$;

2) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$ and $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$;

3) $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$.

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