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A Precise Asymptotic Behaviour of the Large Deviation Probabilities for Weighted Sums

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Abstract

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed positive valued random variables with a common distribution function F. When F belongs to the domain of partial attraction of a semi stable law with index α, $0 < \alpha < 1$, an asymptotic behavior of the large deviation probabilities with respect to properly normalized weighted sums have been studied and in support of this we obtained Chover's form of law of iterated logarithm.

Keywords: Large Deviations, Law of Iterated Logarithm, Semi-Stable Law, Domain of Partial Attraction, Weighted Sums

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d) positive valued random variables (r.v.s) with a common distribution function F. Let BV [0,1] be the set of all continuous bounded variation functions over [0,1]. Set

$$S_n = \sum_{k=1}^{n} X_k, n \geq 1, \text{ and } T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) X_k,$$

where f is a member of BV[0,1]. Let $\{n_k, k \geq 1\}$ be a strictly increasing subsequence of positive integers such that $\frac{n_{k+1}}{n_k} \to r(\geq 1)$ as $k \to \infty$. Kruglov [1] established that, if there exists sequences (a_k) and (b_k) of real constants, $b_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k\to\infty} P\left(\frac{S_{n_k}}{b_k} - a_k \leq x\right) = G_\alpha(x)$$

at all continuity points x of G_α, then G_α is necessarily a semi stable d.f with characteristic exponent α, $0 < \alpha \leq 2$. When $\alpha = 2$, semi-stable becomes normal.

It is known that probabilities of the type $P(|S_n| > x_n)$, or either of the one sided components, are called large deviation probabilities, where $\{x_n, n \geq 1\}$ is a monotone sequence of positive numbers with $x_n \to \infty$ as $n \to \infty$ such that $\frac{S_n}{x_n} \xrightarrow{p} 0$. In fact, under different conditions on sequence of r.v.s, Heyde [2-4] studied the large deviation problems for partial sums. In brief, for the r.v.s which are in the domain of attraction of a stable law and r.v.s which are not belong to the domain of partial attraction of the normal law, Heyde [2] and [3] established the order of magnitude of the larger deviation probabilities, where as in Heyde [3], he obtained the precise asymptotic behavior of large deviation probabilities for r.v.s in the domain of attraction of stable law.

When r.v.s. has i.i.d symmetric stable r.v.s, Chover [5] obtained the law of iterated logarithm (LIL) for partial sums by normalizing in the power and for r.v.s which are in the domain of attraction of a stable law, Peng and Qi [6] obtained Chover's type LIL for weighted sums, where the weights are belongs to BV[0, 1]. Many authors studied the non-trivial limit behavior for different weighted sums. See Peng and Qi [6] and references therein.

Probability of large values plays an important role in studying non-trivial limit behavior for stable like r.v.s. As far as properly normalized partial sums of stable like r.v.s, we can use the asymptotic results of Heyde [2-4]. (See Divanji [7]). However the observations made by Heyde [2-4] on the large deviation probabilities implicitly motivated us to study the large deviation probabilities for weighted sums. In fact, when the underlying i.i.d positive valued r.v.s are in the domain of partial attraction of a semi stable law of Kruglov's [1] setup, denoted as $F \in DP(\alpha)$, $0 < \alpha < 1$, a precise asymptotic behavior of the large deviation probabilities of Heyde [2-4] can be obtained for weighted sums. In support of this can be considered for Chover's type of non-trivial limit behavior for weighted sums.

In the next section we present some lemmas and main result in Section 3. In the last section, we discuss the existence of Chover's form of LIL for weighted sums. In the process i.o, a.s and s.v. mean 'infinitely often', 'almost surely' and 'slowly varying' respectively. C, ε, k and n with or without a super script or subscript denote positive constants with k and n confined to be integers. In the sequel, observe that when $\alpha < 1$, a_k can always be chosen to be zero.

2. Lemmas

Lemma 2.1

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Then there exists s. v. function L, such that

$$\lim_{X\to\infty} \frac{x^{\alpha}\left(1-F(x)\right)}{L(x)} = 1 .$$

Lemma 2.2

Let $F \in DP(\alpha)$, $0 < \alpha < 1$ and let

$B_n = \inf\left\{x > 0:\ 1-F(x) \ge \frac{1}{n}\right\}$. Then $B_n = n^{1/\alpha}\, l(n)$,

where l is a function s. v. at ∞.

The above lemmas can be referred to Divanji and Vasudeva [8].

Lemma 2.3

Let L be any s. v. function and let (x_n) and (y_n) be sequence of real constants tending to ∞ as n→ ∞. Then for any $\delta > 0$, $\lim_{n\to\infty} y_n^{\delta} \frac{L(x_n y_n)}{L(x_n)} = \infty$ and

$$\lim_{n\to\infty} y_n^{-\delta} \frac{L(x_n y_n)}{L(x_n)} = 0 .$$

This lemma can be referred to Drasin and Seneta [9].

Lemma 2.4

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Let (x_n) be a monotone sequence of real numbers tending to ∞ as n→∞ and B_n defined in Lemma 2.2. Then $B_n^{-1} x_n^{-1} S_n \xrightarrow{p} 0$, as n→∞.

This lemma can be referred to Vasudeva and Divanji [10].

Lemma 2.5

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Let (x_n) be a monotone sequence of real numbers tending to ∞ as n→∞ and B_n defined in lemma 2. Then $x_n^{-1} B_n^{-1} T_n \xrightarrow{p} 0$ as $n \to \infty$, with B_n defined in Lemma 2.2.

Proof

Since $f \in BV[0,1]$. Hence there exists a constant C such that $|f(x)| \le C$ and $\sum_{k=1}^{n-1}\left|f\left(\frac{k}{n}\right)-f\left(\frac{k+1}{n}\right)\right| \le C$, for all $n \ge 1$.

Therefore

$$T_n = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) X_k = \sum_{k=1}^{n-1}\left(f\left(\frac{k}{n}\right)-f\left(\frac{k+1}{n}\right)\right) S_k + f(1) S_n \le 2C \max_{1\le k\le n} S_k$$

Dividing on both sides by $x_n B_n$, we have $\frac{T_n}{x_n B_n} \le 2C \max_{1\le k\le n} \frac{S_k}{x_n B_n}$ Observe that X_i's are i.i.d positive valued r.v.s which are in the domain of partial attraction of a semi-stable law and hence $\max_{1\le k\le n} \frac{S_k}{x_n B_n} \le \frac{S_n}{x_n B_n}$ and by lemma 2.4 we have $\frac{S_n}{x_n B_n} \xrightarrow{p} \infty$. This gives $\frac{T_n}{x_n B_n} \xrightarrow{p} 0$, as $n \to \infty$.

3. Main Results

Theorem 3.1

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Let (x_n) be a monotone sequence of real numbers tending to ∞ as n→∞ and B_n defined in lemma 2.2. Then $\lim_{n\to\infty} \frac{P(T_n \ge x_n B_n)}{n\,P(X \ge x_n B_n)} = 1$.

Proof

To prove the assertion, it is enough to show that

$$0 < \liminf_{n\to\infty} \frac{P(T_n \ge x_n B_n)}{n\,P(X \ge x_n B_n)} \le \limsup_{n\to\infty} \frac{P(T_n \ge x_n B_n)}{n\,P(X \ge x_n B_n)} < \infty.$$

Let $\varepsilon > 0$ and define $A_i = \left\{ f\left(\frac{i}{n}\right) X_i \ge (1+\varepsilon) x_n B_n \right\}$ and $B_i = \left\{ \sum_{j=1, j\ne i}^{n} f\left(\frac{j}{n}\right) X_j \le \varepsilon x_n B_n \right\}$, $i = 1,2,\cdots,n$.

Proceeding on the lines of Heyde [4] and Lemma 3.1 of Vasudeva [12], we get,

$$P(T_n \ge x_n B_n) \ge \sum_{i=1}^{n} P(A_i)\left(P(B_i) - \sum_{j=1}^{n} P(A_j)\right) \ge n\,P(A_1)\left[P(B_1) - n\,P(A_1)\right] \tag{1}$$

From Lemma 2.5, we have $\frac{T_n}{x_n B_n} \xrightarrow{p} 0$, as $n \to \infty$ and given $\delta > 0$ with $1 - 2\delta > 0$, we can choose N_1 so

large such that $P(B_i) > 1 - 2\delta$ for all $n \ge N_1$ and for all $i = 1,2,3,\cdots,n$. Further from Lemma 2.5, we see that $nP(A_i) \to 0$ as $n \to \infty$, so that we can choose N_2 so large that $n\,P(A_i) < \delta$, for $n \ge N_2$. Thus for $n \ge N = \max(N_1, N_2)$, we obtain from (1),
$P(T_n \ge x_nB_n) \ge n(1-2\delta)P(X \ge (1+\varepsilon)x_nB_n)$, this implies

$$\frac{P(T_n \ge x_nB_n)}{n\,P(X \ge x_nB_n)} \ge \frac{n(1-2\delta)P(X \ge (1+\varepsilon)x_nB_n)}{nP(X \ge x_nB_n)} \ge (1-2\delta)\frac{P(X \ge (1+\varepsilon)x_nB_n)}{P(X \ge x_nB_n)}.$$

Using Lemma 2.1, we have

$$\frac{P(T_n \ge x_nB_n)}{n\,P(X \ge x_nB_n)} \ge \frac{(1-2\delta)}{(1+\varepsilon)^{\alpha}}\frac{L((1+\varepsilon)x_nB_n)}{x_n^{\alpha}B_n^{\alpha}}\frac{x_n^{\alpha}B_n^{\alpha}}{L(x_nB_n)} \ge \frac{(1-2\delta)}{(1+\varepsilon)^{\alpha}}\frac{L((1+\varepsilon)x_nB_n)}{L(x_nB_n)}$$

Choose $\varepsilon > 0$ sufficiently very small such that $\lim_{n\to\infty}\frac{L((1+\varepsilon)x_nB_n)}{L(x_nB_n)} = 1$, one can find a constant $C_1 > 0$ such that

$$\liminf_{n\to\infty}\frac{P(T_n \ge x_nB_n)}{n\,P(X \ge x_nB_n)} > C_1 > 0.$$

In order to complete the proof, we use truncation method.

Define

$$Y_k = \begin{cases} X_k, & \text{if } f\left(\frac{k}{n}\right)X_k \le x_nB_n \\ 0, & \text{otherwise} \end{cases}$$

Let $R_k = f\left(\frac{k}{n}\right)X_k - f\left(\frac{k}{n}\right)Y_k$,

$T_{1,n} = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)Y_k$ and $T_{2,n} = \sum_{k=1}^{n} R_k$. Notice that $P(T_n \ge x_nB_n) \le P(T_{1,n} > x_nB_n) + P(T_{2,n} \ne 0)$. This implies

$$\frac{P(T_n \ge x_nB_n)}{n\,P(X \ge x_nB_n)} \le \frac{P(T_{1,n} \ge x_nB_n)}{n\,P(X \ge x_nB_n)} + \frac{P(T_{2,n} \ne 0)}{n\,P(X \ge x_nB_n)} \quad (2)$$

Observe that

$P(T_{2,n} \ne 0) \le nP(R_1 \ne 0) = n\,P\left(f\left(\frac{1}{n}\right)X \ge x_nB_n\right)$, for fixed n and f is continuous BV [0,1] and it attains bounds. Hence using Lemma 1, we have,

$$\frac{P(T_{2,n} \ne 0)}{n\,P(X \ge x_nB_n)} = \frac{n\,P\left(f\left(\frac{1}{n}\right)X \ge x_nB_n\right)}{n\,P(X \ge x_nB_n)} \le \frac{L\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{x_n^{\alpha}B_n^{\alpha}}\frac{x_n^{\alpha}B_n^{\alpha}}{L(x_nB_n)}\left(f\left(\frac{1}{n}\right)\right)^{\alpha} \le \left(f\left(\frac{1}{n}\right)\right)^{\alpha}\frac{L\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{L(x_nB_n)}. \quad (3)$$

Using Karamata's representation of s.v. function, one gets that

$$\frac{L\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{L(x_nB_n)} = \frac{a\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{a(x_nB_n)}\exp\left\{\int_0^{\frac{x_nB_n}{f\left(\frac{1}{n}\right)}}\frac{\varepsilon(y)}{y}dy - \int_0^{x_nB_n}\frac{\varepsilon(y)}{y}dy\right\}$$

$$= \frac{a\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{a(x_nB_n)}\exp\left\{\int_{x_nB_n}^{\frac{x_nB_n}{f\left(\frac{1}{n}\right)}}\frac{\varepsilon(y)}{y}dy\right\}.$$

Since $a(x) \to C$ as $x \to C$ and $\varepsilon(y) \to 0$ as $y \to \infty$, there exists $C_0 > 0$ and $\delta_0 < \alpha$, such that

$$\frac{a\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{a(x_nB_n)} \le C_0, \ \varepsilon(y) \le \delta_0, \text{ for } y \ge x_nB_n.$$

This yield

$$\frac{L\left(\frac{x_nB_n}{f\left(\frac{1}{n}\right)}\right)}{L(x_nB_n)} \le C_0\exp\left\{\delta_0\log\left(\frac{1}{f\left(\frac{1}{n}\right)}\right)\right\} \le \frac{C_0}{\left(f\left(\frac{1}{n}\right)\right)^{\delta_0}}. \quad (4)$$

Substituting (4) in (3), one can find some constant C_1 such that the second term in (2) becomes

$\frac{P\left(T_{2,n}\neq 0\right)}{nP\left(X\geq x_n B_n\right)}\leq C_1\left(f\left(\frac{1}{n}\right)\right)^{\alpha-\delta}$. Since $f\in BV[0,1]$ and $f\left(\frac{1}{n}\right)\to f(0)(\in BV[0,1])$, as $n\to\infty$. Therefore we can find some constant C_2 ($>C_1$) such that

$$\lim_{n\to\infty}\frac{P\left(T_{2,n}\neq 0\right)}{n P\left(X\geq x_n B_n\right)}\leq C_2<\infty . \tag{5}$$

Now consider the first term in the right of (2). By Tchebychev's inequality, we get

$$\frac{P\left(T_{1,n}\geq x_n B_n\right)}{nP\left(X\geq x_n B_n\right)}\leq\frac{E\left(T_{1,n}^2\right)}{nx_n^2 B_n^2 P\left(X\geq x_n B_n\right)}.$$

Since

$$E\left(T_{1,n}^2\right)=\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)EY_k^2+\sum_{k=1}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right)f\left(\frac{m}{n}\right)EY_k\, EY_m ,$$

for $k\neq m$. Hence

$$\begin{aligned}\frac{P\left(T_{1,n}\geq x_n B_n\right)}{nP\left(X\geq x_n B_n\right)}&\leq\frac{E\left(T_{1,n}^2\right)}{n x_n^{\alpha} B_n^{\alpha} P\left(X\geq x_n B_n\right)}\\ &=\frac{\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)EY_k^2}{n x_n^{\alpha} B_n^{\alpha} P\left(X\geq x_n B_n\right)}+\frac{\sum_{k=1}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right)f\left(\frac{m}{n}\right)EY_k\, EY_m}{n x_n^{\alpha} B_n^{\alpha} P\left(X\geq x_n B_n\right)}.\end{aligned} \tag{6}$$

By Theorem 1, on page 544, of Feller [12] and Lemma 2.1, one gets that

$$\begin{aligned}&\frac{\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)EY_k^2}{n x_n^2 B_n^2 P\left(X\geq x_n B_n\right)}\\ &\leq\frac{x_n^{\alpha} B_n^{\alpha}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)\frac{x_n^{2-\alpha} B_n^{2-\alpha}}{f^{2-\alpha}\left(\frac{k}{n}\right)} L\left(\frac{x_n B_n}{f\left(\frac{k}{n}\right)}\right)}{n x_n^2 B_n^2 L\left(x_n B_n\right)}\\ &\leq\frac{x_n^{\alpha} B_n^{\alpha}\sum_{k=1}^{n} f^{\alpha}\left(\frac{k}{n}\right)x_n^{2-\alpha} B_n^{2-\alpha} L\left(\frac{x_n B_n}{f\left(\frac{k}{n}\right)}\right)}{n x_n^2 B_n^2 L\left(x_n B_n\right)}\\ &\leq\frac{1}{n}\sum_{k=1}^{n} f^{\alpha}\left(\frac{k}{n}\right)\frac{L\left(\frac{x_n B_n}{f\left(\frac{k}{n}\right)}\right)}{L\left(x_n B_n\right)}.\end{aligned}$$

Using similar steps of (4), one can find some constant C_3 such that

$$\frac{\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)EY_k^2}{n x_n^2 B_n^2 P\left(X\geq x_n B_n\right)}\leq\frac{C_3}{n}\sum_{k=1}^{n} f^{\alpha-\delta_0}\left(\frac{k}{n}\right).$$ Since f is continuous BV[0,1], then there exists C_4($>C_3$) such that

$$\frac{\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)EY_k^2}{n x_n^2 B_n^2 P\left(X\geq x_n B_n\right)}\leq C_4 . \tag{7}$$

Observe that

$$\sum_{k=1}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right)f\left(\frac{m}{n}\right)EY_k\, EY_m\leq\left\{\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left|EY_k\right|\right\}^2 .$$

Now for $0<\alpha<1$,

$$\left|EY_k\right|\leq E\left|Y_k\right|=\int_{|x|\leq\frac{x_n B_n}{f\left(\frac{k}{n}\right)}}|x|\, dP(X<x)\leq\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}} P\left(X\geq x\right)dx$$

Let $A=\frac{\sum_{k=1}^{n}\sum_{m=1}^{n} f\left(\frac{k}{n}\right)f\left(\frac{k}{n}\right)EY_k\, EY_m}{n x_n^2 B_n^2 P(X\geq x_n B_n)}$,

$B=\frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left|EY_k\right|\right)^2}{n x_n^2 B_n^2 P(X\geq x_n B_n)}$ and

$$D=\frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}} P\left(X\geq x\right)dx\right)^2}{n x_n^2 B_n^2 P(X\geq x_n B_n)}.$$

Notice that $A\leq B\leq D$. Again using Lemma 2.1, we have

$$\begin{aligned}D&\leq\frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}} L\left(x\right)x^{-\alpha}dx\right)^2}{n x_n^{2-\alpha} B_n^{2-\alpha} L\left(x_n B_n\right)}\\ &\leq\frac{\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}}\frac{L\left(x\right)}{L\left(x_n B_n\right)} x^{-\alpha}dx\right)^2}{n x_n^{2-\alpha} B_n^{2-\alpha}} L\left(x_n B_n\right).\end{aligned}$$

Following similar steps of (4), we can find some constant C_5 and $\delta_0 > 0$ such that

$$\frac{L(x)}{L(x_n B_n)} \le C_5 (1+\delta_0)\left(\frac{x_n B_n}{x}\right)^{\delta_0}.$$

Hence

$$D \le \frac{\left(C_5 (1+\delta_0)\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}} x^{-\alpha-\delta_0} dx \; x_n^{\delta_0} \; B_n^{\delta_0}\right)^2}{n\, x_n^{2-\alpha} \; B_n^{2-\alpha}}.$$

Also

$$\int_0^{\frac{x_n B_n}{f\left(\frac{k}{n}\right)}} x^{-\alpha-\delta_0} dx = \frac{1}{1-\alpha-\delta_0} x_n^{1-\alpha-\delta_0} \; B_n^{1-\alpha-\delta_0} \; f^{\alpha+\delta_0-1}\left(\frac{k}{n}\right)$$ and

there exists C_6 ($>C_5$) such that

$$D \le \frac{C_6 \left(\sum_{k=1}^{n} f^{\alpha+\delta_0}\left(\frac{k}{n}\right)\right)^2 L(x_n B_n)}{n\, x_n^{\alpha} \; B_n^{\alpha}}.$$

Let $M_n = x_n B_n$, where $x_n \to \infty$ and $B_n \to \infty$ as $n \to \infty$. Since $F \in DP(\alpha)$, $0 < \alpha < 1$, then

We know that

$$\frac{nL(M_n)}{M_n^{\alpha}} \to C_7 \qquad (8)$$

Using (8) one can find some constant C_8 such that $D \le \dfrac{C_8 \left(\sum_{k=1}^{n} f^{\alpha+\delta_0}\left(\frac{k}{n}\right)\right)^2}{n^2}$. Since f is Continuous BV [0,1], therefore there exists C_9 such that $\sum_{k=1}^{n} f^{\alpha+\delta_0}\left(\frac{k}{n}\right) \le n C_9$ and hence

$$D \le C_9 \Rightarrow B \le C_9 \Rightarrow A \le C_9 \qquad (9)$$

From (7) and (9), we claim that

$\dfrac{P(T_{1,n} \ge x_n B_n)}{nP(X \ge x_n B_n)} \to 0$ as $n \to \infty$, *i.e.*, holds.

Substituting (5) and (6) in (2), we get

$\operatorname{Lim\,Sup}_{n\to\infty} \dfrac{P(T_n \ge x_n B_n)}{n\,P(X \ge x_n B_n)} < \infty$. The proof of the theorem is completed.

4. Chover's Form of LIL

Theorem 4.1

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Then

$$\operatorname{Lim\,Sup}_{n\to\infty} \left(\frac{T_n}{B_n}\right)^{\frac{1}{\log\log n}} = e^{\frac{1}{\alpha}} \text{ a.s}.$$

Proof

To prove the assertion, it suffices to show for any $\varepsilon \in (0, 1)$, that

$$P\left(T_n \ge B_n (\log n)^{\frac{1+\varepsilon}{\alpha}} \text{ i.o}\right) = 0 \qquad (10)$$

and

$$P\left(T_n \ge B_n (\log n)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o}\right) = 1 \qquad (11)$$

To prove (10), let

$A_n = \left\{T_n \ge B_n (\log n)^{\frac{1+\varepsilon}{\alpha}}\right\}$ and $x_n = B_n (\log n)^{\frac{1+\varepsilon}{\alpha}}$. By the above Theorem 3.1, one can find a C_{10} such that, $P(A_n) \le C_{10}\, n\, P(X \ge x_n)$. Using Lemma 2.1,

$$P(A_n) \le C_{10}\, n\, x_n^{-\alpha} L(x_n)$$

$$\le C_{10}\, n \frac{L(B_n)}{B_n^{\alpha} (\log n)^{(1+\varepsilon)}} \frac{L(x_n)}{L(B_n)}.$$

Applying Lemma 2.3 with $\delta = \dfrac{\varepsilon}{2}$ and using the boundedness of θ, $P(A_n) \le C_{11} (\log n)^{-\left(1+\frac{\varepsilon}{2}\right)}$ for some $C_{11} > 0$. Consequently $\sum_{n=1}^{\infty} P(A_n) < \infty$ and (3) follows from the Borel-Cantelli Lemma.

Define, for large k,

$$m_k = \min\left\{ j : n_j \ge \beta^{(k-1)^{\delta}} \right\}, \qquad (12)$$

where $\beta > 1$ and $\delta > 0$ and from the relation $T_{n_k} = T_{n_k} - T_{n_{k-1}} + T_{n_{k-1}}$, $k \ge 1$, and in order to establish (11), it is enough if we show that $\varepsilon \in (0, 1)$, that

$$P\left(T_{n_{m_k}} - T_{n_{m_{k-1}}} \ge 2B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o}\right) = 1 \quad (13)$$

and

$$P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o}\right) = 0 \qquad (14)$$

Define $z_n = B_n (\log n)^{\frac{(1-\varepsilon)}{\alpha}}$ and

$D_k = (T_{n_{m_k}} - T_{n_{m_{k-1}}}) \ge z_{n_{m_k}}$, $k \ge 1$. Note that

$T_{n_{m_k}} - T_{n_{m_{k-1}}} \overset{d}{=} T_{n_{m_k} - n_{m_{k-1}}}$, $k \ge 1$. By the above Theorem 3.1, one can find a constant $C_{12} > 0$ and k_1 such that for all k ($\ge k_1$),

$$P(D_k) \ge C_{12} \left(n_{m_k} - n_{m_{k-1}}\right) P\left(X \ge 2\, z_{n_{m_k}}\right)$$

$$= C_{12}\, n_{m_k} \left(1 - \frac{n_{m_{k-1}}}{n_{m_k}}\right) P(X \ge 2 z_{n_{m_k}}).$$

Since $F \in DP(\alpha)$, $0 < \alpha < 1$ and under Kruglov's [9] setup *i.e.*, $\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = r(>1)$ implies that there exists $\lambda = r^{-1}$ (>1) such that

$$\frac{n_{m_{k-1}}}{n_{m_k}} < \lambda < 1 \quad \text{for all } k \ge k_1. \qquad (15)$$

$P(D_k) \ge C_{13} n_{m_k} P\left(X \ge 2z_{n_{m_k}}\right)$, for some $C_{13} > 0$.

Now following the steps similar to those used to get an upper bound of $P(A_n)$, one can find a k_2 such that for all k ($\ge k_2$), $P(D_k) \ge C_{14}(\log n_k)^{-\left(1-\frac{\varepsilon}{2}\right)}$, for some $C_{14} > 0$. Hence $\sum_{k=k_5}^{\infty} P(D_k) = \infty$. In view of the fact that D_k's are mutually independent, by applying the Borel-Cantelli Lemma, (13) is established. Observe that

$$P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}}\right)$$

$$= P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_{k-1}}} \frac{B_{n_{m_k}}}{B_{n_{m_{k-1}}}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}}\right).$$

Again by Theorem 3.1, one can find a constant C_{15} and k_3 such that for all $k \ge k_3$,

$$P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}}\right)$$

$$\le C_{15} n_{m_{k-1}} P\left(X_1 \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}}\right).$$

Again following the steps similar to those used to get an upper bound of $P(A_n)$, one can find a k_4 such that for all k ($\ge k_4$),

$$P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}}\right) \le C_{15} \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{\left(\log n_{m_k}\right)^{\left(1-\frac{3\varepsilon}{2}\right)}}$$

By (12) we have $n_{m_k} \ge \beta^{(k-1)^{\delta}}$ implies $n_{m_{k+1}} \ge \beta^{k^{\delta}} \ge n_{m_k}$ and (15), we have, $n_{k+1} \ge \lambda\, n_k$. Therefore, $n_{m_{k+1}} \ge \beta^{k^{\delta}} \ge n_{m_k} \ge \lambda n_{m_{k-1}} \Rightarrow \lambda n_{m_{k-1}} \le \beta^{k^{\delta}}$ $\Rightarrow n_{m_{k-1}} \le \frac{1}{\lambda}\beta^{k^{\delta}} = \lambda_1 \beta^{k^{\delta}}$, where $\lambda_1 = \frac{1}{\lambda}$. Hence

$$\frac{n_{m_{k-1}}}{n_{m_k}} \le \frac{\lambda_1 \beta^{k^{\delta}}}{\beta^{(k-1)^{\delta}}} \simeq \frac{\lambda_1}{\beta^{k^{\delta_1}}} \quad \text{and}$$

$$\sum_{k=k_5}^{\infty} \frac{n_{m_{k-1}}}{n_{m_k}} \frac{1}{\left(\log n_{m_k}\right)^{\left(1-\frac{3\varepsilon}{2}\right)}} \le \lambda_1 \sum_{k=k_5}^{\infty} \frac{1}{\beta^{k^{\delta_1}} \left(\log n_{m_k}\right)^{\left(1-\frac{3\varepsilon}{2}\right)}} < \infty.$$

Therefore $P\left(T_{n_{m_{k-1}}} \ge B_{n_{m_k}} \left(\log n_{m_k}\right)^{\frac{1-\varepsilon}{\alpha}} \text{ i.o}\right) = 0$,

which implies the proof of (11) follows from (13) and (14) and the proof of the theorem is completed.

Another direct application of Theorem 3.1 is for the Cesàro sums of index r. Here we may write $f\left(\frac{k}{n}\right) = \frac{A^r_{n-k}}{A^r_n}$, where $A^r_n = \frac{\Gamma(n+r+1)}{\Gamma(n+1)\Gamma(r+1)}$. Using Sterling approximation, we get $A^r_n = \frac{n^r}{\Gamma(r+1)}$ so that $f\left(\frac{k}{n}\right) \sim \left(1-\frac{k}{n}\right)^r$. The following result of Vasudeva [11] can be extended to domain of partial attraction of semi stable law and proof follows on similar lines of Theorem 2, we omit the details.

Theorem 4.2

Let $F \in DP(\alpha)$, $0 < \alpha < 1$. Then

$$\lim_{n\to\infty} \sup \left(\frac{T_n}{B_n}\right)^{\frac{1}{\log\log n}} = e^{\frac{1}{\alpha}} \text{ a.s , where}$$

$$T_n = \sum_{k=1}^{n} \left(1-\frac{k}{n}\right)^r X_k \quad \text{and } r > 0.$$

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