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Orbital Stability of Solitary Waves for Generalized Klein-Gordon-Schrödinger Equations

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Abstract

This paper concerns the orbital stability for exact solitary waves of the Generalized Klein-Gordon-Schrödinger equations. Since the abstract results of Grillakis *et al.* [1,2] can not be applied directly, we can extend the abstract stability theory and use the detailed spectral analysis to obtain the stability of the solitary waves.

Keywords: Solitary Waves, Stability, Klein-Gordon-Schrödinger Equations

1. Introduction

In this paper, we consider the the stability for the exact solitary waves of the Generalized Klein-Gordon-Schrödinger equations

$$\begin{cases} i\psi_t + \alpha\psi_{xx} = -|\varphi|^p |\psi|^{p-2}\psi \\ \varphi_{tt} - \varphi_{xx} + M^2\varphi = |\psi|^p |\varphi|^{p-2}\varphi \end{cases} \quad x \in R \qquad (1.1)$$

which describe a classical model of interaction of nucleon field with a meson field [3]. Here ψ is a complex scalar nucleon field, φ is a real meson field, M is the mass of a meson. By applying the abstract stability theory and detailed spectral analysis in [4-6], we obtain the orbital stability of the solitary waves.

This paper is organized as follows: in Section 2, we state the results of the existence of the exact solitary waves; in Section 3, we state the assumptions and the stability results.

2. The Exact Solitary Waves

Consider the following system

$$\begin{cases} i\psi_t + \alpha\psi_{xx} = -|\varphi|^p |\psi|^{p-2}\psi \\ \varphi_{tt} - \varphi_{xx} + M^2\varphi = |\psi|^p |\varphi|^{p-2}\varphi \end{cases} \quad x \in R \qquad (2.1)$$

Let

$$\begin{cases} \psi(x,t) = \mathrm{e}^{i\omega t}\mathrm{e}^{ic(x-ct)}u(x-ct) \\ \varphi(x,t) = v(x-ct) \end{cases} \qquad (2.2)$$

be the solitary waves of (2.1).

Put (2.2) into (2.1) and suppose $u, u'', v, v'' \to 0$,as $x \to \infty$,we obtain

$$\begin{cases} (2\alpha - 1)cu' = 0 \\ \alpha u'' + (c^2 - \omega - \alpha c^2)u + |v|^p |u|^{p-2} u = 0 \\ (c^2 - 1)v'' + M^2 v - |u|^p |v|^{p-2} v = 0 \end{cases} \qquad (2.3)$$

Let

$$\alpha = \frac{1}{2}, u = kv \qquad (2.4)$$

satisfy (2.3) with constant $k \neq 0$ determined later, then we have

$$\begin{cases} u'' + (c^2 - 2\omega)u + \frac{2}{|k|^p}|u|^{2p-2} u = 0 \\ (c^2 - 1)v'' + M^2 v - |k|^p |v|^{2p-2} v = 0 \end{cases} \qquad (2.5)$$

Let $u = c_1 \operatorname{sech}^{\frac{1}{p-1}} c_2 x$ satisfy (2.4)-(2.5) and constants c_1, c_2 will be determined later, then we obtain

$$k^2 = 2(1-c^2),\ c_2^2 = \frac{M^2(1-p)^2}{1-c^2} = (2\omega - c^2)(1-p)^2$$

$$c_1^{2p-2} = 2^{\frac{p-2}{2}} p(2\omega - c^2)(1-c^2)^{\frac{p}{2}}$$

Thus

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$$u(x)=\left[2^{\frac{p-2}{2}}p\left(2\omega-c^2\right)\left(1-c^2\right)^{\frac{p}{2}}\right]^{\frac{1}{2p-2}}\cdot\operatorname{sech}^{\frac{1}{p-1}}\left(\sqrt{2\omega-c^2}\,(p-1)x\right)$$

$$v(x)=\frac{\left[2^{\frac{p-2}{2}}p\left(2\omega-c^2\right)\left(1-c^2\right)^{\frac{p}{2}}\right]^{\frac{1}{2p-2}}}{\sqrt{2\left(1-c^2\right)}}\cdot\operatorname{sech}^{\frac{1}{p-1}}\left(\sqrt{2\omega-c^2}\,(p-1)x\right) \tag{2.6}$$

Finally, we have

Theorem 1. For any real constants *ω*, *c*, *p*, *M* satisfying

$$0<c<1,\ \omega>\frac{1}{2},\ p>1,\ M>0 \tag{2.7}$$

there exist solitary wave of (2.1) in the form of (2.2),with u, v satisfying (2.6).

3. Main Results

Rewrite Equation (2.1) as

$$\begin{cases} i\psi_t+\frac{1}{2}\psi_{xx}+|\varphi|^p|\psi|^{p-2}\psi=0 \\ \varphi_t=n \\ n_t=\varphi_{xx}-M^2\varphi+|\psi|^p|\varphi|^{p-2}\varphi \end{cases},\quad x\in R \tag{3.1}$$

Let $\boldsymbol{u}=\begin{pmatrix}\phi\\ \psi\\ n\end{pmatrix}$, and the function space in which we shall work is $X=H^1_{real}(R)\times H^1_{complex}(R)\times L^2_{real}$, with inner product

$$(\boldsymbol{f},\boldsymbol{g})=\operatorname{Re}\int_R\left(f_1g_1+f_{1x}g_{1x}+f_2\overline{g}_2+f_{2x}\overline{g}_{2x}+f_3g_3\right)\mathrm{d}x,\quad \boldsymbol{f},\boldsymbol{g}\in X \tag{3.2}$$

The dual space of *X* is $X^*=H^{-1}_{real}\times H^{-1}_{complex}\times L^2_{real}$, there is a natural isomorphism $I:X\to X^*$ defined by

$$\langle I\boldsymbol{f},g\rangle=(\boldsymbol{f},\boldsymbol{g}) \tag{3.3}$$

where $<\cdot,\cdot>$ denotes the pairing between *X* and X^*.

$$\langle\boldsymbol{f},\boldsymbol{g}\rangle=\operatorname{Re}\int_R\left(f_1g_1+f_2\overline{g}_2+f_3g_3\right)\mathrm{d}x \tag{3.4}$$

By (3.2)-(3.4), it is obvious

$$I=\begin{pmatrix}1 & & \\ & 1-\frac{\partial^2}{\partial x^2} & \\ & & \end{pmatrix}$$

Because the stability in view here refers to perturbations of the solitary-wave profile itself, a study of the initial-value problem for (1.1) is necessary.

Lemma 1. Let $\boldsymbol{u}_0\in H^1_{real}(R)\times H^1_{complex}(R)\times L^2_{real}$, there exists $T_*=T_*\left(\|\vec{u}_0\|\right)>0$ and a unique solution $\boldsymbol{u}\in C\left([0,T_*);H^1\times H^1\times L^2\right),\boldsymbol{u}(0)=\boldsymbol{u}_0$. In addition, either $T_*=\infty$ or $\|\boldsymbol{u}(x,t)\|_X\to\infty(t\to T_*)$.

Let T_1, T_2 be one-parameter groups of unitary operator on *X* defined by

$$T_1(s_1)\boldsymbol{u}(\cdot)=\boldsymbol{u}(\cdot-s_1),\boldsymbol{u}(\cdot)\in X, s\in R \tag{3.5}$$

$$T_2(s_2)\boldsymbol{u}(\cdot)=\left(\varphi(\cdot),\mathrm{e}^{is_2}\psi(\cdot),n(\cdot)\right),\boldsymbol{u}(\cdot)\in X, s\in R \tag{3.6}$$

Obviously

$$T_1'(0)=\begin{pmatrix}-\frac{\partial}{\partial x} & & \\ & -\frac{\partial}{\partial x} & \\ & & -\frac{\partial}{\partial x}\end{pmatrix},\qquad T_2'(0)=\begin{pmatrix}0 & & \\ & i & \\ & & 0\end{pmatrix}$$

It follows from Theorem 1 and (3.1) that there exist solitary waves $T_1(ct)T_2(\omega t)\left(\varphi_{\omega,c}(x),\psi_{\omega,c}(x),n_{\omega,c}(x)\right)$ with $\varphi_{\omega,c}(x),\psi_{\omega,C}(x),n_{\omega,c}(x)$ defined by

$$\begin{cases}\varphi_{\omega,c}(x)=v(x)\\ \psi_{\omega,c}(x)=e^{icx}u(x)\\ n_{\omega,c}(x)=-cv'(x)\end{cases} \tag{3.7}$$

Let

$$\Phi_{\omega,c}(x)=\left(\varphi_{\omega,c}(x),\psi_{\omega,c}(x),n_{\omega,c}(x)\right)$$

In this and the following sections, we shall consider the orbital stability of solitary waves $T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$ of (3.1). Note that Equation (3.1) is invariant under $T_1(\cdot)$ and $T_2(\cdot)$,we define the orbital stability as follows:

Definition 1. The solitary wave $T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$ is orbitally stable if for all $\varepsilon>0$ there exists $\delta>0$ with the following property. If $\left\|\boldsymbol{u}_0-\Phi_{\omega,c}\right\|_X<\delta$ and $\boldsymbol{u}(t)$ is a solution of (3.1) in some interval $[0,t_0)$ with $\boldsymbol{u}(0)=\boldsymbol{u}_0$, then $\boldsymbol{u}(t)$ can be continued to a solution in

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$0 \le t < +\infty$, and

$$\sup_{0<t<+\infty} \inf_{s_1 \in R} \inf_{s_2 \in R} \left\| \boldsymbol{u}(t) - T_1(s_1) T_2(s_2) \Phi_{\omega,c}(x) \right\|_X < \varepsilon$$

Otherwise $T_1(ct) T_2(\omega t) \Phi_{\omega,c}(x)$ is called orbitally unstable.

So long as ω, c are fixed we write φ, ψ, n for $\varphi_{\omega,c}(x), \psi_{\omega,c}(x), n_{\omega,c}(x)$.

Define

$$E(\boldsymbol{u}) = \int_R \left(\frac{1}{2}|n|^2 + \frac{1}{2}|\varphi_x|^2 + \frac{1}{2}M^2|\varphi|^2 + \frac{1}{4}|\psi_x|^2 - \frac{1}{p}|\varphi|^p |\psi|^p \right) \mathrm{d}x \tag{3.8}$$

$$Q_1(\boldsymbol{u}) = \frac{1}{2}\int_R (n_x \varphi - \varphi_x n) \mathrm{d}x + \frac{1}{2} \mathrm{Im} \int_R \psi_x \bar{\psi} \mathrm{d}x \tag{3.9}$$

$$Q_2(\boldsymbol{u}) = -\frac{1}{2}\int_R |\psi|^2 \mathrm{d}x \tag{3.10}$$

It is easy to verify that $E(\boldsymbol{u})$, $Q_1(\boldsymbol{u})$ and $Q_2(\boldsymbol{u})$ are invariant under T_1, T_2, and formally conserved under the flow of (3.1). Namely

$$\begin{aligned} &E(T_1(s_1)T_2(s_2)\boldsymbol{u}) = E(\boldsymbol{u}), \text{ for any } s_1, s_2 \in R \\ &Q_1(T_1(s_1)T_2(s_2)\boldsymbol{u}) = Q_1(\boldsymbol{u}), \text{ for any } s_1, s_2 \in R \\ &Q_2(T_1(s_1)T_2(s_2)\boldsymbol{u}) = Q_2(\boldsymbol{u}), \text{ for any } s_1, s_2 \in R \end{aligned} \tag{3.11}$$

and for any $t \in R, \boldsymbol{u}(t)$ is a flow of (3.1)

$$E(\boldsymbol{u}(t)) = E(\boldsymbol{u}(0)),\ Q_1(\boldsymbol{u}(t)) = Q_1(\boldsymbol{u}(0)),\ Q_2(\boldsymbol{u}(t)) = Q_2(\boldsymbol{u}(0)) \tag{3.12}$$

Note that Equation (3.1) can be written as the following Hamiltonian system

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = JE'(\boldsymbol{u}) \tag{3.13}$$

where J is a skew-symmetric linear operator, E is a functional (the energy).

However, by (2.4)-(2.6), we have

$$E'(\phi_{\omega,c}) - cQ_1'(\phi_{\omega,c}) - \omega Q_2'(\phi_{\omega,c}) = 0 \tag{3.14}$$

where E', Q' and Q_2' are the Frechet derivatives of E, Q_1 and Q_2, with

$$E'(\boldsymbol{u}) = \begin{pmatrix} -\varphi_{xx} + M^2\varphi - |\psi|^p |\varphi|^{p-2}\varphi \\ -\frac{1}{2}\psi_{xx} - |\varphi|^p |\psi|^{p-2}\psi \\ n \end{pmatrix}$$

$$Q_1'(\boldsymbol{u}) = \begin{pmatrix} n_x \\ -i\psi_x \\ -\varphi_x \end{pmatrix}, \qquad Q_2'(\boldsymbol{u}) = \begin{pmatrix} 0 \\ -\psi \\ 0 \end{pmatrix}$$

Define an operator from X to X^*

$$H_{\omega,c}\boldsymbol{\varphi} = E''(\phi_{\omega,c}) - cQ_1''(\phi_{\omega,c}) - \omega Q_2''(\phi_{\omega,c}) \tag{3.15}$$

with $\boldsymbol{y} = (y_1, y_2, y_3) \in X$, and

$$H_{\omega,c}\boldsymbol{\psi} = \begin{pmatrix} \left(-\frac{\partial}{\partial x^2} + M^2 - (p-1)|\varphi|^{p-2}|\psi|^p\right) y_1 - \frac{p}{2}|\varphi\psi|^{p-2}\varphi(\psi+\bar{\psi}) y_2 - cy_{3x} \\ -p|\varphi\psi|^{p-2}\varphi\psi y_1 + \left[-\frac{1}{2}\frac{\partial}{\partial x^2} - \frac{p}{2}|\varphi|^p|\psi|^{p-2} - \frac{p-2}{2}|\varphi|^p|\psi|^{p-4}\psi^2 + ic\frac{\partial}{\partial x} + \omega\right] y_2 \\ cy_{1x} + y_3 \end{pmatrix} \tag{3.17}$$

Observe that $H_{\omega,c}$ is self-adjoint in the sense that $H_{\omega,c}^* = H_{\omega,c}$ This means that $I^{-1}H_{\omega,c}$ is a bounded self-adjoint operator on X. The spectrum of $H_{\omega,c}$ consists of the real numbers λ such that $H_{\omega,c} - \lambda I$ is not invertible. We claim that $\lambda = 0$ belongs to the spectrum of $H_{\omega,c}$. By (3.11-3.15), it is easy to prove that

$$\begin{aligned} H_{\omega,c}T_1'(0)\Phi_{\omega,c}(x) = 0 \\ H_{\omega,c}T_2'(0)\Phi_{\omega,c}(x) = 0 \end{aligned} \tag{3.16}$$

Let

$$Z = \left\{ k_1 T_1'(0)\Phi_{\omega,c}(x) + k_2 T_2'(0)\Phi_{\omega,c}(x) / k_1, k_2 \in R \right\}$$

By (3.16), Z is contained in the kernel of $H_{\omega,c}$.

Assumption 1. (Spectral decomposition of $H_{\omega,c}$) The space X is decomposed as a direct sum

$$X = N + Z + P \tag{3.18}$$

where Z is defined above, N is a finite-dimensional subspace such that

$$\langle H_{\omega,c}\boldsymbol{u}, \boldsymbol{u} \rangle < 0 \quad \text{for} \quad 0 \ne \boldsymbol{u} \in N \tag{3.19}$$

and P is a closed subspace such that

$$\langle H_{\omega,c}\boldsymbol{u}, \boldsymbol{u} \rangle \ge \delta \|\boldsymbol{u}\|_X^2 \quad \text{for} \quad \boldsymbol{u} \in P \tag{3.20}$$

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with some constant $\delta > 0$ independent of $\boldsymbol{u}$.

We define $d(\omega,c): R\times R \to R$ by

$$d(\omega,c) = E(\phi_{\omega,c}) - cQ_1(\phi_{\omega,c}) - \omega Q_2(\phi_{\omega,c}) \quad (3.21)$$

and define $d''(\omega,c)$ to be the Hessian of function *d*. It is a symmetric bilinear form. In addition, we use $p(d'')$ to express the numbers of positive eigenvalue of d'' and $n(H_{\omega,c})$ to express the numbers of negative eigenvalue of $H_{\omega,c}$.

Theorem 2. Suppose that there exist three function $E(\boldsymbol{u}), Q_1(\boldsymbol{u}), Q_2(\boldsymbol{u})$ satisfying (3.11) and (3.12), and solitary waves $T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$ satisfying (3.14). Moreover, suppose that the operator $H_{\omega,c}$ given by (3.15) satisfies Assumption 1. If $d(\omega,c)$ is non-degenerative, $1<p<3$ and $p(d'') = n(H_{\omega,c})$, then solitary waves $T_1(ct)T_2(\omega t)\Phi_{\omega,c}(x)$ are orbitally stable.

Proof. According to (3.8)-(3.15), we only need to prove that Assumption 1 and $p(d'') = n(H_{\omega,c})$ hold.

First of all, we prove that Assumption 1 hold and $n(H_{\omega,c}) = 1$

For any $\boldsymbol{y} \in X$, let

$$\boldsymbol{y} = (z_1, e^{icx} z_2, z_3), \quad z_2 = z_{21} + iz_{22}, \quad z_{21} = \operatorname{Re} z_2 \quad (3.22)$$

then

$$\begin{aligned}
\langle H_{\omega,c}\boldsymbol{y},\boldsymbol{y}\rangle = \operatorname{Re}\int_R \Big\{ & \Big(-\frac{\partial}{\partial x^2} + M^2 - (p-1)|\varphi|^{p-2}|\psi|^p\Big) z_1^2 \\
& - \frac{p}{2}|\varphi\psi|^{p-2}\varphi(\psi\bar{z}_2 + \bar{\psi} z_2) z_1 - c z_{3x} z_1 \\
& - p|\varphi\psi|^{p-2}\varphi\psi z_1\bar{z}_2 + \Big[-\frac{1}{2}\frac{\partial}{\partial x^2} - \frac{p}{2}|\varphi|^p|\psi|^{p-2} \\
& - \frac{p-2}{2}|\varphi|^p|\psi|^{p-4}\psi^2 + ic\frac{\partial}{\partial x} + \omega\Big] z_2\bar{z}_2 \\
& + c z_{1x} z_3 + z_3^2 \Big\} dx \\
= \int_R \Big[& (1-c^2) z_{1x}^2 + (c z_{1x} + z_3)^2 \\
& + \big(M^2 - (p-1)|\phi|^{p-2}|\psi|^p\big) z_1^2 \\
& - 2p|\varphi\psi|^{p-1} z_1 z_{21}\Big] dx + \langle L_1 z_{21}, z_{21}\rangle + \langle L_2 z_{22}, z_{22}\rangle \\
= \int_R \Big[& (c z_{1x} + z_3)^2 + p|\varphi\psi|^{p-2}\big(|\psi| z_1 - |\phi| z_{21}\big)^2 \\
& + L z_1^2 \Big] dx + \langle \bar{L}_1 z_{21}, z_{21}\rangle + \langle L_2 z_{22}, z_{22}\rangle
\end{aligned}$$

where

$$\begin{aligned}
L &= -(1-c^2)\frac{\partial}{\partial x^2} + M^2 - (2p-1)|\varphi|^{p-2}|\psi|^p \\
L_1 &= -\frac{1}{2}\frac{\partial}{\partial x^2} - (p-1)|\varphi|^p|\psi|^{p-2} + \omega - \frac{c^2}{2} \\
\bar{L}_1 &= -\frac{1}{2}\frac{\partial}{\partial x^2} - (p-1)|\varphi|^p|\psi|^{p-2} + \omega - \frac{c^2}{2} - p|\varphi|^p|\psi|^{p-2} \\
L_2 &= -\frac{1}{2}\frac{\partial}{\partial x^2} - |\varphi|^p|\psi|^{p-2} + \omega - \frac{c^2}{2}
\end{aligned} \quad (3.23)$$

Since $2\omega - c^2 > 0$, note that

$$\begin{aligned}
\bar{L}_1 &= -\frac{1}{2}\frac{\partial}{\partial x^2} + \omega - \frac{c^2}{2} + M_1(x) \\
L_2 &= -\frac{1}{2}\frac{\partial}{\partial x^2} + \omega - \frac{c^2}{2} + M_2(x)
\end{aligned} \quad (3.24)$$

with

$$M_1(x) \to 0, \text{ as } |x| \to \infty; M_2(x) \to 0, \text{ as } |x| \to \infty \quad (3.25)$$

Thus, by Weyl's theorem on the essential spectrum (see [5]), we have

$$\begin{aligned}
\sigma_{ess}(\bar{L}_1) &= [\omega - \frac{c^2}{2}, +\infty),\ \omega - \frac{c^2}{2} > 0 \\
\sigma_{ess}(L_2) &= [\omega - \frac{c^2}{2}, +\infty),\ \omega - \frac{c^2}{2} > 0
\end{aligned} \quad (3.26)$$

Following from (2.3)-(2.5)

$$\bar{L}_1 u' = 0,\ L_2 u = 0 \quad (3.27)$$

By (2.6) and (3.27), we see that u' has a simple zero at $x=0$, then Sturm-Liouvill theorem implies that 0 is the second eigenvalue of $\bar{L}_1$,and $\bar{L}_1$ has exactly one strictly negative eigenvalue $-\lambda^2$, with an eigenfunction χ_1.

In virtue of (3.24)-(3.27), as in [3], we have the following lemma.

Lemma 2. For any real functions $z_{21} \in H^1(R)$, satisfying

$$\langle z_{21}, \chi_1\rangle = \langle z_{21}, u'\rangle = 0 \quad (3.28)$$

there exists a positive number $\delta_1 > 0$ such that

$$\langle \bar{L}_1 z_{21}, z_{21}\rangle \ge \delta_1 \|z_{21}\|_{H^1}^2 \quad (3.29)$$

Lemma 3. For any real functions $z_{22} \in H^1(R)$, satisfying $\langle z_{22}, u\rangle = 0$, there exists a positive number

$\delta_2 > 0$ such that

$$\left\langle L_2 z_{22}, z_{22} \right\rangle \geq \delta_2 \left\| z_{22} \right\|_{H^1}^2 \qquad (3.30)$$

For any $\boldsymbol{y} = \left(z_1, \mathrm{e}^{icx} z_2, z_3\right), z_2 = z_{21} + iz_{22}$ We can simply denote by $\boldsymbol{y} = \left(z_1, z_{21}, z_{22}, z_3\right)$

Choose

$$\boldsymbol{y}_- = \left(\frac{p(uv)^{p-1} + \sqrt{p^2 (uv)^{2p-2} - \left(p(uv)^{p-2} u^2 + L\right)}}{p(uv)^{p-2} u^2 + L} \chi_1, \right.$$

$$\left. \chi_1, 0, -cz_1 \right)$$

then

$$\left\langle H_{\omega,c} \boldsymbol{y}_-, \boldsymbol{y}_- \right\rangle = -\lambda^2 \left\langle \chi_1, \chi_1 \right\rangle < 0 \qquad (3.31)$$

Also note that the kernel of $H_{\omega,c}$ is spanned by the following two vectors:

$$\boldsymbol{y}_{0,1} = \left(-v_x, u_x, cu, -n_x\right), \quad \boldsymbol{y}_{0,2} = \left(0, 0, u, 0\right)$$

Let

$$N = \left\{ k\boldsymbol{y}_- / k \in R \right\}$$

$$Z = \left\{ k_1 \boldsymbol{y}_{0,1} + k_2 \boldsymbol{y}_{0,2} / k_1, k_2 \in R \right\}$$

$$P = \left\{ \boldsymbol{p} \in X / \boldsymbol{p} = \left(p_1, p_2, p_3, p_4\right), \right. \qquad (3.32)$$

$$\left. \left\langle p_2, \chi_1 \right\rangle = \left\langle p_2, u' \right\rangle = \left\langle p_3, u \right\rangle = 0 \right\}$$

Lemma 4. For any $\boldsymbol{p} \in P$, defined by (3.32), there exists a constant $\delta > 0$ such that

$$\left\langle H_{\omega,c} \boldsymbol{p}, \boldsymbol{p} \right\rangle \geq \delta \left\| p \right\|_X^2 \qquad (3.33)$$

with δ independent of $\boldsymbol{p}$.

For any $\boldsymbol{u} \in X, \boldsymbol{u} = \left(z_1, z_{21}, z_{22}, z_3\right)$

$$a = \left\langle z_{21}, z_{21} \right\rangle, b_1 = \frac{\left\langle z_{21}, u' \right\rangle}{\left\langle u', u' \right\rangle}, b_2 = \frac{\left\langle z_{22}, u \right\rangle}{\left\langle u, u \right\rangle} \qquad (3.34)$$

then $\boldsymbol{u} = a\boldsymbol{y}_- + b_1 \boldsymbol{y}_{0,1} + b_2 \boldsymbol{y}_{0,2} + \boldsymbol{p}$.

Thus under the condition of (2.7), Assumption 1 hold and $n\left(H_{\omega,c}\right) = 1$.

In the following, we shall verify that $p\left(d''\right) = n\left(H_{\omega,c}\right) = 1$ under the condition of theorem 1.

From

$$d\left(\omega, c\right) = E\left(\phi_{\omega,c}\right) - cQ_1\left(\phi_{\omega,c}\right) - \omega Q_2\left(\phi_{\omega,c}\right)$$

we have

$$d_\omega = -Q_2\left(\Phi_{\omega,c}\right) = \frac{1}{2} \int_R u^2 \mathrm{d}x = \frac{c_1^2}{2c_2} \int_R \mathrm{sech}^{\frac{2}{p-1}} x \mathrm{d}x$$

$$d_c = -Q_1\left(\Phi_{\omega,c}\right) = -\frac{c}{2} \int_R \left(u^2 + 2v'^2\right) \mathrm{d}x$$

$$= -\frac{c}{2} \left[\frac{c_1^2}{c_2} \int_R \mathrm{sech}^{\frac{2}{p-1}} x \mathrm{d}x + \frac{c_1^2 c_2}{(1-c^2)(p-1)^2} \right.$$

$$\left. \cdot \left(\int_R \mathrm{sech}^{\frac{2}{p-1}} x \mathrm{d}x - \int_R \mathrm{sech}^{\frac{2}{p-1}+2} x \mathrm{d}x \right) \right]$$

Let

$$\int_R \mathrm{sech}^{\frac{2}{p-1}} x \mathrm{d}x = A > 0$$

then $\int_R \mathrm{sech}^{\frac{2}{p-1}+2} x \mathrm{d}x = \frac{2}{p+1} A > 0$.

Thus

$$d_{\omega c} = A \frac{\partial}{\partial c} \left(\frac{c_1^2}{2c_2} \right), \quad d_{\omega\omega} = A \frac{\partial}{\partial \omega} \left(\frac{c_1^2}{2c_2} \right)$$

$$d_{cc} = \left[\frac{\partial}{\partial c} \left(\frac{c_1^2}{2c_2} \right) \right] (-c) \left(1 + \frac{(p-1)\left(2\omega - c^2\right)}{\left(1-c^2\right)(p+1)} \right) A$$

$$+ \left(A \frac{c_1^2}{2c_2} \right) \left(-1 - \frac{(p-1)\left(2\omega - c^2\right)}{\left(1-c^2\right)(p+1)} \right.$$

$$\left. - \frac{2c^2 (p-1)(2\omega - 1)}{(p+1)\left(1-c^2\right)^2} \right)$$

$$d_{c\omega} = \left[\frac{\partial}{\partial \omega} \left(\frac{c_1^2}{2c_2} \right) \right] (-c) \left(1 + \frac{(p-1)\left(2\omega - c^2\right)}{\left(1-c^2\right)(p+1)} \right) A$$

$$- \left(A \frac{c_1^2}{2c_2} \right) \frac{2c(p-1)}{\left(1-c^2\right)(p+1)}$$

Therefore, we obtain

$$d'' = \begin{pmatrix} d_{\omega\omega} & d_{\omega c} \\ d_{c\omega} & d_{cc} \end{pmatrix}$$

For $1 < p < 3$,

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$$\det\left(d''\right)=d_{\omega\omega}d_{cc}-d_{\omega c}d_{c\omega}$$

$$=\left[A\frac{\partial}{\partial\omega}\left(\frac{c_1^2}{2c_2}\right)\right]\left[\left(A\frac{c_1^2}{2c_2}\right)\right.$$

$$\left.\cdot\left(-1-\frac{(p-1)\left(2\omega-c^2\right)}{\left(1-c^2\right)(p+1)}-\frac{2c^2(p-1)(2\omega-1)}{(p-1)\left(1-c^2\right)^2}\right)\right]$$

$$+\left[A\frac{\partial}{\partial c}\left(\frac{c_1^2}{2c_2}\right)\right]\left[\left(A\frac{c_1^2}{2c_2}\right)\frac{2c(p-1)}{\left(1-c^2\right)(p+1)}\right]$$

$$\therefore p\left(d''\right)=1.$$

$$\therefore n\left(H_{\omega,c}\right)=p\left(d''\right)=1$$

Thus, theorem 2 is proved completely.

4. References

[1] M .Grillakis, J. Shatah and W. Strauss, "Stability Theory of Solitary Waves in the Presence of Symmetry, Ⅰ," *Journal of Functional Analysis*, Vol. 74, No. 1, 1987, pp. 160-197. doi:10.1016/0022-1236(87)90044-9

[2] M. Grillakis, J. Shatah and W. Strauss, "Stability Theory of Solitary Waves in the Presence of Symmetry, Ⅱ," *Journal of Functional Analysis*, Vol. 94, No. 2, 1990, pp. 308-348.

[3] I. Fukuda and M. Tsumi, "On Coupled *Klein-Gordon-Schrödinger* Equations," *Journal of Applied Mathematics*, Vol. 66, 1978, pp. 358-378.

[4] J. P. Albert and J. L. Bona, "Total Positivity and the Stability of Internal Waves in Stratified Fluids of Finite Depth," *IMA Journal of Applied Mathematics*, Vol. 46, No. 1-2, 1991, pp. 1-19. doi:10.1093/imamat/46.1-2.1

[5] M. Reed and B. Simon, "Methods of Modern Mathematical Physics: Fourier Analysis, Self-Adjointness," Academic Press, Waltham, 1975.

[6] B. L. Guo and L. Chen, "Orbital Stability of Solitary Waves of the Long Wave-Short Wave Resonance Equations," *Mathematical Methods in the Applied Sciences*, Vol. 21, No. 10, 1998, pp. 883-894.