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Refinements to Hadamard's Inequality for Log-Convex Functions

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Abstract

In this paper we show that a log-convex function satisfies Hadamard's inequality, as well as we give an extension for this result in several directions.

Keywords: Log-Convex Functions, Hadamard's Inequality, Integral Inequality

1. Introduction

Let $f: I \subset \Re \to \Re$ be a convex mapping of the interval I of real numbers and a, $b \in I$ with a < b. The following double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality. In [1], Fejer generalized the inequality (1.1) by proving that if $g:[a,b]\to\Re$ is nonnegative, integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on [a,b], then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

A positive function *f* is log-convex on a real interval $[a,b]$ if for all $x, y \in [a,b]$ and $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \le f^{\lambda}(x) f^{1-\lambda}(y). \tag{1.3}$$

If the above inequality reversed, then *f* is termed log-concave. We define for $x, y > 0$

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y}, & x \ne y \\ x, & x = y \end{cases}$$

In [2] the following result is achieved:

Theorem 1.1. *Let f be a positive log-convex function on [a, b]. Then*

$$\frac{1}{b-a}\int_a^b f(a)\,dt \le L(f(a), f(a)). \tag{1.4}$$

For f a positive log-concave function, the inequality is reversed.

2. Lemmas

The following lemmas are needed for our aim.

Lemma 2.1. *Let* $0 < t < 1$, *then the following inequality holds*

$$t^t (1-t)^{1-t} \ge 1/2. \tag{2.1}$$

Proof. Set

$$f(t) = \ln\left(t^t (1-t)^{1-t}\right) - \ln 1/2.$$

We have

$$f'(t) = \ln t - \ln(1-t) = 0, \quad \text{for} \quad t = 1/2.$$

$$f''(t) = \frac{1}{t} + \frac{1}{1-t} > 0.$$

Therefore *f* attains its minimum at $t = 1/2$ which is $1/2$ Hence $f(t) > 0$ which implies $e^{f(t)} > 0$, and (2.1) follows.

Lemma 2.2. *For* $0 < a < b, 0 \le t \le 1$, *the following inequality holds*

$$\sqrt{ab} \ge \begin{cases} a^{1-t} b^t, & t \le 1/2 \\ a^t b^{1-t}, & t \ge 1/2, \end{cases} \tag{2.2}$$

and for $a > 0,\ b > 0,\ 0 \le t \le 1$, *the following inequality holds*

$$2\sqrt{ab} \le a^t b^{1-t} + a^{1-t} b^t \le a + b. \tag{2.3}$$

Proof. For $t \le 1/2$, we have $(b/a)^{1/2} \ge (b/a)^t$, which implies $\sqrt{ab} \ge a^{1-t}b^t$, and for $t \ge 1/2$, $(a/b)^{1/2} \ge (b/a)^t$, which implies $\sqrt{ab} \ge a^t b^{1-t}$. We also have

$$\left(a^{\frac{t}{2}}b^{\frac{1-t}{2}} - a^{\frac{1-t}{2}}b^{\frac{t}{2}}\right)^2 \ge 0, \text{ which implies}$$

$$2\sqrt{ab} \le a^t b^{1-t} + a^{1-t}b^t.$$

Set $f(t) = a + b - a^t b^{1-t} - a^{1-t}b^t$. Then, on keeping b fixed, we have

$$f'(a) = 1 - ta^{t-1}b^{1-t} - (1-t)a^{-t}b^t = 0, \text{ for } a = b.$$

$$f''(a) = -t(t-1)a^{t-2}b^{1-t} + t(1-t)ta^{-t-1}b^t.$$

As $[f''(a)]_{a=b} = 2t(1-t)a^{-1} \ge 0$, f attains its minimum at $a = b$ which is 0, therefore $f(a) \ge 0$, and (2.3) is satisfied.

Although some of the coming results (Lemma 2.3 and theorem 3.1) are known, but we prove them by new simple method.

Lemma 2.3. *Let* $a, b > 0$, *then the following inequality holds*

$$\sqrt{ab} \le \frac{a-b}{\ln a - \ln b} \le \frac{a+b}{2}, \tag{2.4}$$

Proof. Left inequality. Let us assume that $b > a$. Set

$$f(x) = x^{1/2} - x^{-1/2} - \ln x, \quad x \ge 1. \tag{2.5}$$

$$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2} - x^{-1} \ge 0 \quad \text{as}$$

$$\left(x^{-1/4} - x^{-3/4}\right)^2 \ge 0 \Rightarrow x^{-1/2} + x^{-3/2} \ge 2x^{-1}.$$

Therefore f is non-decreasing, and that implies $f(x) \ge f(1) = 0$. The result follows by putting $x = b/a$ in (2.5).

Right inequality. Let $a \ge b$, and let $x = a/b$. Set

$$f(x) = \ln x - 2\frac{x-1}{x+1}, \quad x \ge 1. \tag{2.6}$$

We have

$$f'(x) = \frac{1}{x} - \frac{4}{(x+1)^2} \ge 0 \quad \text{as} \quad (x-1)^2 \ge 0 \Rightarrow \frac{1}{x} \ge \frac{4}{(x+1)^2}.$$

Then f is non-decreasing, and hence $f(x) \ge f(1) = 0$. The result follows by putting $x = a/b$ in (2.6).

Lemma 2.4. The function

$$f(x) = \frac{x-1}{\ln x}, \quad x \ge 1 \tag{2.7}$$

is non-decreasing.

Proof.

$$f'(x) = \frac{\ln x + x^{-1} - 1}{(\ln x)^2} = \frac{g(x)}{(\ln x)^2}.$$

$$g'(x) = \frac{1}{x} - \frac{1}{x^2} \ge 0,$$

therefore g is non-decreasing. Since $g(1) = 0$, then $g(x) \ge 0$, and hence $f'(x) \ge 0$, that is f is non-decreasing.

3. Theorems

Theorem 3.1. *Let f be a positive log-convex function on* $[a,b]$, *then f satisfies* (1.1).

Proof. This can be achieved immediately as the log-convex function is convex which follows from the fact that "Every increasing convex function of a convex function is convex" which implies that $f(x) = e^{\ln f(x)}$ is convex. Or the proof can be achieved by following the definition:

Making use of lemma 2.2, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \frac{1}{b-a}\int_a^b f\left(\frac{a+b-x+x}{2}\right)\mathrm{d}x \le \frac{1}{b-a}\int_a^b f^{1/2}(a+b-x)f^{1/2}(x)\mathrm{d}x \\ &\le \frac{1}{b-a}\left(\int_a^b f(a+b-x)\mathrm{d}x\right)^{1/2}\left(\int_a^b f(x)\mathrm{d}x\right)^{1/2} \\ &= \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{1}{2}\int_0^1 f(ta+(1-t)b)\mathrm{d}t + \frac{1}{2}\int_0^1 f((1-t)a+tb)\mathrm{d}t \\ &\le \int_0^1 \frac{f^t(a)f^{1-t}(b) + f^{1-t}(a)f^t(b)}{2}\mathrm{d}t \le \frac{f(a)+f(b)}{2}\int_0^1 \mathrm{d}t = \frac{f(a)+f(b)}{2}. \end{aligned}$$

The following giving a refinement to theorem 3.1.

Theorem 3.2. *Let f be a log convex function. Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}\mathrm{d}x\right)^2 \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(b)-f(a)}{\ln f(b) - \ln f(a)} \le \frac{f(a)+f(b)}{2} \tag{3.1}$$

Proof.

$$\sqrt{f\left(\frac{a+b}{2}\right)}=\frac{1}{(b-a)}\int_a^b\sqrt{f\left(\frac{a+b}{2}\right)}\mathrm{d}x=\frac{1}{(b-a)}\int_a^b\sqrt{f\left(\frac{a+b-x+x}{2}\right)}\mathrm{d}x$$

$$\le\frac{1}{(b-a)}\int_a^b\sqrt{\sqrt{f(a+b-x)}\sqrt{f(x)}}\mathrm{d}x\le\frac{1}{b-a}\left(\int_a^b\sqrt{f(a+b-x)}\mathrm{d}x\right)^{1/2}\left(\int_a^b\sqrt{f(x)}\mathrm{d}x\right)^{1/2}=\frac{1}{b-a}\int_a^b\sqrt{f(x)}\mathrm{d}x,$$

which implies

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{(b-a)^2}\left(\int_a^b\sqrt{f(x)}\mathrm{d}x\right)^2\le\frac{1}{(b-a)}\int_a^b f(x)\mathrm{d}x\le\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\le\frac{f(a)+f(b)}{2}$$

in view of [2] and Lemma 2.3.

The following presents an extension to Fejer's generalization (1.2) for log-convex functions

Theorem 3.3. *Let f be log convex, g is positive, integrable and symmetric to* $x=(a+b)/2$ *Then the following inequality holds*

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b\sqrt{g(x)}\mathrm{d}x\right)^2\le\left(\int_a^b\sqrt{f(x)g(x)}\mathrm{d}x\right)^2\le\int_a^b f(x)g(x)\mathrm{d}x$$
$$\le(b-a)\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\int_a^b g(x)\mathrm{d}x\le(b-a)\frac{f(a)+f(b)}{2}\int_a^b g(x)\mathrm{d}x. \tag{3.2}$$

Proof.

$$\sqrt{f\left(\frac{a+b}{2}\right)}\int_a^b\sqrt{g(x)}\mathrm{d}x=\int_a^b\sqrt{f\left(\frac{a+b-x+x}{2}\right)}\sqrt{g(x)}\mathrm{d}x\le\int_a^b\sqrt{f^{1/2}(a+b-x)f^{1/2}(x)}\sqrt{g(x)}\mathrm{d}x$$

$$=\int_a^b\sqrt{f^{1/2}(a+b-x)g^{1/2}(a+b-x)}\sqrt{f^{1/2}(x)g^{1/2}(x)}\mathrm{d}x\le\left(\int_a^b\sqrt{f(a+b-x)g(a+b-x)}\mathrm{d}x\right)^{1/2}\left(\int_a^b\sqrt{f(x)g(x)\mathrm{d}x}\right)^{1/2}$$

$$=\int_a^b\sqrt{f(x)g(x)}\mathrm{d}x\le\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^{1/2}\left(\int_a^b\mathrm{d}x\right)^{1/2}=\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^{1/2}(b-a)^{1/2}$$

which implies

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b\sqrt{g(x)}\mathrm{d}x\right)^2\le\left(\int_a^b\sqrt{f(x)g(x)}\mathrm{d}x\right)^2\le(b-a)\int_a^b f(x)g(x)\mathrm{d}x$$

Now, for $0\le t\le 1/2$, we have

$$\int_a^b f(x)g(x)\mathrm{d}x=(b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t\le(b-a)\int_0^1 f^t(a)f^{1-t}(b)g(ta+(1-t)b)\mathrm{d}t$$
$$\le(b-a)\sqrt{f(a)f(b)}\int_0^1 g(ta+(1-t)b)\mathrm{d}t\le\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\int_a^b g(x)\mathrm{d}x,$$

in view of Lemmas 2.2 and 2.3. Also, we have for $t\ge 1/2$,

$$\int_a^b f(x)g(x)\mathrm{d}x=(b-a)\int_0^1 f((1-t)a+tb)g((1-t)a+tb)\mathrm{d}t\le(b-a)\int_0^1 f^{1-t}(a)f^{1-t}(b)g((1-t)a+tb)\mathrm{d}t$$
$$\le(b-a)\sqrt{f(a)f(b)}\int_0^1 g((1-t)a+tb)\mathrm{d}t\le\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\int_a^b g(x)\mathrm{d}x,$$

in view of Lemmas 2.2 and 2.3. Consequently, we obtain, by Lemma 2.2,

$$\int_a^b f(x)g(x)\mathrm{d}x \le \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}\int_a^b g(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\mathrm{d}x.$$

This completes the proof of the theorem.

The following is another refinement of theorem 3.1.

Theorem 3.4. *Assume that* $f: I \to \Re$ *be an increasing log-convex function. Then for all* $t \in [0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right) \le w(a,b) \le \int_a^b f(x)\mathrm{d}x \le W(t) \le \frac{f(a)-f(b)}{\ln f(a)-\ln f(b)} \le \frac{f(a)+f(b)}{2}, \tag{3.3}$$

where

$$w(a,b) = \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)}, \tag{3.4}$$

$$W(t) = (1-t)\frac{f(ta+(1-t)b)-f(a)}{\ln f(ta+(1-t)b)-\ln f(a)} + t\frac{f(b)-f(ta+(1-t)b)}{\ln f(b)-\ln f(ta+(1-t)b)}. \tag{3.5}$$

Proof. We have via Lemmas 2.3 and 2.4

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{2}\frac{3a+b}{4}+\frac{1}{2}\frac{a+3b}{4}\right) \le \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} = w(t) \le \frac{2}{b-a}\sqrt{\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x \int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x}$$

$$\frac{2}{b-a}\frac{1}{2}\left(\int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x \int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x\right) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{1}{b-a}\left(\int_a^{ta+(1-t)b} f(x)\mathrm{d}x + \int_{ta+(1-t)b}^b f(x)\mathrm{d}x\right)$$

$$= (1-t)\left(\frac{1}{(1-t)(b-a)}\int_a^{ta+(1-t)b} f(x)\mathrm{d}x\right) + t\left(\frac{1}{t(b-a)}\int_{ta+(1-t)b}^b f(x)\mathrm{d}x\right)$$

$$\le (1-t)\frac{f(ta+(1-t)b)-f(a)}{\ln f(ta+(1-t)b)-\ln f(a)} + t\frac{f(b)-f(ta+(1-t)b)}{\ln f(b)-\ln f(ta+(1-t)b)} = W(t)$$

$$\le (1-t)\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} + t\frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} = \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} \le \frac{f(a)+f(b)}{2}.$$

Theorem 3.5. *Let* f *is log-convex and* g *is non-negative, integrable,* $(1/p)+(1/q)=1$, $p>1$, *then the following inequality holds*

$$\int_a^b f(x)g(x)\mathrm{d}x \le \left(\frac{b-a}{p}\frac{f^p(a)-f^p(b)}{\ln f(a)-\ln f(b)}\right)^{1/p}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q}. \tag{3.6}$$

Proof. We have, via Holder's inequality

$$\int_a^b f(x)g(x)\mathrm{d}x \le \left(\int_a^b f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q} = \left((b-a)\int_0^1 f^p(ta+(1-t)b)\mathrm{d}t\right)^{1/p}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q}$$

$$\le \left((b-a)\int_0^1 f^{pt}(a)f^{p(1-t)}(b)\mathrm{d}t\right)^{1/p}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q} = \left((b-a)f^p(b)\int_0^1\left(\frac{f(a)}{f(b)}\right)^{pt}\mathrm{d}t\right)\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q}$$

$$= \left(\frac{b-a}{p}\frac{f^p(a)-f^p(b)}{\ln f(a)-\ln f(b)}\right)^{1/p}\left(\int_a^b g^q(x)\mathrm{d}x\right)^{1/q}$$

4. References

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