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# On Certain Theta Function Identities Analogous to Ramanujan's *P*-*Q* Eta Function Identities

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## Abstract

The purpose of this paper is to provide direct proofs of certain theta function identities analogous to Ramanujan's *P*-*Q* eta functions identities.

**Keywords:** Eta Function Identities, Theta Function, *P*-*Q* Modular Equations

## 1. Introduction

In the unorganized pages of his second notebook [1,2], Ramanujan recorded 23 identities involving ratio of Dedekind's eta function of which have been proved by B. C. Berndt and L.-C. Zhang [3] by employing Ramanujan's modular equations of various degree, or via his mixed modular equations or via the theory of modular forms. Similar 14 identities involving ratio of Dedekind's eta function found on page 55 of Ramanujan's lost notebook [4] have been proved by Berndt [5] employing the above methods. Berndt and H. H. Chan [6], Berndt, Chan and Zhang [7], have employed some of the above mentioned *P*-*Q* modular equations for the explicit evaluation of Rogers-Ramanujan's continued fractions, and Ramanujan-weber-class invariants. Motivated by their works, several new *P*-*Q* eta functions identities have been discovered and employed them in finding the explicit evaluation of continued fractions, class invariant, and ratio of theta functions by many mathematics. For example see [8-20].

The purpose of this paper is to provide direct proofs of some of *P*-*Q* eta functions identities. In Section 2 of this paper, we found convenient to gather some definitions and lemmas which are required to prove *P*-*Q* eta function identities. In Section 3, we derive some *P*-*Q* eta function identities.

## 2. Preliminary Results

First we shall provide some useful notations and definition. In Chapter 16 of his second notebook [2,21,22] Ramanujan develops theory of theta function and his theta function is defined by theta function and his theta function defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty, |ab| < 1,$$

where we employ the customary notation

$$(a;b)_\infty = \prod_{k=1}^{\infty} \left(1 - aq^{k-1}\right).$$

If we set $a = qe^{2iz}$, $b = qe^{-2iz}$ and $q = e^{\pi ij}$, where $z$ is complex and Im($j$) > 0, then $f(a, b) = \vartheta_3(z, j)$, where $\vartheta_3(z, j)$ denote the classical theta function in its standard notation [23]. Following Ramanujan, we define

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(-q;q^2\right)_\infty^2 \left(q^2;q^2\right)_\infty,$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \left(q;q^2\right)_\infty^{-1} \left(q^2;q^2\right)_\infty,$$

$$f(-q) := f\left(-q,-q^2\right) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty$$

and

$$\chi(-q) := \left(q;q^2\right)_\infty.$$

For convenience, we denote $f\left(-q^n\right)$ by $f_n$ for positive integer $n$. It is easy to see that

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$$\varphi(-q)=\frac{f_1^2}{f_2},\psi(q)=\frac{f_2^2}{f_1},\varphi(q)=\frac{f_2^5}{f_1^2 f_4^2},$$

$$\psi(-q)=\frac{f_1 f_4}{f_2}, \tag{2.1}$$

$$\chi(-q)=\frac{f_1}{f_2} \text{ and } f(q)=\frac{f_2^3}{f_1 f_4}$$

**Lemma 2.1.** We have

$$\varphi(-q)\varphi(q)=\varphi^2(-q^2), \tag{2.2}$$

$$\varphi(q)\psi(q^2)=\psi^2(q), \tag{2.3}$$

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2) \tag{2.4}$$

and

$$\varphi^4(q)-\varphi^4(-q)=16q\psi^4(q^2). \tag{2.5}$$

The identities (2.2)-(2.5) are due to Ramanujan [2], and for a proof see [22].

**Lemma 2.2.** We have

$$\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)=4q\psi(q^2)\psi(q^6), \tag{2.6}$$

$$\varphi^2(q)+\varphi^2(q^3)=\frac{2\varphi(-q^2)\varphi(-q^3)\varphi(-q^6)}{\varphi(-q)}, \tag{2.7}$$

$$\varphi^2(q)-3\varphi^2(q^3)=-2\frac{\varphi(q^3)\varphi^3(-q^2)}{\varphi(q)\varphi(-q^6)}, \tag{2.8}$$

$$\psi^2(q^2)+q\psi^2(q^6)=\frac{\varphi^3(-q^3)\psi(-q)\psi^2(q^6)}{\varphi(-q)\psi^3(-q^2)}, \tag{2.9}$$

and

$$\psi^2(q^2)-3q\psi^2(q^6)=\frac{\varphi(-q)\psi(-q)\psi(-q^3)}{\varphi(-q^3)}. \tag{2.10}$$

The identity (2.6) is due to Berndt [22]. The (2.7) and (2.9) are due to N. D. Baruah and R. Barman [24]. Recently K. R. Vasuki, G. Sharath and K. R. Rajanna [25] have deduced (2.7)-(2.10) by employing the following theta function identities due to Ramanujan [4,21,22]:

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d) = f(ac,bd)f(ad,bc)$$

and

$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d) = 2af\left(\frac{b}{c},ac^2d\right)f\left(\frac{b}{d},acd^2\right).$$

**Lemma 2.3.** We have

$$\varphi^2(q)-\varphi^2(q^5)=4q\chi(q)\chi(q^5)\psi^2(-q^5), \tag{2.11}$$

$$\varphi^2(-q)-\varphi^2(-q^5)=-4q\chi(-q)\chi(-q^5)\psi^2(q^5), \tag{2.12}$$

$$\varphi^2(q)-5\varphi^2(q^5)=-4\frac{f_2^2\chi(q^5)}{\chi(q)}, \tag{2.13}$$

$$\varphi^2(-q)-5\varphi^2(-q^5)=-4\frac{f_2^2\chi(-q^5)}{\chi(-q)}, \tag{2.14}$$

$$\psi^2(q)-q\psi^2(q^5)=\frac{\varphi^2(-q^5)}{\chi(-q)\chi(-q^5)}, \tag{2.15}$$

$$\psi^2(-q)+q\psi^2(-q^5)=\frac{\varphi^2(q^5)}{\chi(q)\chi(q^5)}, \tag{2.16}$$

$$\psi^2(q)-5q\psi^2(q^5)=f_1^2\frac{\chi(-q)}{\chi(-q^5)} \tag{2.17}$$

and

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)}+\frac{\psi^2(q^5)}{\psi^2(q)}-q\frac{\psi^2(-q^5)}{\psi^2(-q)}=1. \tag{2.18}$$

The identities (2.11) and (2.15) are due to Ramanujan [2,22], S. Y. Kang [26], has given proof of (2.11)-(2.17) by employing the theta function identities. Recently S. Bhargava, Vasuki and Rajanna [27] deduce (2.11)-(2.17) from Ramanujan ${}_1\psi_1$ summation formula. The identity (2.18) is due to Berndt [22] and he given a direct and interesting proof of the same by employing only simply deducible theta function identity.

The following lemma is due to Berndt [22]. In fact Berndt, obtained it from a modular equation of Ramanujan, and expressed that a direct proof has not been given.

**Lemma 2.4.** We have

$$\varphi^2(q)\varphi^2(q^5)-\varphi^2(-q)\varphi^2(-q^5) -16q^3\psi^2(q^2)\psi^2(q^{10})=8qf_2^2f_{16}^2. \tag{2.19}$$

**Proof.** Squaring both sides of (2.12), we obtain

$$\varphi^4(-q)+\varphi^4(-q^5)-2\varphi^2(-q)\varphi^2(-q^5) =16q^2\chi^2(-q)\chi^2(-q^5)\psi^4(q^5). \tag{2.20}$$

Squaring both sides of (2.15), and then replacing $q$ to $q^2$ and then multiplying by $16q$, we obtain

$$16q\psi^4(q^2)+16q^5\psi^4(q^{10})-32q^3\psi^2(q^2)\psi^2(q^{10}) =16q\frac{\varphi^4(-q^{10})}{\chi^2(-q^2)\chi^2(-q^{10})}. \tag{2.21}$$

Adding (2.20) and (2.21) and then employing (2.5), we deduce that

$$\begin{aligned}&\varphi^4(q)+\varphi^4(q^5)-2\varphi^2(-q)\varphi^2(-q^5)\\&-32q^3\psi^2(q^2)\psi^2(q^{10})\\&=16q\left\{q\chi^2(-q)\chi^2(-q^5)\psi^4(q^5)\right.\\&\left.+\frac{\varphi^4(-q^{10})}{\chi^2(-q^2)\chi^2(-q^{10})}\right\}\\&=16qf_2^2f_{10}^2\left\{q\frac{\psi^2(q^5)}{\psi^2(q)}+\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)}\right\}\end{aligned}$$

upon using (1.1). Now employing (2.18) and then using (2.16), in the right side of the above, we obtain

$$\begin{aligned}&\varphi^4(q)+\varphi^4(q^5)-2\varphi^2(-q)\varphi^2(-q^5)\\&-32q^3\psi^2(q^2)\psi^2(q^{10})\\&=16q\frac{f_2^2f_5f_{20}\varphi^2(q^5)}{f_1f_4}.\end{aligned}\tag{2.22}$$

From (2.11) and (2.13), it follows that

$$16qf_2^2f_{10}^2=\left[\varphi^2(q)-\varphi^2(q^5)\right]\left[5\varphi^2(q^5)-\varphi^2(q)\right],$$

which is equivalent to

$$\begin{aligned}&16qf_2^2f_{10}^2-2\varphi^2(q)\varphi^2(q^5)\\&=-\varphi^4(q)-5\varphi^4(q^5)+4\varphi^2(q)\varphi^2(q^5).\end{aligned}\tag{2.23}$$

From (2.23) and (2.22), we deduce that

$$\begin{aligned}&16qf_2^2f_{10}^2+2\varphi^2(-q)\varphi^2(-q^5)\\&+32q^3\psi^2(q^2)\psi^2(q^{10})-2\varphi^2(q)\varphi^2(q^5)\\&=4\varphi^2(q)\varphi^2(q^5)-4\varphi^4(q^5)-16q\frac{f_2^2f_5f_{20}\varphi^2(q^5)}{f_1f_4}\\&=4\varphi^2(q^5)\left\{\varphi^2(q)-\varphi^2(q^5)-4q\frac{f_2^2f_5f_{20}}{f_1f_4}\right\}=0\end{aligned}$$

upon using (2.11) and (1.1), This completes the proof.

## 3. *P*-*Q* Eta Function Identities

**Lemma 3.1.** Let

$$P:=\frac{\varphi(-q)}{\varphi(-q^2)},\text{ and }\ Q:=\frac{\varphi(q)}{\varphi(q^2)}.$$

Then,

$$P^4+1=\frac{2}{Q^2}.\tag{3.1}$$

**Proof.** We have

$$\begin{aligned}P^4+1&=\frac{\varphi^4(-q)+\varphi^4(-q^2)}{\varphi^4(-q^2)}\\&=\frac{\varphi^2(-q)}{\varphi^4(-q^2)}\left[\varphi^2(-q)+\varphi^2(q)\right]\\&=\frac{2\varphi^2(-q)\varphi^2(q^2)}{\varphi^4(-q^2)}\\&=\frac{2\varphi^2(q^2)}{\varphi^2(q)}\\&=\frac{2}{Q^2}\end{aligned}$$

where we have used (2.2) and (2.4). This completes the proof.

**Theorem 3.1.** Let

$$P:=\frac{\varphi(-q)}{\varphi(-q^2)},\text{ and }\ Q:=\frac{\varphi(-q^3)}{\varphi(-q^6)}.$$

Then,

$$\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4+6=4\left[(PQ)^2+\frac{1}{(PQ)^2}\right].\tag{3.2}$$

**Proof.** By (2.2) and (2.6),

$$\begin{aligned}1-(PQ)^2&=1-\frac{\varphi^2(-q)\varphi(-q^3)}{\varphi^2(-q^2)\varphi^2(-q^6)}=1-\frac{\varphi(-q)\varphi(-q^3)}{\varphi(q)\varphi(q^3)}\\&=\frac{\varphi(q)\varphi(q^3)-\varphi(-q)\varphi(-q^3)}{\varphi(q)\varphi(q^3)}\\&=\frac{4q\psi(q^2)\psi(q^6)}{\varphi(q)\varphi(q^3)}.\end{aligned}$$

Taking forth power on both sides and then employing (2.5), we have

$$\begin{aligned}\left[1-(PQ)^2\right]^4&=\left\{16q\frac{\psi^4(q^2)}{\varphi^4(q)}\right\}\left\{16q^3\frac{\psi^4(q^6)}{\varphi^4(q^3)}\right\}\\&=\left\{1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right\}\left\{1-\frac{\varphi^4(-q^3)}{\varphi^4(q^3)}\right\}\\&=\left\{1-\frac{\varphi^8(-q)}{\varphi^8(q^2)}\right\}\left\{1-\frac{\varphi^8(-q^3)}{\varphi^8(-q^6)}\right\}\\&=(1-P)(1-Q^8),\end{aligned}$$

where, we have used (2.2). Thus, we have

$$\left[1-(PQ)^2\right]^4=\left(1-P^8\right)\left(1-Q^8\right).$$

Now expanding both sides and then dividing throughout by $(PQ)^4$, we obtain the required result.

**Corollary 3.1.** Let

$$P:=\frac{\varphi(q)}{\varphi(q^2)},\text{and}Q:=\frac{\varphi(q^3)}{\varphi(q^6)}.$$

Then,

$$\left(\frac{P}{Q}\right)^4+\left(\frac{Q}{P}\right)^4-36\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]$$
$$+48\left[PQ+\frac{2}{PQ}\right]\left[\frac{P}{Q}+\frac{Q}{P}\right]=16\left[(PQ)^2+\frac{4}{(PQ)^2}\right]+138.$$

**Proof.** Squaring both sides of (3.2) and then employing (3.1) and after some simplification, we obtain the required result.

**Corollary 3.2.** Let

$$P:=\frac{\varphi(-q)}{\varphi(q)}\text{ and }Q:=\frac{\varphi(-q^3)}{\varphi(q^3)}.$$

Then,

$$\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2+6=4\left[PQ+\frac{1}{PQ}\right].$$

**Proof.** We have

$$\frac{\varphi^2(-q)}{\varphi^2(-q^2)}=\frac{\varphi(-q)}{\varphi(q)}.$$

Using this in (3.2), we obtain the required result.

Corollary 3.1 and 3.2 are due to Vasuki and Srivatsakumar [19].

**Theorem 3.2.** Let

$$P:=\frac{\varphi(-q)}{\varphi(-q^3)},\text{ and }\ Q:=\frac{\varphi(-q^2)}{\varphi(-q^6)}.$$

Then,

$$\frac{3}{Q^2}-Q^2=\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2. \tag{3.3}$$

**Proof.** By (2.7), (2.8) and (2.2),

$$P^2\left(\frac{3-P^2}{1+P^2}\right)=\frac{\varphi^2(-q)}{\varphi^2(-q^3)}\left[\frac{3\varphi^2(-q^3)-\varphi^2(-q)}{\varphi^2(-q)+\varphi^2(-q^3)}\right]$$
$$=\frac{\varphi^2(-q)\varphi^2(q)}{\varphi^2(-q^3)\varphi^2(q^3)}=\frac{\varphi^4(-q^2)}{\varphi^4(-q^6)}=Q^4. \tag{3.4}$$

The required result follows from the above after some algebraic manipulations.

**Corollary 3.3.** Let

$$P:=\frac{\varphi(-q)}{\varphi(-q^3)}\text{ and }Q:=\frac{\varphi(q)}{\varphi(q^3)}.$$

Then,

$$\frac{3}{PQ}-PQ=\frac{Q}{P}+\frac{P}{Q}. \tag{3.5}$$

**Proof.** We have from (2.2)

$$\frac{\varphi(-q^2)}{\varphi(-q^6)}=PQ.$$

Using this in (3.3), we obtain (3.5).

**Theorem 3.3.** Let

$$P:=\frac{\varphi(q)}{\varphi(q^3)}\text{ and }Q:=\frac{\varphi(q^2)}{\varphi(q^6)}.$$

Then,

$$(PQ)^2+\frac{9}{(PQ)^2}+2\left[PQ+\frac{3}{PQ}\right]\left[\frac{P}{Q}-\frac{Q}{P}\right]$$
$$=\left(\frac{P}{Q}\right)^2-\left(\frac{Q}{P}\right)^2+12. \tag{3.6}$$

**Proof.** By (2.2), (2.7) and (2.8),

$$\frac{3-P^2}{1+P^2}=\frac{3\varphi^2(q^3)-\varphi^2(q)}{\varphi^2(q^3)+\varphi^2(q)}$$
$$=\frac{\varphi(-q)\varphi^2(-q^2)\varphi(q^3)}{\varphi(q)\varphi(-q^3)\varphi^2(-q^6)}$$
$$=\frac{\varphi^2(-q)}{\varphi^2(-q^2)}. \tag{3.7}$$

Changing $q$ to $q^2$ in (3.7), we obtain

$$\frac{3-Q^2}{1+Q^2}=\frac{\varphi^2(-q^2)}{\varphi^2(-q^6)}. \tag{3.8}$$

From (3.7), (2.2) and (3.8), we deduce that

$$P^2\left(\frac{3-P^2}{1+P^2}\right)=\frac{\varphi^2(q)\varphi^2(-q)}{\varphi^2(q^3)\varphi^2(-q^3)}$$
$$=\left(\frac{\varphi^2(-q^2)}{\varphi^2(-q^6)}\right)^4$$
$$=\left(\frac{3-Q^2}{1+Q^2}\right)^2.$$

Thus, we have

$$P^2\left(\frac{3-P^2}{1+P^2}\right)=\left(\frac{3-Q^2}{1+Q^2}\right)^2,$$

which is equivalent to (3.6).

Identities (3.3), (3.5) and (3.6) are due to Bhargava *et al.* [12].

**Theorem 3.4.** Let

$$P:=\frac{\psi(q)}{q^{\frac{1}{8}}\psi(q^2)} \text{ and } Q:=\frac{\psi(q^3)}{q^{\frac{3}{8}}\psi(q^6)}.$$

Then,

$$(PQ)^2+\frac{16}{(PQ)^2}=4\left[\left(\frac{P}{Q}\right)^4-\left(\frac{Q}{P}\right)^4\right]+9. \quad (3.9)$$

**Proof.** By (2.3) and (2.5),

$$P^8-16=\frac{\psi^8(q)}{q\psi^8(q^2)}-16$$
$$=\frac{\varphi^4(q)}{q\psi^4(q^2)}-16 \quad (3.10)$$
$$=\frac{\varphi^4(-q)}{q\psi^4(q^2)}.$$

Changing $q$ to $q^3$ in (3.10), we have

$$Q^8-16=\frac{\varphi^4(-q^3)}{q^3\psi^4(q^6)}. \quad (3.11)$$

By (2.3) and (2.6),

$$(PQ)^2-4=\frac{\psi^2(q)\psi^2(q^3)}{q\psi^2(q^2)\psi^2(q^6)}-4$$
$$=\frac{\varphi(q)\varphi(q^3)}{q\psi(q^2)\psi^2(q^6)}-4$$
$$=\frac{\varphi(-q)\varphi(-q^3)}{q\psi(q^2)\psi^2(q^6)}$$

From (3.10), (3.11) and the above, it is easy to see that

$$(P^8-16)(Q^8-16)=\left[(PQ)^2-4\right]^4,$$

which equivalent to (3.9).

**Corollary 3.4.** Let

$$P:=\frac{\varphi(q)}{q^{\frac{1}{4}}\psi(q^2)} \text{ and } Q:=\frac{\varphi(q^3)}{q^{\frac{3}{4}}\psi(q^6)}.$$

Then,

$$PQ+\frac{16}{PQ}=4\left[\left(\frac{P}{Q}\right)^2+\left(\frac{Q}{P}\right)^2\right]+9. \quad (3.12)$$

**Proof.** By from (2.3), we have

$$\frac{\psi^2(q)}{q^{\frac{1}{4}}\psi^2(q^2)}=\frac{\varphi(q)}{q^{\frac{1}{4}}\psi(q^2)},$$

using this in (3.9), we obtain (3.12).

The identity (3.12) is due to Vasuki and Srivatsakumar [18].

**Theorem 3.5.** Let

$$P:=\frac{\psi(q)}{q^{\frac{1}{4}}\psi(q^3)} \text{ and } Q:=\frac{\psi(q^2)}{q^{\frac{1}{2}}\psi(q^6)}.$$

Then,

$$P^2+\frac{3}{P^2}=\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2. \quad (3.13)$$

**Proof.** By (2.3), (2.9) and (2.10), we have

$$Q^2\left(\frac{Q^2+3}{Q^2-1}\right)=\frac{\psi^2(q^2)}{q\psi^2(q^6)}\left[\frac{\varphi^2(q)\psi^4(q^3)}{\varphi^4(q^3)\psi^2(q^6)}\right]$$
$$=\frac{\psi^2(q^2)}{q\psi^2(q^6)}\frac{\varphi^2(q)}{\varphi^2(q^3)}$$
$$=\frac{\psi^4(q)}{q\psi^4(q^3)}$$
$$=P^4.$$

This is equivalent to (3.13).

**Theorem 3.6.** Let

$$P:=\frac{\varphi(-q)}{\varphi(-q^2)} \text{ and } Q:=\frac{\varphi(-q^5)}{\varphi(-q^{10})}.$$

Then,

$$4\left[(PQ)^2-\frac{1}{(PQ)^2}\right]=\left(\frac{P}{Q}\right)^3-\left(\frac{Q}{P}\right)^3+5\left[\frac{P}{Q}-\frac{Q}{P}\right]. \quad (3.14)$$

**Proof.** By (2.2) and (2.5),

$$1-P^8=1-\frac{\varphi^8(-q)}{\varphi^8(-q^2)}$$
$$=1-\frac{\varphi^4(-q)}{\varphi^4(q)} \quad (3.15)$$
$$=\frac{16q\psi^4(q^2)}{\varphi^4(q)}.$$

Changing $q$ to $q^2$ in (3.15), we obtain

$$1-Q^8=\frac{16q^5\psi^4\left(q^{10}\right)}{\varphi^4\left(q^5\right)}. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\sqrt{\left(1-P^8\right)\left(1-Q^8\right)}=\frac{16q^3\psi^2\left(q^2\right)\psi^2\left(q^{10}\right)}{\varphi^2\left(q\right)\varphi^2\left(q^5\right)}. \tag{3.17}$$

By (2.2), we have

$$\left(PQ\right)^4=\frac{\varphi^4\left(-q\right)\varphi^4\left(-q^5\right)}{\varphi^4(-q^2)\varphi^4(q^{10})}=\frac{\varphi^2\left(-q\right)\varphi^2\left(-q^5\right)}{\varphi^2\left(q\right)\varphi^2\left(q^5\right)}. \tag{3.18}$$

By (2.1), (3.17) and (3.18), we deduce that

$$\left(PQ\right)^4\sqrt{\left(1-P^8\right)\left(1-Q^8\right)}=16q^3\frac{\left(f_1f_4f_5f_{20}\right)^{12}}{\left(f_2f_{10}\right)^{24}}. \tag{3.19}$$

By (2.1), (2.19), (3.17), and (3.18), we also deduce that

$$\begin{aligned}&1-\left(PQ\right)^4-\sqrt{\left(1-P^8\right)\left(1-Q^8\right)}\\&=\frac{\varphi^2\left(q\right)\varphi^2\left(q^5\right)-\varphi^2\left(-q\right)\varphi^2\left(-q^5\right)-16q^3\psi^2\left(q^2\right)\psi^2\left(q^{10}\right)}{\varphi^2(q)\varphi^2(q^5)}\\&=\frac{8qf_2^2f_{10}^2}{\varphi^2(q)\varphi^2(q^5)}\\&=8q\frac{\left(f_1f_4f_5f_{20}\right)^4}{\left(f_2f_{10}\right)^8}.\end{aligned} \tag{3.20}$$

From (3.19) and (3.20), we obtain

$$\begin{aligned}&\left[1-\left(PQ\right)^4-\sqrt{\left(1-P^8\right)\left(1-Q^8\right)}\right]^3\\&=32\left(PQ\right)^4\sqrt{\left(1-P^8\right)\left(1-Q^8\right)},\end{aligned}$$

which implies

$$\frac{A^3+3AB^2}{32\left(PQ\right)^4+B^2+3A^2}=B,$$

where

$$A=1-\left(PQ\right)^4$$

and

$$B=\sqrt{\left(1-P^8\right)\left(1-Q^8\right)}.$$

Squaring the above on both sides and then factoring using maple, we obtain

$$C\left(P,Q\right)D\left(P,Q\right)E\left(P,Q\right)=0,$$

where

$$C\left(P,Q\right)=P^6-5P^2Q^4+5P^4Q^2+4PQ-4\left(PQ\right)^5-Q^6$$

$$D\left(P,Q\right)=P^6-5P^2Q^4+5P^4Q^2-4PQ+4\left(PQ\right)^5-Q^6,$$

and

$$\begin{aligned}E\left(P,Q\right)=&P^{12}-10P^{10}Q^2+16\left(PQ\right)^{10}+15P^8Q^4\\&+20\left(PQ\right)^6+15P^4Q^8-10P^2Q^{10}\\&+16\left(PQ\right)^2+Q^{12}.\end{aligned}$$

It is easy to see that, $P$ and $Q$ have the following series expansion

$$P=1-2q+2q^2-4q^3+6q^4-8q^5+12q^6+\cdots$$

and

$$Q=1-2q^5+2q^{10}-4q^{15}+6q^{20}-8q^{25}+12q^{30}+\cdots$$

Using these in $C(P,Q)$, $D(P,Q)$ and $E(P,Q)$, we obtain

$$\begin{aligned}&C\left(P,Q\right)\\&=2560q^{11}-13760q^{12}-49600q^{13}+149760q^{14}+\cdots\end{aligned}$$

$$\begin{aligned}&D\left(P,Q\right)\\&=-64q+284q^2-1408q^3+4352q^4-12096q^5+\cdots\end{aligned}$$

and

$$\begin{aligned}&E\left(P,Q\right)\\&=64-768\text{q}+5376q^2-28672q^3+129024q^4+\cdots\end{aligned}$$

One can see that $q^{-1}$ $D(P,Q)$ and $q^{-1}$ $E(P,Q)$ does not tends to 0 as $q$ tends to 0, whereas $q^{-1}C(P,Q)$ tends to 0 as $q$ tends to 0. Hence $q^{-1}C(P,Q)=0$ in some neighborhood $q=0$. By analytic continuation $q^{-1}C(P,Q)=0$ in $|q|<1$. Thus we have

$$C\left(P,Q\right)=0\cdot$$

Dividing the above throughout by $\left(PQ\right)^3$, we obtain (3.14).

**Corollary 3.5.** Let

$$P:=\frac{\varphi\left(-q\right)}{\varphi\left(q\right)}\text{ and }Q:=\frac{\varphi\left(-q^5\right)}{\varphi\left(q^5\right)}.$$

Then,

$$4\left[\left(PQ\right)-\frac{1}{\left(PQ\right)}\right]=\left(\frac{P}{Q}\right)^{\frac{3}{2}}-\left(\frac{Q}{P}\right)^{\frac{3}{2}}+5\left[\left(\frac{P}{Q}\right)^{\frac{1}{2}}-\left(\frac{Q}{P}\right)^{\frac{1}{2}}\right].$$

Corollary 3.5 follows from (2.2) and (3.14). The following theorem is due to
Adiga *et al*. [9].

**Theorem 3.7.** Let

$$P:=\frac{\varphi\left(-q\right)}{\varphi\left(-q^2\right)}\text{ and }Q:=\frac{\varphi\left(-q^2\right)}{\varphi\left(-q^{10}\right)}.$$

Then,

$$Q^4P^2 - Q^4 - 4(PQ)^2 - P^4 + 5P^2 = 0.$$

Or

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = Q^2 + \frac{5}{Q^2}.$$

**Proof.** By (2.1), (2.2), (2.3), (2.12) and (2.14),

$$\begin{aligned}\frac{P^2-5}{P^2-1} &= \frac{\varphi^2(-q)-5\varphi^2(-q^5)}{\varphi^2(-q)-\varphi^2(-q^5)}\\ &= \frac{f_2^2}{q\chi^2(-q)\psi^2(q^5)}\\ &= \frac{f_2^4}{qf_1^2\psi^4(q^5)}\\ &= \frac{\psi^2(q)}{q\psi^2(q^5)}\\ &= \frac{\varphi(q)\psi(q^2)}{q\varphi(q^5)\psi(q^{10})}\\ &= \frac{\varphi^2(-q^2)\varphi(-q^2)\psi(q^2)}{q\varphi^2(-q)\varphi^2(-q^{10})\psi(q^{10})}\\ &= \frac{Q^2}{P}\frac{\psi(q^2)}{q\psi(q^{10})}.\end{aligned}$$

Thus

$$\frac{P^2-5}{P^2-1} = \frac{Q^2}{P}\frac{\psi(q^2)}{q\psi(q^{10})}. \quad (3.22)$$

Changing $q$ to $q^2$ in the above, we obtain

$$\frac{Q^2-5}{Q^2-1} = \frac{1}{Q}\frac{\psi(q^4)}{q^2\psi(q^{20})}\frac{\varphi^2(-q^4)}{q\varphi^2(-q^{20})}. \quad (3.23)$$

From (2.3), (3.22), and (3.23), we have

$$\begin{aligned}\frac{\left[(P^2-5)/(P^2-1)\right]^2}{\left[(Q^2-5)/(Q^2-1)\right]} &= \frac{Q^5}{P^2}\frac{\psi^2(q^5)\psi(q^{20})\psi^2(q^{20})}{\psi(q^4)\psi^2(q^{10})\psi^2(-q^4)}\\ &= \frac{Q^5}{P^2}\frac{\varphi(-q^{10})}{\varphi(-q^2)} = \frac{Q^4}{P^2}.\end{aligned}$$

Thus,

$$P^2\left[\frac{P^2-5}{P^2-1}\right]^2 = Q^4\left[\frac{Q^2-5}{Q^2-1}\right].$$

This is equivalent to (3.21).

**Theorem 3.8.** Let

$$P := \frac{\psi(q)}{q^{\frac{1}{8}}\psi(q^2)} \text{ and } Q := \frac{\psi(q^5)}{q^{\frac{5}{8}}\psi(q^{10})}.$$

Then,

$$\begin{aligned}&\left(\frac{Q}{P}\right)^6 + \left(\frac{P}{Q}\right)^6 + 10\left[\left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^4\right]\\ &+15\left[\left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2\right] \quad (3.24)\\ &= (PQ)^4 + \frac{256}{(PQ)^4} + 20.\end{aligned}$$

**Proof.** By (2.1), (2.2), (2.3), (2.15), and (2.18),

$$\begin{aligned}P^2 - Q^2 &= \frac{\psi^2(q)}{q^{\frac{1}{4}}\psi^2(q^2)} - \frac{\psi^2(q^5)}{q^{\frac{5}{4}}\psi^2(q^{10})}\\ &= \frac{1}{q^{\frac{5}{4}}}\left[q\frac{\varphi(q)}{\psi(q^2)} - \frac{\varphi(q^5)}{\psi(q^{10})}\right]\\ &= \frac{1}{q^{\frac{5}{4}}}\left[q\frac{\varphi^2(-q^2)}{\varphi(-q)\psi(q^2)} - \frac{\varphi^2(-q^{10})}{\varphi(-q^5)\psi(q^{10})}\right]\\ &= \frac{1}{q^{\frac{5}{4}}}\left[q\frac{\varphi^2(-q^2)}{\psi^2(-q)} - \frac{\varphi^2(-q^{10})}{\psi^2(-q^5)}\right]\\ &= \frac{\varphi^2(-q^2)}{q^{\frac{5}{4}}\psi^2(-q^5)}\left[q\frac{\psi^2(-q^5)}{\psi^2(-q)} - \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)}\right]\\ &= \frac{-\varphi^2(-q^2)}{q^{\frac{5}{4}}\psi^2(-q^5)}\left[1 - q\frac{\psi^2(q^5)}{\psi^2(q)}\right]\\ &= \frac{-\varphi^2(-q^2)}{q^{\frac{5}{4}}\psi^2(-q^5)\psi^2(q)}\frac{f_2 f_5^3}{f_1 f_{10}} \quad (3.25)\\ &= -\frac{f_1 f_2 f_5 f_{10}}{q^{\frac{5}{4}} f_4^2 f_{20}^2}\end{aligned}$$

Changing $q$ to $q^5$ in (3.10), we have

$$Q^8 - 16 = \frac{\varphi^4(-q^5)}{q^5\psi^4(q^{10})}.$$

Thus, from (2.1), (3.10) and the above, we have

$$P^2Q^2(16-P^8)(16-Q^8) = \left(\frac{f_1 f_2 f_5 f_{10}}{q^{\frac{5}{4}} f_4^2 f_{20}^2}\right)^6. \quad (3.26)$$

Comparing (3.25) and (3.26), we obtain

$$\left(P^2-Q^2\right)^6=P^2Q^2\left(16-P^8\right)\left(16-Q^8\right),$$

which is equivalent to the required result.

The following corollary is due to Vasuki and SrivatsaKumar [18].

**Corollary 3.6.** Let

$$P:=\frac{\varphi(q)}{q^{\frac{1}{4}}\psi\left(q^2\right)} \text{ and } Q:=\frac{\varphi\left(q^5\right)}{q^{\frac{5}{4}}\psi\left(q^{10}\right)}.$$

Then,

$$\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3+10\left[\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2\right]+15\left[\left(\frac{Q}{P}\right)+\left(\frac{P}{Q}\right)\right]$$
$$=\left(PQ\right)^2+\frac{256}{\left(PQ\right)^2}+20.$$

Corollary 3.6 follows from (2.3) and (3.24). The following theorem is due to

Adiga *et al.* [9].

**Theorem 3.9.** Let

$$P:=\frac{\psi(q)}{q^{\frac{1}{2}}\psi\left(q^5\right)} \text{ and } Q:=\frac{\psi\left(q^2\right)}{q\psi\left(q^{10}\right)}.$$

Then,

$$P^4Q^2-4P^2Q^2+5Q^2-P^4-Q^4=0. \quad (3.27)$$

**Proof.** By (2.15), (2.17) and (2.1),

$$\frac{P^2-5}{P^2-1}=\frac{\psi^2(q)-q\psi^2\left(q^5\right)}{\psi^2(q)-5q\psi^2\left(q^5\right)}$$
$$=\frac{\varphi^2\left(-q^5\right)}{f_1^2\chi^2(-q)} \quad (3.28)$$
$$=\frac{\varphi^2\left(-q^5\right)}{\varphi^2(-q)}.$$

Changing $q$ to $q^2$ in (3.28), we obtain

$$\frac{Q^2-1}{Q^2-5}=\frac{\varphi^2\left(-q^{10}\right)}{\varphi^2\left(-q^2\right)}. \quad (3.29)$$

By (2.2), (2.3), (3.28), and (3.29),

$$\left(\frac{P^2-1}{P^2-5}\right)\left(\frac{Q^2-5}{Q^2-1}\right)^2=\frac{\varphi^2\left(-q^5\right)}{\varphi^2(-q)}\frac{\varphi^4\left(-q^2\right)}{\varphi^4\left(-q^{10}\right)}$$
$$=\frac{\varphi^2(q)}{\varphi^2\left(q^5\right)}=\frac{P^4}{Q^2}.$$

Thus, we have

$$Q^2\left(P^2-1\right)\left(Q^2-5\right)^2-P^4\left(P^2-5\right)\left(Q^2-1\right)^2=0.$$

Factorizing the above, we find that

$$C(P,Q)\cdot D(P,Q)=0,$$

where

$$C(P,Q)=P^4Q^2-P^4-4P^2Q^2-Q^4+5Q^2$$

and

$$D(P,Q)=P^2Q^2-Q^2+5-P^2.$$

If $D(P,Q)=0$, then

$$Q^2=\frac{P^2-5}{P^2-1}=\frac{\varphi^2(-q)}{\varphi^2\left(-q^5\right)}$$

by (3.28). This is not true by the definition of Q. Hence $D(P,Q)\neq 0$. Thus we most have $C(P,Q)=0$. This completes the proof.

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