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An Alternative Method of Stochastic Optimization: The Portfolio Model

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Abstract

We provide a new simple approach to stochastic dynamic optimization. In doing so, we derive the existing (standard) results using a far simpler technique than the duality and the variational methods.

Keywords: Stochastic Optimization, Investment, Portfolio

1. Introduction

Previous studies in stochastic optimization relied on the duality approach and/or variational techniques such as using the Feynman Kac formula and the Hamilton-Jacobi-Bellman partial differential equations. Examples include [1-3], among many others.

In this paper, we offer a new simple approach to stochastic dynamic optimization. That is, we prove the previous results using a simpler method than the duality or the Hamilton-Jacobi-Bellman partial differential equations methods. We apply our method to the standard investment model. Our approach is based on dividing the time horizon into sub-horizons and applying Stein's lemma.

2. The Portfolio Model

We use the standard investment model (see, for example, [3], among many others). Similar to previous models, we consider a risky asset and a risk-free asset. The risk-free asset price process is given by $S_0 = e^{\int_t^T r ds}$, where $r \in C_b^2(R)$ is the rate of return.

The dynamics of the risky asset price are given by

$$\mathrm{d}S_s = S_s\left\{\mu \mathrm{d}s + \sigma \mathrm{d}W_s\right\}, \tag{1}$$

where μ and σ are the deterministic rate of return and the volatility, respectively, and W_s is a standard Brownian motion.

The wealth process is given by

$$X_T^{\pi} = x + \int_t^T \left\{rX_s^{\pi} + \left(\mu_s - r_s\right)\pi_s\right\}\mathrm{d}s + \int_t^T \pi_s\sigma_s \mathrm{d}W_s, \tag{2}$$

where x is the initial wealth, $\left\{\pi_s\right\}_{t\le s\le T}$ is the risky portfolio process with $E\int_t^T \pi_s^2 \mathrm{d}s < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x)$ is admissible (that is, $X_s^{\pi} \ge 0$).

The investor's objective is to maximize the expected utility of the terminal wealth

$$V(t,x) = \operatorname*{Sup}_{\pi} E\left[U\left(X_T^{\pi}\right)\middle|\mathcal{F}_t\right] = E\left[U\left(\pi^*\right)\middle|\mathcal{F}_t\right], \tag{3}$$

where $V(.)$ is the (smooth) value function, $U(.)$ is continuous, bounded and strictly concave utility function, and $\mathcal{F}$ is the filtration.

We rewrite (2) as

$$\begin{aligned} X_T^{\pi} &= x + r_u X_u^{\pi} + \left(\mu_u - r_u\right)\pi_u + \pi_u\sigma_u W_u \\ &+ \int_{\bar{t}}^T \left\{r_s X_s^{\pi} + \left(\mu_s - r_s\right)\pi_s\right\}\mathrm{d}s + \int_{\bar{t}}^T \pi_s\sigma_s \mathrm{d}W_s \\ &+ \int_t^T \left\{r_s X_s^{\pi} + \left(\mu_s - r_s\right)\pi_s\right\}\mathrm{d}s + \int_t^T \pi_s\sigma_s \mathrm{d}W_s; \\ &\bar{t} < u < t, u \notin \left[\bar{t},T\right]\cup\left[t,T\right], u \in \left[t,T\right]. \end{aligned} \tag{4}$$

Substituting the above equation into (3) and differentiating with respect to π_u^* (and setting the derivative equal to zero) yields

$$\left(\mu_u - r_u\right)E\left[U'(.)\middle|\mathcal{F}_t\right] + \sigma_u E\left[U'(.)W_u\middle|\mathcal{F}_t\right] = 0. \tag{5}$$

By Stein's lemma

$$\begin{aligned} E\left[U'(.)W_u\middle|\mathcal{F}_t\right] &= Cov\left(X_u, W_u\right)E\left[U''(.)\middle|\mathcal{F}_t\right] \\ &= \pi_u^*\sigma_u E\left[U''(.)\middle|\mathcal{F}_t\right]. \end{aligned} \tag{6}$$

Substituting this into (5) yields

$$\pi_u^* = -\frac{(\mu_u - r_u) E\left[U'(.) \middle| \mathcal{F}_t\right]}{\sigma_u^2 E\left[U''(.) \middle| \mathcal{F}_t\right]} = -\frac{(\mu_u - r_u) V_x(.)}{\sigma_u^2 V_{xx}(.)}. \quad (7)$$

This solution can be generalized to any point on time *s*

$$\pi_s^* = -\frac{(\mu_s - r_s) V_x(.)}{\sigma_s^2 V_{xx}(.)}. \quad (8)$$

This is exactly the solution obtained by the previous literature, but its derivation is far simpler. Furthermore, this approach can be applied to many other stochastic models.

3. References

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