Applied Mathematics, **2011, 2, 705-710**
doi:10.4236/am.2011.26093 Published Online June 2011 (http://www.SciRP.org/journal/am)

On Eccentric Digraphs of Graphs

Medha Itagi Huilgol*, Syed Asif Ulla S., Sunilchandra A. R.
Department of Mathematics, Bangalore University, Bangalore, India
E-mail: medha@bub.ernet.in
Received March 11, 2011; *revised April* 9, 2011; *accepted April* 12, 2011

Abstract

The eccentricity *e*(*u*) of a vertex *u* is the maximum distance of *u* to any other vertex of *G*. A vertex *v* is an eccentric vertex of vertex *u* if the distance from *u* to *v* is equal to *e*(*u*). The eccentric digraph *ED*(*G*) of a graph (digraph) *G* is the digraph that has the same vertex as *G* and an arc from *u* to *v* exists in *ED*(*G*) if and only if *v* is an eccentric vertex of *u* in *G*. In this paper, we have considered an open problem. Partly we have characterized graphs with specified maximum degree such that *ED*(*G*) = *G*.

Keywords: Eccentric Vertex, Eccentric Degree, Eccentric Digraph, Degree Sequence, Eccentric Degree Sequence

1. Introduction

A directed graph or digraph *G* consists of a finite nonempty set $V(G)$ called *vertex set* with vertices and *edge set* $E(G)$ of ordered pairs of vertices called arcs; that is $E(G)$ represents a binary relation on $V(G)$. Throughout this paper, a graph is a symmetric digraph; that is, a digraph *G* such that $(u,v)\in E(G)$ implies $(v,u)\in E(G)$. If (u,v) is an arc, it is said that *u* is adjacent to *v* and also that *v* is adjacent from *u*. The set of vertices which are from (to) a given vertex *v* is denoted by $N^{+}(u)\left[N^{-}(u)\right]$ and its cardinality is the out-degree of *v* [in-degree of *v*]. A *walk* of length *k* from a vertex *u* to a vertex *v* in *G* is a sequence of vertices $u=u_0,u_1,u_2,\cdots,u_{k-1},u_k=v$ such that each pair (u_{i-1},u_i) is an arc of *G*. A digraph *G* is strongly connected if there is a *u* to *v* walk for any pair of vertices *u* and *v* of *G*. The *distance d*(*u*,*v*) from *u* to *v* is the length of a shortest *u* to *v* walk. The *eccentricity e*(*v*) of *v* is the distance to a farthest vertex from *v*. If $dist(u,v)=e(u)(v\neq u)$ we say that *v* is an eccentric vertex of *u*. We define $dist(u,v)=\infty$ whenever there is no path joining the vertices *u* and *v*. The *radius rad*(*G*) and *diameter diam*(*G*) are minimum and maximum eccentricities, respectively. As in [2], the sequential join $G_1+G_2+G_3+\cdots+G_k$ of graphs $G_1,G_2,\cdots,G_k$ is the graph formed by taking one copy of each of the graphs $G_1,G_2,\cdots,G_k$ and adding in additional edges from each vertex of G_i to each vertex in G_{i+1}, for $1\le i\le k-1$. Throughout this paper, $'G=H'$ means *G* and *H* are isomorphic. The reader is referred to Buckley and Harary [2] and Chartrand and Lesniak [3] for additional, undefined terms.

Buckley [4] defines the eccentric digraph $ED(G)$ of a graph *G* as having the same vertex set as *G* and there is an arc from *u* to *v* if *v* is an eccentric vertex of *u*. The paper [4] presents the eccentric digraphs of many classes of graphs including complete graphs, complete bipartite graphs, antipodal graphs and cycles and gives various interesting general structural properties of eccentric digraphs of graphs. The antipodal digraph of a digraph *G* denoted by $A(G)$, has the vertex set as *G* with an arc from vertex *v* in $A(G)$ if and only if *v* is an antipodal vertex of *u* in *G*; that is $dist(u,v)=diam(G)$. This notion of antipodal digraph of a digraph was introduced by Johns and Sleno [5] as an extension of the definition of the antipodal graph of a graph given by Aravamudhan and Rajendran [6]. It is clear that $A(G)$ is a subgraph of $ED(G)$, and $A(G)=ED(G)$ if and only if *G* is self centered.

In [7] Akiyama *et al.* have defined eccentric graph of a graph *G*, denoted by G_e, has the same set of vertices as *G* with two vertices *u* and *v* being adjacent in G_e if and only if either *v* is an eccentric vertex of *u* in *G* or *u* is an eccentric vertex of *v* in *G*, that is $dist_G(u,v)=\min\left\{e_G(u),e_G(v)\right\}$. Note that G_e is the underlying graph of $ED(G)$.

In [8] Boland and Miller introduced the concept of the

*Part of this paper [1] was presented at the International Conference on Emerging Trends in Mathematics and Computer Applications, MEPCO Schlenk Engineering College, Sivakasi, India (Dec. 2010) and had appeared in the Proceedings of the same.

eccentric digraph of a digraph. In [9] Gimbert *et al.* have proved that $G_e = ED(G)$ if and only if *G* is self-centered. In the same paper, the authors have characterized eccentric digraphs in terms of complement of the reduction of *G*, denoted by $\overline{G^-}$ Given a digraph *G* of order *n*, a reduction of *G*, denoted by G^-, is derived from *G* by removing all its arcs incident from vertices with out-degree $n-1$. Note that $ED(G)$ is a subgraph of $\overline{G^-}$.

In [9] , Gimbert *et al.* have studied on the behaviour of sequences of iterated eccentric digraphs. Given a positive integer $k \geq 2$, the k^{th} iterated eccentric digraph of *G* is written as $ED^k(G) = ED(ED^{k-1}(G))$, where $ED^0(G) = G$ and $ED^1(G) = ED(G)$. The iterated sequence of eccentric digraphs concerns with the smallest integer numbers $\dot{p} > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. We call *p the period of G* and *t the tail of G*; these quantities are denoted *p*(*G*) and *t*(*G*) respectively. In [8,10] Boland *et al.* have discussed many interesting results about eccentric digraphs. Also they have listed open problems about these graphs. One of these open problems is being discussed mainly in this paper. We have characterized graphs with specified maximum degree such that $ED(G) = G$.

2. Basic Results

In this section we list some results which are quite evident for eccentric digraphs of graphs.

Remark 1. *Since every vertex in a graph has at least one vertex at eccentric distance it follows that every vertex in an eccentric digraph will have out degree at least one.*

Remark 2. *There exists no directed cycle in an eccentric digraph.*

Let C be a directed cycle with edge *uv* being directed from $u \to v$ as shown below in **Figure 1**.

The other edges can be bidirectional. If all the other edges except $u \to v$ are bidirectional then a symmetric edge vy_1 indicates the equality of eccentric values of v and y_1. Likewise $ecc(y_1) = ecc(y_2) = \cdots = ecc(y_n) = ecc(x)$. Also edge xu is symmetric. Hence, $ecc(u) = ecc(x)$. So also, $ecc(v) = ecc(y_1) = ecc(x) = ecc(u)$. This contradicts the existence of the directed arc $u \to v$ as u being tail has less eccentricity as compared to that of v.

The same argument can be extended to a directed cycle with more than one directed arc, as the eccentricities go on increasing in the same direction.

The above two conditions are not sufficient for a graph to be an eccentric digraph.

For example consider a symmetric cycle having a pendant vertex adjacent to one of the vertices on the cycle. The pendant vertex having in-degree zero and out-degree one as in **Figure 2**.

Vertex x_i is at eccentric distance from u. Let v be adjacent to u and lying on the eccentric path connecting u and x_i. All the vertices in the graph except x_i are at distance atmost $n-1$ from u, where n is the eccentricity of u. This implies v being adjacent to u will have eccentricity n. But v lying on the symmetric circle can have eccentricity $\geq n+1$. Therefore the above graph cannot be an eccentric digraph.

Also we give a counter example for a problem given in [2] , as follows:

Problem 3, Ex. 2.2 (p. 41) [2]: If *G* is self-centered with radius 2 , then $\overline{G}$ is self-centered with radius 2 .

Counter Example: Consider C_7, join the vertices at distance 2 in C_7. Let *G* be the resulting graph with $rad(G) = 2$. Considering $\overline{G}$, we observe that $\overline{G} \cong C_7$; that is $\overline{G}$ is self-centered of radius 3 .

3. Graphs with Isomorphic Eccentric Digraph

In case of undirected graphs, Buckley [4] proved that the

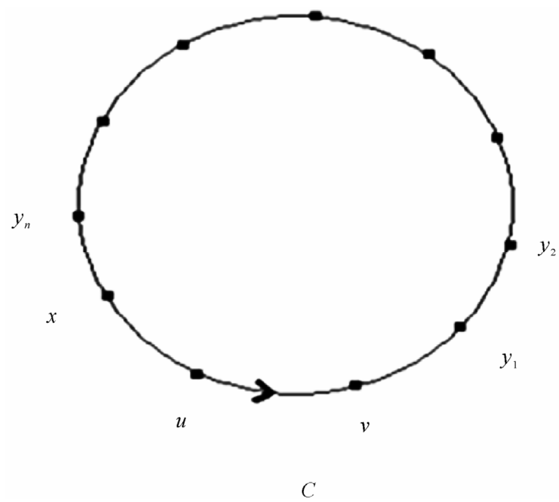

Figure 1. Directed cycle *C*.

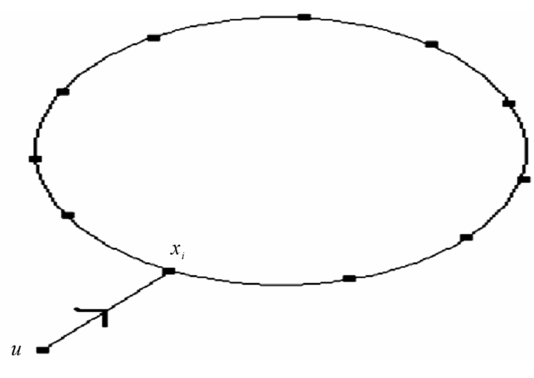

Figure 2. Directed cycle with unidirectional edge u-x_i.

eccentric digraph of a graph G is equal to its complement, $ED(G)=\overline{G}$, if and only if G is either a self-centered graph of radius two or G is the union of $k \geq 2$ complete graphs. In [9], Gimbert *et al.* have proved that the eccentric digraph $ED(G)$ is symmetric if and only if G is self centered.

Here we are looking at graphs which have their eccentric digraphs isomorphic to themselves. So by Gimbert's result these graphs are self-centered graphs. In this section we consider self-centered, undirected graphs. The following observations are easily justified.

Remark 3. *Odd cycles is a class of graphs for which* $ED(G)=G$.

Remark 4. *Odd cycles are graphs with minimum number of edges and maximum eccentricity on given number of vertices such that* $ED(G)=G$.

Remark 5. *For a self-centered graph G with radius* $\geq$ 3, *the complement* $\overline{G}$ *is self-centered with radius equal to two. Hence* $G \subset \overline{G}$, *and* $G \not\cong \overline{G}$, *and* $ED(G)$ *is isomorphic to a subgraph of* $\overline{G}$. *Further, by using Buckley's result* [4], *we can say that* $ED(\overline{G})=\overline{\overline{G}}=G$. *That is if* $ED(\overline{G})=ED(G)$, *then* $G=ED(G)$.

Remark 6. *Complete graphs is another class of graphs for which* $ED(G)=G$.

Remark 7. *It is easy to see that for graphs upto order* 7, *the only graphs for which* $ED(G)=G$, *are* $K_2, K_3, K_4, K_5, C_5, K_6, K_7, C_7$.

Remark 8. *Two isomorphic graphs have their eccentric digraphs isomorphic*, *but the converse need not be true always.*

As an example, as shown inn **Figure 3**, we give a pair of non-isomorphic, self-centered graphs with same eccentricity having one eccentric digraph.

Lemma 9. *Let* G *be a self-centered graph with radius* 2, *then* $ED(G)=G$ *if and only if* G *is self-complementary.*

Proof. Given self-centered graph G be self-complementary with radius 2. Then by Buckley's characterization theorem [4], $ED(G)=\overline{G}=G$. Conversley, consider a self-centered graph of radius 2, with $ED(G)=G$. Then, $ED(G)=\overline{G}$, that is $G=ED(G)=\overline{G}$. Hence the result. □

Lemma 10. *All self-centered graphs* G *with eccentricity greater than or equal to* 3 *with* $\overline{G}$ *having* $period=1$, $tail=1$, *satisfies the condition* $ED(G)=G$.

Proof. Let G be a self-centered graph with eccentricity ≥ 3 .Then $\overline{G}$ is self-centered graph with eccentricity equal to 2. Hence, $ED(\overline{G})=\overline{\overline{G}}=G$; that is $ED^2(\overline{G}) \cong ED(G)$. If $\overline{G}$ has $period=1$ and $tail=1$, that is $ED^2(\overline{G}) \cong ED(\overline{G})$ then $ED^2(\overline{G}) \cong ED(\overline{G}) \cong G$. But $ED^2(\overline{G}) \cong ED(G)$ implies

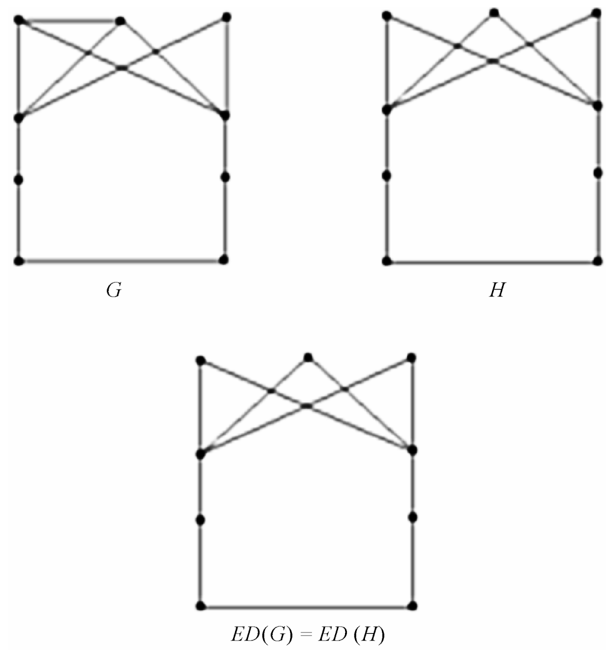

Figure 3. ***ED*(*G*) = *ED*(*H*).**

$ED(G) \cong ED^2(\overline{G}) = ED(\overline{G}) \cong G$. Hence the result. □

For connected graph G to be isomorphic to $ED(G)$ the necessary condition is that the graph should not be unique eccentric node graph as defined by Parthasarathy and Nandakumar [11]. Also, for $G=ED(G)$, the necessary condition is that the for every vertex of degree say k, there must exist another vertex with k number of eccentric vertices. This can be defined as eccentric degree of a vertex.

Definition 11. *For a vertex v of a graph* G *is defined to be the number of vertices at eccentric distance from v. Also the eccentric degree sequence of a graph is defined as a listing of eccentric degrees of vertices written in non-increasing order*.

So for $ED(G)=G$, the eccentric degree sequence of G should be equal to the degree sequence of G. But this condition is not sufficient as seen in the example below, depicted as **Figure 4**. Here both G and $ED(G)$ have their degree seqence and eccentric degree sequences as $(3^4, 2^9)$, but $ED(G) \not\cong G$.

Next, we consider self-centered graphs with given maximum degree $\Delta(G)$. By [2], $\Delta(G) \leq p-2r+2$, for a self-centered graph G with radius r. Our next result shows that there is no possibility of having a graph with $ED(G)=G$, with $\Delta(G)=p-2r+2$.

Proposition 12. *There does not exist a graph* G *with* $\Delta(G)=p-2r+2$, *such that* $ED(G)=G$.

Proof. Let G be a self-centered graph with $\Delta(G)=p-2r+2$ Let $u \in V(G)$ such that *deg* $u = p-2r+2$. Partition the vertex set into sets lying

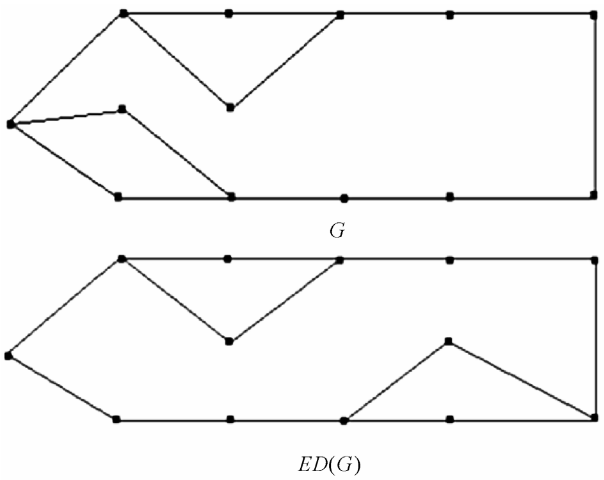

Figure 4. *G* and *ED*(*G*) having same eccentric degree sequence and degree sequence, but *ED*(*G*) ≇ *G*.

at distance i from u and name them as A_i, $1 \le i \le r$. Since $deg\, u = p-2r+2$, $|A_1| = p-2r+2$. As G is self-centered, it cannot contain a cut-vertex and hence $|A_i| \ge 2,\ 2 \le i \le r$. Hence, $2r-2$ vertices are needed to satisfy the conditions of the graph under consideration, but we have $p-(p-2r+2) = 2r-3$ vertices. Hence, it is not possible to construct a graph with $ED(G) = G$, with $\Delta(G) = p-2r+2$.

Theorem 13. *A connected self-centered graph G with $\Delta(G) = p-2r+1$ is isomorphic to its eccentric digraph if and only if its degree sequence is of the form $(p-2r+1)^2, 2^{p-2}$ with structure*

$$K_1 + \overline{K_{p-2r+1}} \cdot F \cdot \overline{K_2} + (\overline{K_2} - H) + (\overline{K_2} - H) + \cdots + (\overline{K_2} - H)\{r - times\},$$

where F is the graph obtained by joining one vertex of $\overline{K_{p-2r+1}}$ to one vertex of $\overline{K_2}$ and remaining $p-2r$ vertices to one vertex of $\overline{K_2}$, and H is the $1-$ factor removed from successive $\overline{K_2} + \overline{K_2}$.

Proof. Let G be a self-centered graph with $\Delta(G) = p-2r+1$. Let u be a vertex of G with $deg\, u = p-2r+1$. As seen in the above Lemma, each A_i should contain at least two vertices each, for G to satisfy $G = ED(G)$, with self-centeredness and not being unique eccentric node graph. Since the remaining $2r-2$ vertices are to be distributed into $r-1$ sets with at least two in each set, it follows that each A_i has exactly two vertices.

Let the vertices in the set A_k be labelled x_k^1 and x_k^2 for all k, $2 \le k \le r$. Since G has no cutvertex, x_{r-1}^1 should be adjacent to at least one vertex of A_r, let us say, to x_r^1 and similarly let x_{r-1}^2 be adjacent to x_r^2. Degree of x_r^1 is at least two, so it can be adjacent to any of x_r^2, x_{r-1}^2 or both. Suppose, x_r^1 is adjacent to x_{r-1}^2, then x_r^1 will have u as the only eccentric vertex, a contradiction.

Similar contradiction is arrived if x_r^1 is adjacent to both x_r^2 and x_{r-1}^2 Same argument can be applied in case of x_r^2. Therefore, x_r^1 and x_r^2 are mutually adjacent to have degree at least two. There are two paths P_1 and P_2^{ij} where

$$P_1 = u, e_1^i, x_1^i, e_2^j, x_2^1, e_3^1, x_3^1, \cdots, e_{k-1}^1, x_k^1, e_{k+1}^1, x_k^1, e_{k+1}^1, x_{k+1}^1, \cdots, x_{r-2}^1, e_{r-1}^1, x_{r-1}^1, e_r^1, x_r^1; i, j = 1$$

and

$$P_2^{ij} = u, e_1^i, x_1^i, e_2^j, x_2^2, e_3^2, x_3^2, \cdots, e_{k-1}^2, x_{k-1}^2, e_k^2, x_k^2, e_{k+1}^2, x_{k+1}^2, \cdots, x_{r-2}^2, e_{r-1}^2, x_{r-1}^2, e_r^2, x_r^2; 2 \le i, j \le p-2r+1,$$

i need not be equal to j.

Other cases are proved as follows:

Claim 1: The vertices x_{k+1}^1 are adjacent to either x_k^1 or x_k^2, but not both; for all k, $2 \le k \le r-1$.

Proof of the claim: Since G has no cut vertex each vertex belonging to A_k, $2 \le k \le r-1$, is adjacent to at least one vertex in A_{k-1} and A_{k+1}. Without loss generality let each vertex x_{k+1}^1 be adjacent to x_k^1, for all k, $2 \le k \le r-1$. Let us consider A_k, with vertices x_k^1 and x_k^2. Let x_{k+1}^1 be adjacent to x_k^2 along with x_k^1. Then eccentricity of x_{k+1}^1 can be at most $r-1$ as any vertex lying on the path P_1 or P_2^{ij} can be at most at a distance $r-1$ from x_{k+1}^1. And any vertex on the path P_2^{ij} can be at most at a distance of $r-2$ from x_k^2. In any case if x_{k+1}^1 is adjacent to both x_k^2 and x_k^1 then G ceases to be self-centered, a contradiction and hence the proof of the claim.

Claim 2: Each A_i, $2 \le k \le r-1$, is independent, that is, $\langle A_i \rangle = \overline{K_2}$.

Proof of the claim: First we prove the result for A_{r-1}. If the vertices x_{r-1}^1 and x_{r-1}^2 belonging to A_{r-1} are adjacent then x_{r-1}^1 will have u as the only eccentric vertex, a contradiction proves that $\langle A_{r-1} \rangle = \overline{K_2}$.

For any other A_k, $2 \le k \le r-2$, if x_k^1 and x_k^2 are adjacent then, eccentricity of x_k^1 and x_k^2 will be at most $r-2$ as vertices on P_1 or P_2^{ij} will be at distance at most $r-2$ from x_k^1 or x_k^2 and hence for all k, $2 \le k \le r-1$, $\langle A_i \rangle = \overline{K_2}$.

Claim 3: x_2^1 and x_2^2 do not have common neighbors in A_1.

Proof of the claim: As in case of the vertices of A_{r-1}, the vertices x_2^1 and x_2^2 of A_2 will have their eccentricity equal to $r-1$ if they have a common neighbor in A_1, hence the claim.

Claim 4: x_2^1 is adjacent to x_1^1 and x_2^2 is adjacent to all other p-2r vertices of A_1.

Proof of the claim: If x_2^1 is adjacent to

$\left\{x_1^k\right\}_{k=1}^{k=i}$, $1 < i < p-2r$, then two vertices x_r^2 and x_{r-1}^2 have eccentric degree $k + 1$, but only one vertex x_2^1 has degree equal to $k+1$, a contradiction.

If x_2^1 and x_2^2 have degree k each ,that is if $2k = p-2r+1$, then x_r^1, x_r^2, x_{r-1}^1 and x_{r-1}^2 have eccentric degree equal to $k + 1$, but only two vertices x_2^1 and x_2^2 have degree $k + 1$.

Hence, x_2^1 is adjacent to x_1^1 and x_2^1 not adjacent to x_1^i, $1 < i < p-2r$.

Finding the eccentric degree of all vertices of G we see that, ecc. $\deg\left(x_k^1\right) = 2,\ 1 \le k \le r-1$ and ecc.deg $\left(y_k^1\right) = 2$, except for x_{r-1}^1, where $y_k^1 \in N\left(x_k^1\right)$.

Similarly, ecc. $\deg\left(x_k^2\right) = 2,\ 1 \le k \le r-1$ and ecc.deg $\left(y_k^2\right) = 2$, where $y_k^2 \in N\left(x_k^2\right)$. Hence, the eccentric degree sequence of G is $\left(p-2r+1\right)^2, 2^{p-2}$, which is same as that of degree sequence of G.

Claim 5: $\left\langle A_1 \right\rangle = \overline{K_{p-2r+1}}$.

Proof of the claim: We show that for any two vertices x_1^t and x_1^s where $t \ne s,\ 1 \le t,\ s \le p-2r+1,$ are not adjacent. Here we need to consider two possibilities:

Case 1): $x_1^t \in N\left(x_2^1\right)$ and $x_1^s \in N\left(x_2^2\right)$

Case 2): $x_1^t \in N\left(x_2^2\right)$ and $x_1^s \in N\left(x_2^2\right)$

In case 1), x_1^t will have only one eccentric vertex, a contradiction.

In case 2), $\deg\left(x_1^t\right), \deg\left(x_1^s\right) \ge 3$ then only the vertices x_r^1 or x_r^2 of A_r and x_{r-1}^1 or x_{r-1}^2 of A_{r-1} can have vertices of A_1 as eccentric vertices, along the paths P_1 or P_2^{ij}, since x_2^1 and x_2^2 do not share a common neighbor in A_1, as claimed in 3.

By Claim 4, we see that x_2^1 is adjacent to x_1^1 and x_2^2 is adjacent to all other $p - 2r$ vertices of A_1, implies that when degrees of x_1^t and x_1^s are changed by making them adjacent, the eccentric degree of any other vertex of does not change, hence we will not be able to get $ED(G) = G$, a contradiction proves the claim.

Referring all the claims we conclude that G is of the form defined in the statement of the theorem.

Converse is easy to observe that if G is as given in the statement of the theorem, then $ED(G) = G$. □

In the next result we consider a particular case of graphs with $ED(G) = G$, that is, odd cycles.

Remark 14. *In a labelled* $C_{2n+1}, n \ge 1$, *two vertices* v_i, v_j *are at eccentric distance in* $ED\left(C_{2n+1}\right)$, *if and only if* $d_G\left(v_i, v_j\right) = \frac{n}{2}$ *or* $\frac{n+1}{2}$.

Remark 15. *For unlabelled odd cycles, iterations of* $ED\left(C_{2n+1}\right)$ *can be packed into* K_n, *since there are* $\frac{n-1}{2} - 1 = \frac{n-3}{2}, ED(G)'s$, *whereas,* $rad\left(C_{2n+1}\right) = n$.

In case of labelled odd cycles the sequence of $ED(G)$'s can be packed into K_p, if the permutation on p number of vertices defined by

$$f(1) = 1,\ f(2i) = r+2-i, i = 1,2,3,\cdots,r;$$
$$f(2i+1) = 2r+2-i, i = 1,2,3,\cdots,r$$

is a product of three cyclic permutations of length $1, r, r$, respectively.

Proposition 16. *There exists a self-centered graph* G, *such that* $ED(G) = G$, *containing an odd cycle.*

Proof. Let be C_p be a cycle, whose vertices are labeled as $1,2,3,\cdots,p$. Let $S = \{1,2,3,\cdots,p\}$. Define a function on the set of vertices of C_p as

$$f(1) = 1, f(2i) = \left((p-2i+3)/2\right), 1 \le i \le \left((p-1)/2\right),$$
$$f(2i+1) = p+1-i, 1 \le i \le \left((p-1)/2\right).$$

Now, we partition of set of vertices of C_p into $\{S_1, S_2, S_3, \cdots, S_m\}$ where,

$$S_1 = (1), S_2 = \left\{2, f(2), f^2(2), \cdots, f^n(2)\right\}, 2 \notin S_1,$$

where n is the least positive integer such that $f^{n+1}(2) = 2$ whereas $f^n(2)$ is obtained by applying f on 2, n times. Similarly,

$$S_m = \left\{l, f(l), f^2(l), \cdots, f^{m-1}(l)\right\},$$
$$l \notin S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_{m-1},$$

where m is the least positive integer such that $f^{m+1}(l) = l$. It is clear that $S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_m = S$ and $S_i \cap S_j = \phi, 1 \le i, j \le m, i \ne j$. Now, for each vertex of S_i we define sets of vertices not in C_p by $S_{i_j} = \left\{f_{i_j}(l),\ f_{i_j}^2(l), \cdots, f_{i_j}^{n_i}(l)\right\}$ for each i, and $j = 1,2,3,\cdots$ whose adjacencies are to the vertices of C_p defined by: If $f^g(l)$, when $1 \le l \le p$ is adjacent to $f^h(l)$, then, $f_{i_j}^g(l)$ is adjacent to $N\left(f^g(l)\right)$, $f_{i_j}^h(l)$ and $f_{i_j}^h(l)$ is adjacent to $N\left(f^h(l)\right), f_{i_j}^g(l)$; otherwise, $f_{i_j}^g(l)$ is adjacent to $N\left(f^g(l)\right)$ and $f_{i_j}^h(l)$ is adjacent to $N\left(f^h(l)\right)$. So we get a self-centered graph with radius $(p-1)/2$, satisfying $ED(G) = G$, as vertices l on C_p and respective $f_{i_j}^m(l),\ 1 \le l \le p$, have same distance from other vertices in the graph. By Remark 15, we get $ED(G) \cong G$. □

Following is an example of a self-centered graph with radius 4, as shown in **Figure 5**, satisfying $ED(G) \cong G$, with C_9, as base. So $S = \{1,2,\cdots,9\}$ and $V\left(C_p\right) = \left\{S_1, S_2, S_3, S_4, S_5\right\}$, where $S_1 = \{1\}, S_2 = \{2,5,8\}, S_3 = \{4\}, S_4 = \{7\}, S_5 = \{3,6,9\}$.

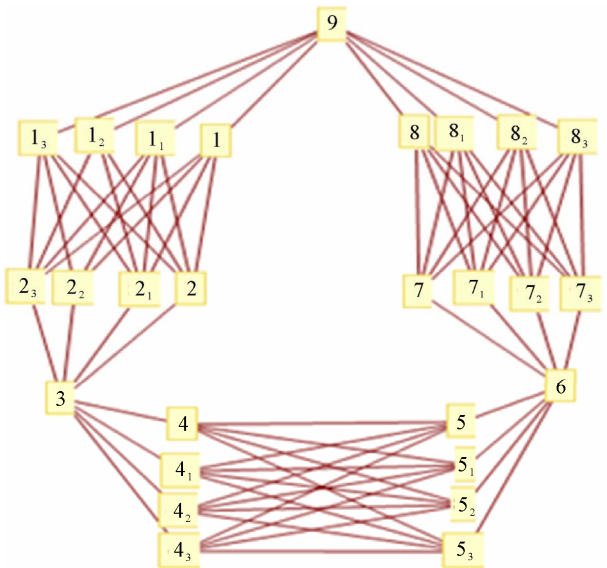

Figure 5. $ED(G) \cong G$.

For S_i, $1 \le i \le 4$, we have

$$S_{1_1} = \{1_1\},\ S_{1_2} = \{1_2\}, S_{1_3} = \{1_3\}; S_{2_1} = \{2_1, 5_1, 8_1\},$$
$$S_{2_2} = \{2_2, 5_2, 8_2\},\ S_{2_3} = \{2_3, 5_3, 8_3\}; S_{3_1} = \{4_1\},$$
$$S_{3_2} = \{4_2\}, S_{3_3} = \{4_3\}; S_{4_1} = \{7_1\}, S_{4_2} = \{7_2\}, S_{4_3} = \{7_3\}.$$

4. References

[1] M. I. Huilgol, S. A. S. Ulla and A. R. Sunilchandra, "On Eccentric Digraphs of Graphs," *Proceedings of the International Conference on Emerging Trends in Mathematics and Computer Applications*, MEPCO Schlenk Engineering College, Sivakasi, 16-18 December 2010, pp. 41-44.

[2] F. Buckley and F. Harary, "Distance in Graphs," Addison-Wesley, Redwood City, 1990.

[3] G. Chartrand and L. Lesniak, "Graphs and Digraphs," 3rd Edition, Chapman & Hall, London, 1996.

[4] F. Buckley, "The Eccentric Digraph of a Graph," *Congressus Numerantium*, Vol. 149, 2001, pp. 65-76.

[5] G. Johns and K. Sleno, "Antipodal Graphs and Digraphs," *International Journal of Mathematics and Mathematical Sciences*, Vol. 16, No. 3, 1993, pp. 579-586. doi:10.1155/S0161171293000717

[6] R. Aravamudhan and B. Rajendran, "On Antipodal Graphs," *Discrete Mathematics*, Vol. 49, No. 1, 1984, pp. 193-195. doi:10.1016/0012-365X(84)90117-1

[7] J. Akiyama, K. Ando and D. Avis, "Eccentric Graphs," *Discrete Mathematics*, Vol. 56, No. 1, 1985, pp. 1-6. doi:10.1016/0012-365X(85)90188-8

[8] J. Boland and M. Miller, "The Eccentric Digraph of a Digraph," *Proceedings of the* 12*th Australasian Workshop of Combinatorial Algorithms* (*AWOCA* 2001), Freiburg, 3-7 September 2001, pp. 66-70.

[9] J. Gimbert, M. Miller, F. Ruskey and J. Ryan, "Iterations of Eccentric Digraphs," *Bulletin of the Institute of Combinatorics and Its Applications*, Vol. 45, 2005, pp. 41-50.

[10] J. Boland, F. Buckley and M. Miller, "Eccentric Digraphs," *Discrete Mathematics*, Vol. 286, No. 1-2, 2004, pp. 25-29. doi:10.1016/j.disc.2003.11.041

[11] R. Nandakumar and K. R. Pathasarathy, "Unique Eccentric Point Graphs," *Discrete Mathematics*, Vol. 46, No. 1, 1983, pp. 69-74. doi:10.1016/0012-365X(83)90271-6

