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Soliton Resonances of the Nonisospectral Modified Kadomtsev-Petviashvili Equation

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Abstract

Many equations possess soliton resonances phenomenon, this paper studies the soliton resonances of the nonisospectral modified Kadomtsev-Petviashvili (mKP) equation by asymptotic analysis.

Keywords: Soliton, Resonances, Hirota Bilinear Method, Nonisospectral mKP Equation

1. Introduction

In the process of searching for explicit solutions, quite a few systematic methods have been developed, such as inverse scattering transformation [1], Darboux transformations [2], Hirota's bilinear method [3-5], and so on. Among them, the bilinear method first proposed by Hirota provides us with a comprehensive approach to construct exact solutions of nonlinear evolution equations (NEEs). Meanwhile, as the interacting of the solution, soliton resonance has been studied in many papers. Miles obtained resonantly interacting solitary waves of KP equation [6], these solutions are coherent structures that describe the diffraction of a soliton at a corner, and suggest that, under certain conditions, a KP soliton can't turn at a convex corner without separating or otherwise losing its identity. Thus, these structures provide a solution of the problem of "Mach reflection" in water waves, and this phenomenon is now known as soliton resonance. Asymptotic analysis is a very important tool in studying the behaviors of soliton solutions, we call the asymptotic line soliton solutions as $y \to -\infty$ and as $y \to -\infty$ the incoming and outgoing line soliton solutions, respectively. The amplitudes, directions and even the number of incoming solitons are in general different from those of the outgoing ones, when resonance occurs two soliton solutions under certain condition resonate and create a new soliton solution.

Multisoliton solutions exhibiting nontrivial spatial structures and interaction patterns were found in many well-known soliton equations. Hirota studied resonances of solitons in one-dimensional space theoretically taking the Sawada-Kotera equation with a nonvanishing boundary condition as an example by his bilinear method [7], in which he pointed out that two solitons at the resonant state fused after colliding with each other, or a soliton splited into two solitons. Other (1 + 1)-dimensional space equations like KdV-SK and Hirota-Satsuma equations [8] and Boussinesq equation [9].

However more emphases are placed on (2 + 1)-dimensional ones, the most relevant with ours like the following: Wadati clarified the fundamental properties of the soliton in KP equation [10], Medina then went further in this equation [11], Pashaev created four virtual soliton resonance solution for KP-II [12], Biondini made use of tau-function in Wronskian to study it [13], after that Isojima studied the parameter regions for resonance and also study the "spider web"-like solution for cKP system [14,15], the approach of the Reference [16] for MKP-II equation allows audiences to interpret the resonance soliton as a composite object of two dissipative solitons in (1 + 1) dimensions, Hao investigated the resonance of two line solitons of the nonisospectral KP equation [17] which classified the resonance condition clearly. Resonance can also occur in (3 + 1)-dimensional system [18] and even multi-dimensional space [19,20].

In recent years, much attention has been paid to the study of nonisospectral systems [21], as nonisospectral evolution equations are of physical and mathematical importance, which can be used to describe solitary waves in a certain type of non-uniform media with a relaxation effect. The aim of this paper is to clarify the fundamental properties of the soliton resonances in the (2 + 1)-dimensional nonisospectral mKP equation

$$\begin{aligned}&4u_t + y\left(u_{xxx} - 6u^2 u_x + 6u_x \partial^{-1} u_y + 3\partial^{-1} u_{yy}\right)\\&+2xu_y - u^2 + 3\partial^{-1} u_y = 0\end{aligned} \tag{1.1}$$

whose Wronskian and Grammian type solutions have been studied by Deng [22] and Zhang [23] respectively.

This letter is organized as follows: in Section 2, the 2- and 3-soliton solution of Equation (1.1) will be presented using Hirota's bilinear method. Then 2- and 3-soliton resonances will be studied in Sections 3 and 4 respectively. Finally, concluding remarks are given in Section 5.

2. 2- and 3-Soliton Solutions of the Nonisospectral mKP Equation

Through the transformation $u = \left(\log \frac{g}{f}\right)_x$, Equation (1.1) can be transformed into the bilinear form

$$D_y g \cdot f - D^2{}_x g \cdot f = 0 \tag{2.1a}$$

$$\begin{aligned}&4D_t g \cdot f + y\left(D^3{}_x g \cdot f + 3D_x D_y g \cdot f\right)\\&+2xD_y g \cdot f + g_x f + g f_x = 0\end{aligned} \tag{2.1b}$$

where D is the well-known Hirota bilinear operator

$$\begin{aligned}&D^l{}_x D^m{}_y D^n{}_t a \cdot b\\&= \left(\partial x - \partial x'\right)^l \left(\partial y - \partial y'\right)^m \left(\partial t - \partial t'\right)^n\\&\quad \cdot a(x,y,t) b(x',y',t') \big| x'\\&= x, y' = y, t' = t\end{aligned}$$

If we note the N-soliton solution as $u_N \underline{\underline{\Delta}} \left(\log \frac{g_N}{f_N}\right)_x$ and

$$\begin{aligned}g_N &= \sum_{\varepsilon=0,1} \exp\left[\sum_{j=1}^{N} \varepsilon_j \left(\theta_j + \log b_j\right) + \sum_{1\le j<l} \varepsilon_j \varepsilon_l A_{jl}\right],\\ f_N &= \sum_{\varepsilon=0,1} \exp\left[\sum_{j=1}^{N} \varepsilon_j \left(\theta_j + \log a_j\right) + \sum_{1\le j<l} \varepsilon_j \varepsilon_l A_{jl}\right],\end{aligned} \tag{2.2}$$

where the sum is taken over all possible combinations of $\varepsilon_j = (0,1)$ $(j = 1,2,\cdots,N)$, then the first three soliton solutions are

$$g_1 = 1 + a_1 e^{\theta_1},\ f_1 = 1 + b_1 e^{\theta_1}, \tag{2.3a}$$

$$\begin{aligned}g_2 &= 1 + a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_1 a_2 e^{\theta_1 \theta_2 - \Delta_{12}},\\ f_2 &= 1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + b_1 b_2 e^{\theta_1 \theta_2 - \Delta_{12}},\end{aligned} \tag{2.3b}$$

$$\begin{aligned}g_3 &= 1 + a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_3 e^{\theta 3} + a_1 a_2 e^{\theta_1 \theta_2 - \Delta_{12}} + a_1 a_3 e^{\theta_1 \theta_3 - \Delta_{13}}\\&\quad + a_2 a_3 e^{\theta_2 \theta_3 - \Delta_{23}} + a_1 a_2 a_3 e^{\theta_1 + \theta_2 + \theta_3 - \Delta_{12} - \Delta_{13} - \Delta_{23}},\\ f_3 &= 1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + b_3 e^{\theta 3} + b_1 b_2 e^{\theta_1 \theta_2 - \Delta_{12}} + b_1 b_3 e^{\theta_1 \theta_3 - \Delta_{13}}\\&\quad + b_2 b_3 e^{\theta_2 \theta_3 - \Delta_{23}} + b_1 b_2 b_3 e^{\theta_1 + \theta_2 + \theta_3 - \Delta_{12} - \Delta_{13} - \Delta_{23}},\end{aligned}, \tag{2.3c}$$

where

$$A_{ij} \underline{\underline{\Delta}} \frac{\left(k_i - k_j\right)\left(q_i - q_j\right)}{\left(k_i + q_j\right)\left(k_j + q_i\right)} \quad (1 \le i < j \le 3),$$

$$\theta_i = \left(k_i + q_i\right) x - \left(k_i{}^2 - q_i{}^2\right) y + \delta_i,\ e_i^{\delta} = \omega_i,$$

$e^{-\Delta_{ij}} = A_{ij} > 0$, k_i, q_i, a_i, b_i and ω_i are all functions corresponding to t, which satisfy the following dispersion relations:

$$\begin{aligned}&k_{i,t} = \frac{1}{2} k_i{}^2, q_{i,t} = -\frac{1}{2} q_i{}^2, a_i = q_i, b_i = -k_i,\\&\omega_{i,t} = \frac{1}{4}\left(q_i - k_i\right)\omega_i,\ (i = 1,2,3).\end{aligned}$$

What's more, in order to avoid the divergence of u, we suppose f_i and g_i are all positive. Let $k_i + q_i = \mu_i$ and $k_i - q_i = \nu_i$, then θ_i can be rewritten as $\theta_i = \mu_i\left(x - \nu_i y\right) + \delta_i$ and without lose of generality we suppose $\nu_i > \nu_j\ (i > j)$.

3. 2-Solitons

In general, a soliton is observed when the following two conditions are satisfied:

1) Two terms of Equation (2.3b) are so large that other two terms are neglected.

2) Under the condition 1), the large two terms are of the same order. Under these two conditions, the peak of the soliton is on the line $\theta_i = cons\tan t$.

3.1. Pure 2-Soliton

When $0 < A_1 < \infty$ and $\ne 1$, for the limit $y \to -\infty$, $\theta_1 > \theta_2$ the condition 1) and 2) are satisfied in two regions:

$$\begin{cases} u^{(1)} = \left(\log \dfrac{1 + a_1 e^{\theta_1}}{1 + b_1 e^{\theta_1}}\right) x, & \theta_1 \approx 0, \theta_2 \to -\infty \\ u^{(2)} = \left(\log \dfrac{a_1 e^{\theta_1} + a_1 a_2 e^{\theta_1 \theta_2 - \Delta_{12}}}{b_1 e^{\theta_1} + b_1 b_2 e^{\theta_1 \theta_2 - \Delta_{12}}}\right) x, & \theta_1 \to +\infty, \theta_2 \approx \Delta_{12} \end{cases} \tag{3.1}$$

so when $y \to -\infty$, $u = u^{(1)} + u^{(2)}$. As

$$\left(\log \frac{a_1 e^{\theta_1} + a_1 a_2 e^{\theta_1 \theta_2 - \Delta_{12}}}{b_1 e^{\theta_1} + b_1 b_2 e^{\theta_1 \theta_2 - \Delta_{12}}}\right)_x = \left(\log \frac{1 + a_2 e^{\theta_2 - \Delta_{12}}}{1 + b_2 e^{\theta_2 - \Delta_{12}}}\right)_x,$$

we will use the simplified for later convenience.

Similarly, when, $y \to +\infty$,
$u = \left(\log \frac{1+a_1 e^{\theta_1 - \Delta_{12}}}{1+b_1 e^{\theta_1 - \Delta_{12}}}\right)_x + \left(\log \frac{1+a_2 e^{\theta_2}}{1+b_2 e^{\theta_2}}\right)_x$ Above all, both of them have four arms and displays the regular interaction, that means two soliton solutions maintain their original amplitudes and velocities during the interaction (See **Figure 1**).

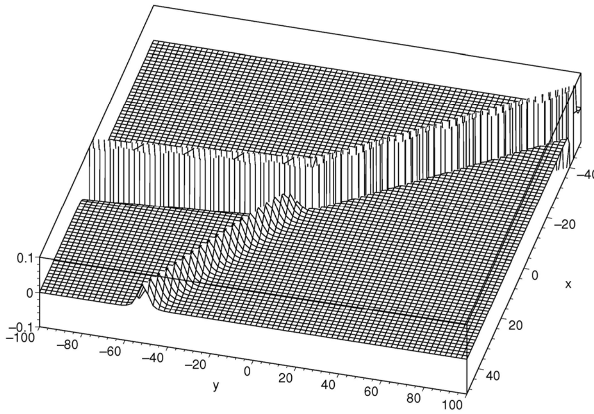


Figure 1. Pure 2-soliton solution.

3.2. Soliton Resonances

When $A_{12} \to +0$, or $A_{12} \to +\infty$, the phase shift $|\Delta_{12}|$, becomes $+\infty$, the length of the intermediate region becomes infinite, this may be thought as "soliton resonance", and the dispersion relation plays a major role in producing the soliton resonance. Further more, as

$$\begin{cases} A_{12} \to +0 \Leftrightarrow (k_1 - k_2)(q_1 - q_2) \to +0 \\ A_{12} \to +\infty \Leftrightarrow (k_1 + q_2)(k_2 + q_1) \to +0 \end{cases}, \quad (3.2)$$

we call them as "minus resonance" and "plus resonance" respectively.

3.2.1. Minus Resonance

Case 1. By taking $A_{12} \to +0$, Equation (2.3b) becomes $g_2 = 1 + a_1 e^{\theta_1} + a_2 e^{\theta_2}$, $f_2 = 1 + b_1 e^{\theta_1} + b_2 e^{\theta_2}$, from which we have the asymptotic forms (see Equation (3.3)).

The solution has three arms each of which are exact 1-solitons.

Case 2. Substituting

$$\begin{gathered} g_2 \to A_{12}^{-1} g_2, f_2 \to A_{12}^{-1} f_2, \\ e^{\theta_1} \to A_{12}^{-1} e^{\theta_1}, e^{\theta_2} \to A_{12}^{-1} e^{\theta_2} \end{gathered} \quad (3.4)$$

into Equation (2.4b), get

$$A_{12} g_2 = 1 + a_1 A_{12}^{-1} e^{\theta_1} + a_2 A_{12}^{-1} e^{\theta_2} + a_1 a_2 A_{12}^{-1} e^{\theta_1 + \theta_2} \quad (3.5)$$

by taking $A_{12} \to +0$, Equation (3.5) becomes

$$g_2 = a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_1 a_2 e^{\theta_1 + \theta_2} \quad (3.6a)$$

Similarly

$$f_2 = b_1 e^{\theta_1} + b_2 e^{\theta_2} + b_1 b_2 e^{\theta_1 + \theta_2} \quad (3.6a)$$

The above substitutions are nothing but only a translation of the coordinates.

The corresponding asymptotic forms are Equation (3.7).

The solution has three arms again.

3.2.2. Plus Resonance

Case 1. Substituting $e^{\theta_1} \to A_{12}^{-1} e^{\theta_1}$ into Equation (2.3 b), then taking $A_{12} \to +\infty$ we get

$$\begin{gathered} g_2 = 1 + a_2 e^{\theta_2} + a_1 a_2 e^{\theta_1 + \theta_2}, \\ f_2 = 1 + b_2 e^{\theta_2} + b_1 b_2 e^{\theta_1 + \theta_2} \end{gathered} \quad (3.8)$$

$$u = \begin{cases} u^{(1)} = \left(\log \frac{1+a_1 e^{\theta_1}}{1+b_1 e^{\theta_1}}\right)_x, & y \to -\infty, \theta_1 \approx 0, \theta_2 \to -\infty \\ u^{(2)} = \left(\log \frac{1+a_2 e^{\theta_2}}{1+b_1 e^{\theta_1}}\right)_x, & y \to +\infty, \theta_1 \to -\infty, \theta_2 \approx 0 \\ u^{(1-2)} = \left(\log \frac{a_1 e^{\theta_1} + a_2 e^{\theta_2}}{b_1 e^{\theta_1} + a_2 e^{\theta_2}}\right)_x, & y \to +\infty, \theta_1 \to +\infty, \theta_2 \to +\infty \end{cases}. \quad (3.3)$$

$$u = \begin{cases} u^{(2)} = \left(\log \frac{1+a_2 e^{\theta_2}}{1+b_2 e^{\theta_2}}\right)_x, & y \to -\infty, \theta_1 \to +\infty, \theta_2 \approx 0 \\ u^{(1)} = \left(\log \frac{1+a_1 e^{\theta_2}}{1+b_1 e^{\theta_1}}\right)_x, & y \to +\infty, \theta_1 \approx \Delta_{12}, \theta_2 \to +\infty \\ u^{(1-2)} = \left(\log \frac{a_1 e^{\theta_1} + a_2 e^{\theta_2}}{b_1 e^{\theta_1} + a_2 e^{\theta_2}}\right)_x, & y \to -\infty, \theta_1 \to -\infty, \theta_2 \to +\infty \end{cases}. \quad (3.7)$$

$$u=\begin{cases} u^{(1+2)}=\left(\log\dfrac{1+a_1a_2\mathrm{e}^{\theta_1+\theta_2}}{1+b_1b_2\mathrm{e}^{\theta_1+\theta_2}}\right)_x, & y\to-\infty,\theta_1+\theta_2\approx 0,\theta_2\to-\infty \\ u^{(1)}+u^{(2)}=\left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x+\left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2}}{1+b_2\mathrm{e}^{\theta_2}}\right)_x, & \begin{matrix} y\to+\infty,\theta_1\approx 0,\theta_2\to+\infty \\ \theta_1\to-\infty,\theta_2\approx 0\end{matrix}\end{cases} \tag{3.9}$$

Case 2. Substituting $\mathrm{e}^{\theta_2}\to A_{12}^{-1}\mathrm{e}^{\theta_2}$ into (2.3b), then taking $A_{12}\to+\infty$ we get

$$g_2=1+a_1\mathrm{e}^{\theta_1}+a_1a_2\mathrm{e}^{\theta_1+\theta_2},\ f_2=1+b_1\mathrm{e}^{\theta_1}+b_1b_2\mathrm{e}^{\theta_1+\theta_2} \tag{3.10}$$

$$u=\begin{cases} u^{(1)}+u^{(2)}=\left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x+\left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2}}{1+b_2\mathrm{e}^{\theta_2}}\right)_x, & \begin{matrix} y\to-\infty,\theta_1\approx 0,\theta_2\to-\infty \\ \theta_1\to-\infty,\theta_2\approx 0\end{matrix} \\ u^{(1+2)}=\left(\log\dfrac{1+a_1a_2\mathrm{e}^{\theta_1+\theta_2}}{1+b_1b_2\mathrm{e}^{\theta_1+\theta_2}}\right)_x, & y\to+\infty,\theta_1\to-\infty,\theta_1+\theta_2\approx 0\end{cases} \tag{3.11}$$

The above asymptotic analysis discusses the 2-soliton solution and it's two type of resonances, minus resonance and plus resonance, by which we know that they all possess three arms, this theory can be illustrated by **Figure 2**, and furthermore, they show that when resonance occurs, interaction of two high and steep waves can produce a new weak one.

We have assumed that $\nu_1>\nu_2$, the case of $\nu_2>\nu_1$ is similar, however it is different in the case of $\nu_1=\nu_2$. Let $x-\nu_1 y\underline{\underline{\Delta}}Z$, then

$$\theta_1=\mu_1Z+\delta_1,\ \theta_2=\mu_2Z+\delta_2,\quad A_{12}=\frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2} \tag{3.12}$$

where two soliton lie in parallel, this solution is similar to 2-soliton solution of the KdV equation.

4. 3-Solitons

In this section, we analyze the behaviors in asymptotic regions about typical four types of solutions.

When $0<A_{12},A_{13},A_{23}<\infty$ and $\neq 1$, for the limit $y\to-\infty,\theta_1>\theta_2>\theta_3$, the condition 1) and 2) are satisfied in three regions:

$$\begin{cases} u^{(1)}=\left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta 1}}\right)_x, \\ \qquad \theta_1\approx 0,\theta_2\to-\infty,\theta_3\to-\infty \\ u^{(2)}=\left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2-\Delta_{12}}}{1+b_2\mathrm{e}^{\theta_2-\Delta_{12}}}\right)_x, \\ \qquad \theta_1\to+\infty,\theta_2\approx\Delta_{12},\theta_3\to+\infty \\ u^{(3)}=\left(\log\dfrac{1+a_3\mathrm{e}^{\theta_3}-\Delta_{13}-\Delta_{23}}{1+b_3\mathrm{e}^{\theta_3}-\Delta_{13}-\Delta_{23}}\right)_x, \\ \qquad \theta_1\to+\infty,\theta_2\to+\infty,\theta_3\approx\Delta_{13}\end{cases} \tag{4.1}$$

so when $y\to-\infty, u=u^{(1)}+u^{(2)}+u^{(3)}$.

Similarly, when $y\to+\infty$

$$\begin{aligned} u=&\left(\log\frac{1+a_1\mathrm{e}^{\theta_1}-\Delta_{13}-\Delta_{12}}{1+b_1\mathrm{e}^{\theta_1}-\Delta_{13}-\Delta_{12}}\right)_x \\ &+\left(\log\frac{1+a_2\mathrm{e}^{\theta_2}-\Delta_{12}}{1+b_2\mathrm{e}^{\theta_2}-\Delta_{12}}\right)_x \\ &+\left(\log\frac{1+a_3\mathrm{e}^{\theta_3}}{1+a_3\mathrm{e}^{\theta_3}}\right)_x.\end{aligned} \tag{4.2}$$

The above limit analysis can prove that 3-soliton solution has 6 arms on theory, **Figure 3** can illustrate it too.

The soliton resonance occurs when one or two or even three of $\Delta_{ij}\to\pm\infty$, we call them 1-, 2-, 3-resonance solution respectively, each of which include minus resonance and plus resonance, in the following, we will discuss them all.

4.1. 1-Resonance

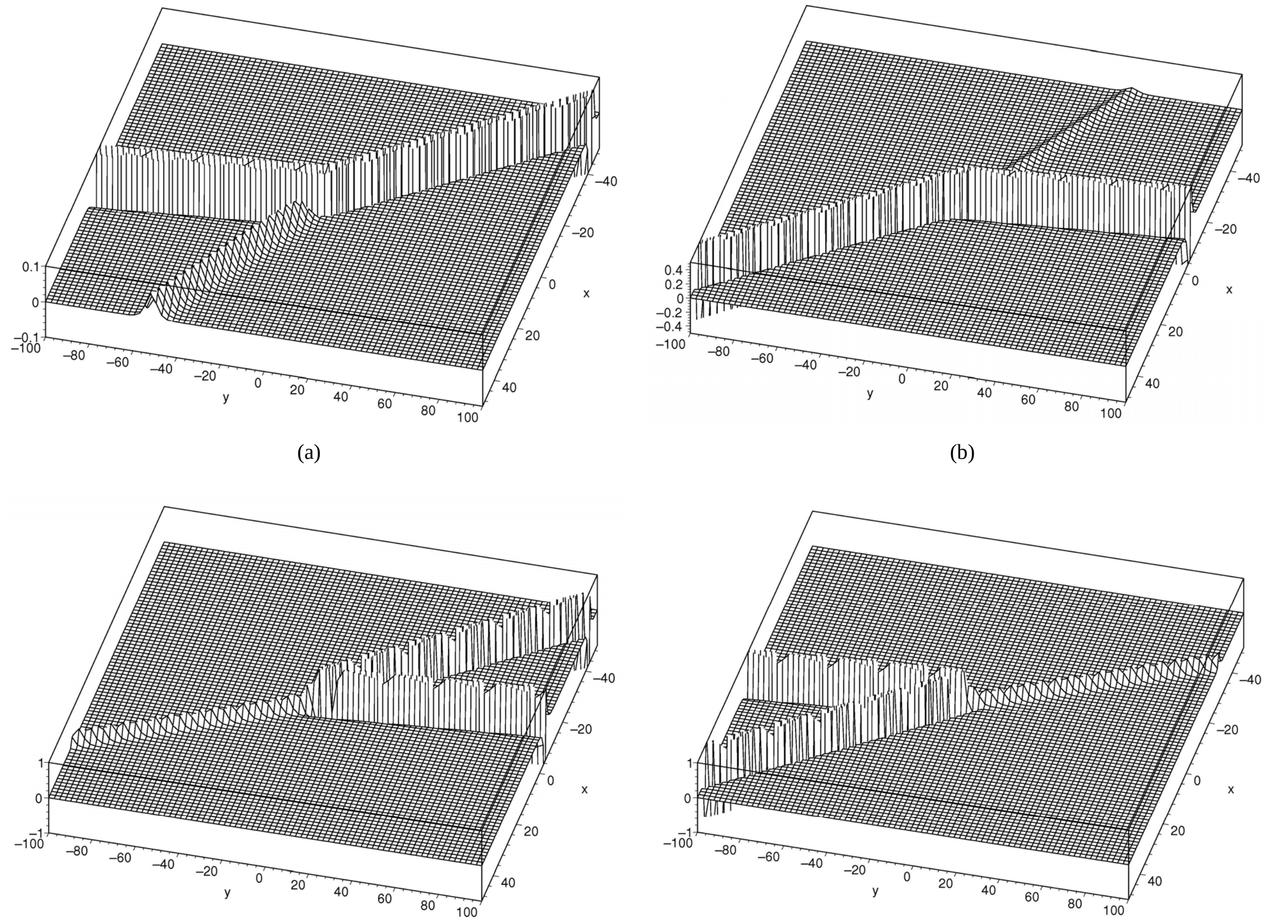
In this case, one of $\Delta_{ij}\to\pm\infty$, we suppose $\Delta_{13}\to\pm\infty$ without lose of generality, that is equal to $A_{13}\to\pm 0$ (minus 1-resonance) and $A_{13}\to\pm\infty$ (plus 1-resonance).

4.1.1. Minus 1-Resonance

Taking the limit of $A_{13}\to\pm 0$, Equation (2.3c) becomes

$$\begin{aligned} g_3=&1+a_1\mathrm{e}^{\theta_1}+a_2\mathrm{e}^{\theta_2}+a_3\mathrm{e}^{\theta_{31}} \\ &+a_1a_2\mathrm{e}^{\theta_1+\theta_2-\Delta_{12}}+a_2a_3\mathrm{e}^{\theta_2+\theta_3-\Delta_{23}} \\ f_2=&1+b_1\mathrm{e}^{\theta_1}+b_2\mathrm{e}^{\theta_2}+b_3\mathrm{e}^{\theta_2} \\ &+b_1b_3\mathrm{e}^{\theta_1+\theta_2-\Delta_{12}}+b_2b_3\mathrm{e}^{\theta_2+\theta_3-\Delta_{23}}\end{aligned}, \tag{4.3}$$

consequently, the asymptotic forms of the solution are given by



(a) (b) (c) (d)

Figure 2. Minus and plus resonance of 2-soliton solution, (a) 2-soliton minus 1; (b) 2-soliton minus case 2; (c) 2-soliton plus 1; (d) 2-soliton plus case 2.

$$u=\begin{cases} u^{(1)}+u^{(2)}=\left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x+\left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2-\Delta_{12}}}{1+b_2\mathrm{e}^{\theta_2-\Delta_{12}}}\right)_x, \\ \qquad y\to-\infty \\ u^{(2)}+u^{(3)}=\left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2-\Delta_{23}}}{1+b_2\mathrm{e}^{\theta_2-\Delta_{23}}}\right)_x+\left(\log\dfrac{1+a_3\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x, \\ \qquad y\to+\infty \\ u^{(1-3)}=\left(\log\dfrac{a_1\mathrm{e}^{\theta_1-\Delta_{12}}+a_3\mathrm{e}^{\theta_3-\Delta_{23}}}{b_1\mathrm{e}^{\theta_1-\Delta_{12}}+b_3\mathrm{e}^{\theta_3-\Delta_{23}}}\right)_x, x\to+\infty \end{cases} \tag{4.4}$$

So minus 1-resonance of 2-soliton solution has five arms (See **Figure 4(a)**).

4.1.2. Plus 1-Resonance

Taking the limit of $A_{13}\to+\infty$, Equation (2.3c) becomes

$$\begin{aligned} g_3&=a_1a_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}}+a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{12}-\Delta_{13}-\Delta_{23}}, \\ f_3&=b_1b_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}}+b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{12}-\Delta_{13}-\Delta_{23}}, \end{aligned} \tag{4.5}$$

$$u=u^{(2)}=\left(\log\frac{1+a_2\mathrm{e}^{\theta_2-\Delta_{12}-\Delta_{23}}}{1+b_2\mathrm{e}^{\theta_2-\Delta_{12}-\Delta_{23}}}\right)_x. \tag{4.6}$$

It is clearly that plus 1-resonance of 2-soliton solution

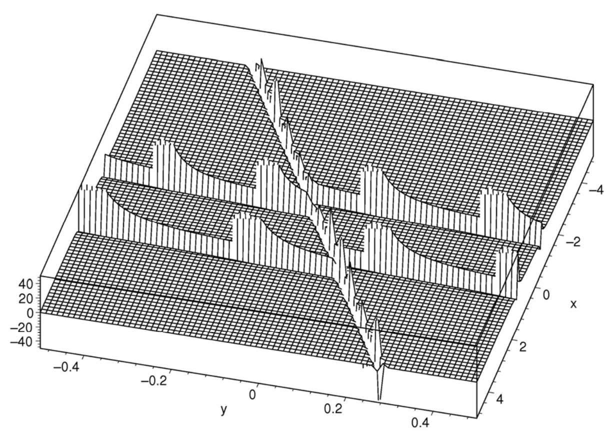


Figure 3. Pure 3-soliton solution.

only has one arm, and the figure is similar to that of 1-solition solution.

4.2. 2-Resonance

In this case, two of $\Delta_{ij} \to \pm\infty$, we suppose $\Delta_{12} \to \pm\infty$, $\Delta_{23} \to \pm\infty$ without lose of generality, which are equal to $\Delta_{12} \to +0,\ \Delta_{23} \to +0$ (minus 2-resonance) and $\Delta_{12} \to +\infty,\ \Delta_{23} \to +\infty$ (plus 2-resonance).

4.2.1. Plus 2-Resonance

Case 1. Substituting $A_{12}\mathrm{e}^{\theta_2} \to \mathrm{e}^{\theta_2}$, $A_{23}\mathrm{e}^{\theta_3} \to \mathrm{e}^{\theta_3}$ into Equation (2.3c), and taking the limit of A_{12} and A_{13}, we get

$$\begin{aligned} g_3 &= 1 + a_1\mathrm{e}^{\theta_1} + a_1a_2\mathrm{e}^{\theta_1+\theta_2} + a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \\ f_3 &= 1 + b_1\mathrm{e}^{\theta_1} + b_1b_2\mathrm{e}^{\theta_1+\theta_2} + b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \end{aligned} \tag{4.7}$$

Then

$$u = \begin{cases} u^{(3)} + u^{(2)} + u^{(1)} = \left(\log\dfrac{1+a_3\mathrm{e}^{\theta_3-\Delta_{13}}}{1+b_3\mathrm{e}^{\theta_3-\Delta_{13}}}\right)_x \\ \quad + \left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2}}{1+b_2\mathrm{e}^{\theta_2}}\right)_x + (\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}})_x, y \to -\infty, \\ u^{(1+2+3)} = \left(\log\dfrac{1+a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}}}{1+b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}}}\right)_x, y \to +\infty. \end{cases} \tag{4.8}$$

Case 2. Substituting $A_{12}\mathrm{e}^{\theta_1} \to \mathrm{e}^{\theta_1}, A_{23}\mathrm{e}^{\theta_2} \to \mathrm{e}^{\theta_2}$ into Equation (2.3c), and taking the limit of A_{12} and A_{23}, we get

$$\begin{aligned} g_3 &= 1 + a_3\mathrm{e}^{\theta_3} + a_2a_3\mathrm{e}^{\theta_2+\theta_3} + a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \\ f_3 &= 1 + b_3\mathrm{e}^{\theta_3} + b_2b_3\mathrm{e}^{\theta_2+\theta_3} + b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \end{aligned}, \tag{4.9}$$

Then

$$u = \begin{cases} u^{(3)} + u^{(2)} + u^{(1)} = \left(\log\dfrac{1+a_3\mathrm{e}^{\theta_3}}{1+b_3\mathrm{e}^{\theta_3}}\right)_x + \left(\log\dfrac{1+a_2\mathrm{e}^{\theta_2}}{1+b_2\mathrm{e}^{\theta_2}}\right)_x \\ \quad + \left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}-\Delta_{13}}{1+b_1\mathrm{e}^{\theta_1}-\Delta_{13}}\right)_x, y \to +\infty \\ u^{(1+2+3)} = \left(\log\dfrac{1+a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}}}{1+b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}}}\right)_x, y \to -\infty \end{cases} \tag{4.10}$$

Case 3. Substituting $A_{12}A_{23}\mathrm{e}^{\theta_2} \to \mathrm{e}^{\theta_2}$ into Equation (2.3c), and taking the limit of A_{12} and A_{23}, we get

$$\begin{aligned} g_3 &= 1 + a_1\mathrm{e}^{\theta_1} + a_3\mathrm{e}^{\theta_3} + a_1a_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}} \\ &\quad + a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \\ f_3 &= 1 + b_1\mathrm{e}^{\theta_1} + b_3\mathrm{e}^{\theta_3} + +b_1b_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}} \\ &\quad + b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3-\Delta_{13}} \end{aligned} \tag{4.11}$$

Then

$$u = \begin{cases} u^{(3)} + u^{(1+2)} = \left(\log\dfrac{1+a_3\mathrm{e}^{\theta_3}}{1+b_3\mathrm{e}^{\theta_3}}\right)_x \\ \quad + \left(\log\dfrac{1+a_1a_2\mathrm{e}^{\theta_1+\theta_2-\Delta_{13}}}{1+b_1b_2\mathrm{e}^{\theta_1+\theta_2-\Delta_{13}}}\right)_x, \quad y \to +\infty \\ u^{(1)} + u^{(2+3)} = \left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x \\ \quad + \left(\log\dfrac{1+a_2a_3\mathrm{e}^{\theta_2+\theta_3-\Delta_{13}}}{1+b_2b_3\mathrm{e}^{\theta_2+\theta_3-\Delta_{13}}}\right)_x, \quad y \to -\infty \end{cases} \tag{4.12}$$

4.2.2. Minus 2-Resonance

In the limit of $A_{12} \to +0,\ A_{23} \to +0$, Equation (2.3c) can be rewritten as

$$\begin{aligned} g_3 &= 1 + a_1\mathrm{e}^{\theta_1} + a_2\mathrm{e}^{\theta_2} + a_3\mathrm{e}^{\theta_3} + a_1a_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}} \\ f_3 &= 1 + b_1\mathrm{e}^{\theta_1} + b_2\mathrm{e}^{\theta_2} + b_3\mathrm{e}^{\theta_3} + b_1b_3\mathrm{e}^{\theta_1+\theta_3-\Delta_{13}} \end{aligned}, \tag{4.13}$$

Then

$$u = \begin{cases} u^{(2-3)} + u^{(1)} = \left(\log\dfrac{a_2\mathrm{e}^{\theta_2} + a_3\mathrm{e}^{\theta_3}}{b_2\mathrm{e}^{\theta_2} + b_3\mathrm{e}^{\theta_3}}\right)_x \\ \quad + \left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1-\Delta_{13}}}{1+b_1\mathrm{e}^{\theta_1-\Delta_{13}}}\right)_x, y \to +\infty \\ u^{(3)} + u^{(1)} = \left(\log\dfrac{1+a_3\mathrm{e}^{\theta_3-\Delta_{13}}}{1+b_3{}^{\theta_3-\Delta_{13}}}\right)_x \\ \quad + \left(\log\dfrac{1+a_1\mathrm{e}^{\theta_1}}{1+b_1\mathrm{e}^{\theta_1}}\right)_x, \quad y \to -\infty \end{cases}, \tag{4.14}$$

The case of condition $\Delta_{12} \to \pm\infty,\ \Delta_{13} \to \pm\infty$ and $\Delta_{13} \to \pm\infty,\ \Delta_{23} \to \pm\infty$ are similar.

By the asymptotic analysis above, we know that two types of 2-resonance 3-soliton solution possess four arms (See **Figures 4(b)-(d)**), the 2-soliton solution has also four arms, but differently, the behaviors of the former in the intermediate region are not stationary.

4.3. 3-Resonance

For the plus 3-resonance, substituting $A_{12}\mathrm{e}^{\theta_1} \to \mathrm{e}^{\theta_1}$, $A_{23}\mathrm{e}^{\theta_2} \to \mathrm{e}^{\theta_2}, A_{13}\mathrm{e}^{\theta_3} \to \mathrm{e}^{\theta_3}$ into Equation (2.3c), and taking the limit of $A_{12} \to \infty, A_{13} \to \infty, A_{23} \to \infty$, we get

$$\begin{aligned} g_3 &= 1 + a_1a_2a_3\mathrm{e}^{\theta_1+\theta_2+\theta_3} \\ f_3 &= 1 + b_1b_2b_3\mathrm{e}^{\theta_1+\theta_2+\theta_3} \end{aligned} \tag{4.15}$$

this case is like 1-soliton solution, which only has one arm.

For the minus 3-resonance, by taking the limit

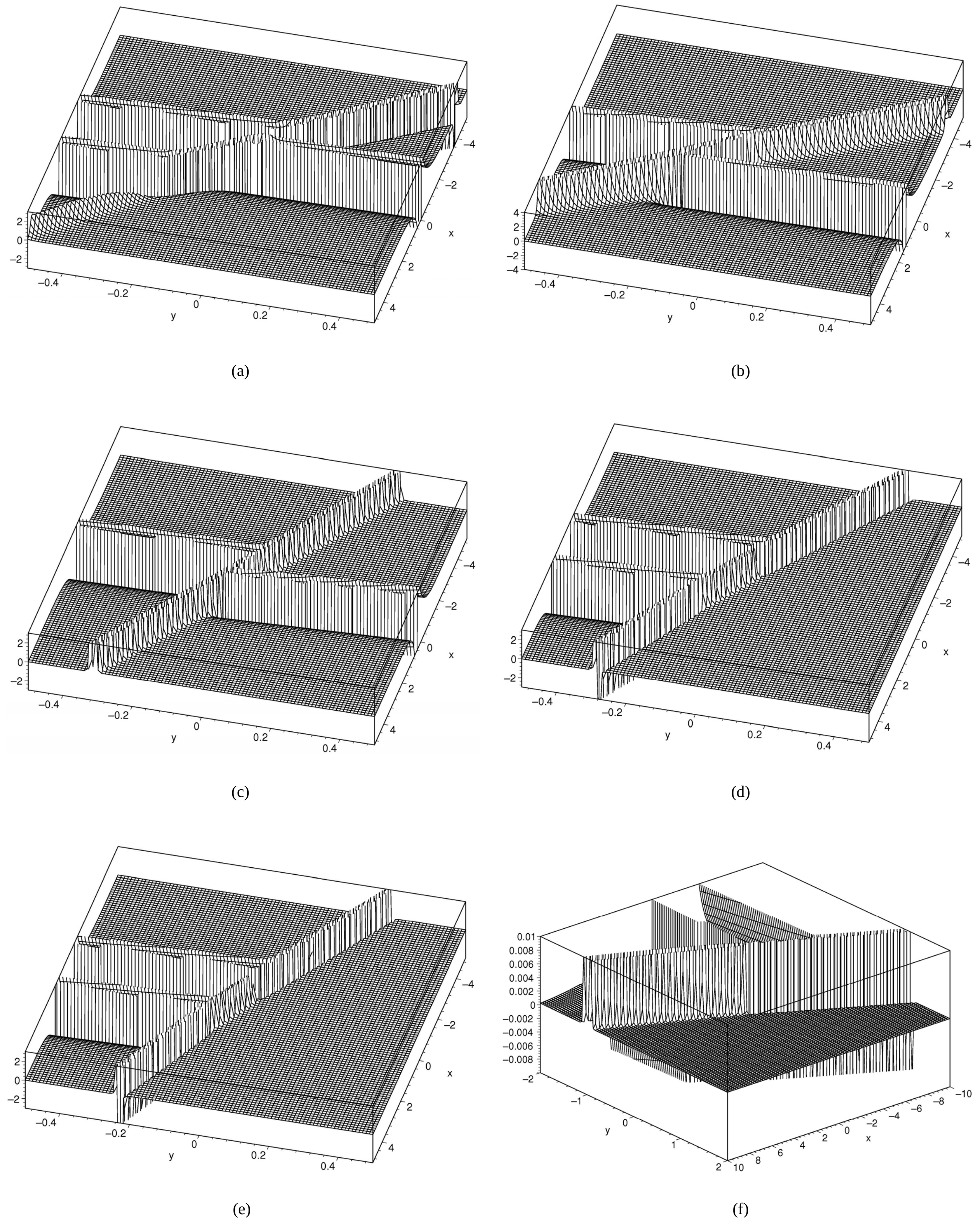


Figure 4. Minus and plus resonance of 3-soliton solution, (a) 3-soliton minus 1-resonance; (b) 3-soliton plus 2-resonance case 1; (c) 3-soliton plus 2-resonance case 2; (d) 3-soliton plus 2-resonance case 3; (e) 3-soliton minus 2-resonance; (f) 3-soliton minus 3-resonance.

$A_{12} \to +0, A_{13} \to +0, A_{23} \to +0$ of the Equation (2.3c), we have

$$\begin{aligned} g_3 &= 1 + a_1 e^{\theta_1} + a_2 e^{\theta_2} + a_3 e^{\theta_3} \\ f_3 &= 1 + b_1 e^{\theta_1} + b_2 e^{\theta_2} + b_3 e^{\theta_3} \end{aligned} \tag{4.16}$$

Then

$$u = \begin{cases} u^{(3)} + u^{(2-3)} = \left(\log \dfrac{1 + a_3 e^{\theta_3}}{1 + b_3 e^{\theta_3}}\right)_x \\ \qquad + \left(\log \dfrac{a_2 e^{\theta_2} + a_3 e^{\theta_3}}{b_2 e^{\theta_2} + b_3 e^{\theta_3}}\right)_x, \quad y \to +\infty \\ u^{(1-2)} + u^{(1)} = \left(\log \dfrac{a_1 e^{\theta_1} + a_2 e^{\theta_2}}{b_1 e^{\theta_1} + b_2 e^{\theta_2}}\right)_x \\ \qquad + \left(\log \dfrac{1 + a_1 e^{\theta_1}}{1 + b_1 e^{\theta_1}}\right)_x, \qquad y \to -\infty, \end{cases} \tag{4.17}$$

which has four arms (See **Figure 4(f)**).

5. Conclusions

In this work, we have primarily focused on the asymptotic behavior of the \$2\$- and \$3\$-soliton solution as $x.y \to \pm\infty$ and their interactions in the xy plane. Generally, in the case of multi-soliton, saying N-soliton solutions, it has $1\text{-}, 2\text{-}, \cdots, C_N^2\text{-}$ resonance N-soliton solutions, and all of them have minus and plus ones. The condition will be more complicated with the increase of N. A full characterization of interaction patterns of the general ones is an important open problem, which is left for further study. It is pointed out that the amplification of the amplitude has been experimentally observed and has practical in maritime security and coastal engineering. It has been found out that many soliton equations have resonance phenomenon which will be helpful in making further investigation on the interaction and energy distribution of gravity waves, and evaluating the impact on the ship traffic on the surface of water. We expect that the results presented in this work will be useful to study solitonic solutions in a variety of integrable systems.

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