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L^p Inequalities for Polynomials

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Abstract

In this paper we consider a problem of investigating the dependence of $\left\|P(Rz)-\beta P(rz)\right\|_p$ on $\left\|P(z)\right\|_p$ for every real or complex number β with $|\beta|\le 1$, $R>r\ge 1$, $p>0$ and present certain compact generalizations which, besides yielding some interesting results as corollaries, include some well-known results, in particular, those of Zygmund, Bernstein, De-Bruijn, Erdös-Lax and Boas and Rahman as special cases.

Keywords: L^p-Inequalities, Polynomials, Complex Domain

1. Introduction

Let $P_n(z)$ denote the space of all complex polynomials $P(z)=\sum_{j=0}^{n}a_j z^j$ of degree at most n. For $P\in P_n$, define

$$\left\|P(z)\right\|_p := \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p\right\}^{1/p}, 1\le p<\infty$$

and

$$\left\|P(z)\right\|_\infty := \max_{|z|=1}\left|P(z)\right|.$$

A famous result known as Bernstein's inequality(for reference,see[1] or [2]) states that if $P\in P_n$,then

$$\left\|P'(z)\right\|_\infty \le n\left\|P(z)\right\|_\infty \tag{1}$$

whereas concerning the maximum modulus of $P(z)$ on the circle $|z|=R>1$, we have

$$\left\|P(Rz)\right\|_\infty \le R^n\left\|P(z)\right\|_\infty, \tag{2}$$

(for reference, see [3]). Inequalities (1) and (2) can be obtained by letting $p\to\infty$ in the inequalities

$$\left\|P'(z)\right\|_p \le n\left\|P(z)\right\|_p, p\ge 1 \tag{3}$$

and

$$\left\|P(Rz)\right\|_p \le R^n\left\|P(z)\right\|_p, R>1, p>0, \tag{4}$$

respectively. Inequality (3) was found by Zygmund [4] whereas inequality (4) is a simple consequence of a result of Hardy [5] (see also [6]). Since Inequality (3) was deduced from M.Riesz's interpolation formula [7] by means of Minkowski's inequality,it was not clear, whether the restriction on p was indeed essential. This question was open for a long time. Finally Arestov [8] proved that (3) remains true for $0<p<1$ as well. Both the Inequalities (3) and (4) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z|<1$. In fact, if $P\in P_n$ and $P(z)\ne 0$ in $|z|<1$, then Inequalities (3) and (4) can be respectively replaced by

$$\left\|P'(z)\right\|_p \le n\frac{\left\|P(z)\right\|_p}{\left\|1+z\right\|_p}, p>0 \tag{5}$$

and

$$\left\|P(Rz)\right\|_p \le \frac{\left\|R^n z+1\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p, R>1, p>0. \tag{6}$$

Inequality (5) is due to De-Bruijn [9] for $p\ge 1$ and Rahman and Schmeisser [10] extended it for $0<p<1$ whereas the Inequality (6) was proved by Boas and Rahman [11] for $p\ge 1$ and later it was extended for $0<p<1$ by Rahman and Schmeisser[12]. For $p=\infty$, the Inequality (5) was conjectured by Erdös and later verified by Lax [13] whereas Inequality (6) was proved by Ankeny and Rivlin [14].

Recently the Authors in [12] (see also [15]) investigated the dependence of

$$\left\|P(Rz)-P(z)\right\|_p \text{ on } \left\|P(z)\right\|_p$$

for $R>1$, $p\ge 1$. As a compact generalization of Inequalities (3) and (4), they have shown that if $P\in P_n$, then for every $R>1$ and $p\ge 1$,

$$\left\|P(Rz)-P(z)\right\|_p \le \left(R^n-1\right)\left\|P(z)\right\|_p . \quad (7)$$

It is natural to seek the corresponding analog of (7) for polynomials $P \in P_n$ having no zero in $|z|<1$ and which is a compact generalization of Inequalities (5) and (6). In the present paper we consider a more general problem of investigating the dependence of

$$\left\|P(Rz)-\beta P(rz)\right\|_p \text{ on } \left\|P(z)\right\|_p$$

for every real or complex number β with $|\beta|\le 1$, $R>r\ge 1$, $p>0$ and develop a unified method for arriving at these results. We first present the following interesting result and a compact generalization of Inequalities (3) and (4), which also extends Inequality (7) for $0<p<1$ as well.

Theorem 1. *If* $P \in P_n$, *then for every real or complex number* β with $|\beta|\le 1$, $R>r\ge 1$ and $p>0$,

$$\left\|P(Rz)-\beta P(rz)\right\|_p \le \left|R^n-\beta r^n\right|\left\|P(z)\right\|_p . \quad (8)$$

The result is best possible and equality in (8) *holds for* $P(z)=az^n, a\ne 0.$

Remark 1. For $\beta=0$, Theorem 1 reduces to Inequality (4) and for $\beta=1$, $r=1$, it validates Inequality (7) for each $p>0$.

If we set $\beta=1$ in Inequality (8), we immediately get the following generalization of Inequality (7).

Corollary 1. *If* $P \in P_n$, *then for* $R>r\ge 1$ and $p>0$

$$\left\|P(Rz)-P(rz)\right\|_p \le \left(R^n-r^n\right)\left\|P(z)\right\|_p . \quad (9)$$

The result is best possible and equality in (9) *holds for* $P(z)=az^n, a\ne 0.$

If we divide the two sides of Inequality (9) by $(R-r)$ and let $R\to r$, we get:

Corollary 2. *If* $P \in P_n$, *then for* $r\ge 1$ and $p>0$,

$$\left\|P'(rz)\right\|_p \le nr^{n-1}\left\|P(z)\right\|_p . \quad (10)$$

Remark 2. For $r=1$, Corollary 2 reduces to Zygmund's Inequality (3) for each $p>0$.

The following result which is a compact generalization of Inequalities of (1) and (2) follows from Theorem 1 by letting $p\to\infty$ in Inequality (8).

Corollary 3. *If* $P \in P_n$, *then for every real or complex number* β with $|\beta|\le 1$ *and* $R>r\ge 1$,

$$\left|P(Rz)-\beta P(rz)\right| \le \left|R^n-\beta r^n\right|\max_{|z|=1}\left|P(z)\right| \text{ for } |z|=1. \quad (11)$$

The result is best possible and equality in (11) *holds for* $P(z)=az^n, a\ne 0.$

Remark 3. For $\beta=0$, Corollary 3 reduces to Inequality (2) and for $\beta=1$, if we divide the two sides of (11) by $R-r$ and let $R\to r$, it follows that *if* $P\in P_n$, *then for* $r\ge 1$,

$$\left|P'(rz)\right| \le nr^{n-1}\max_{|z|=1}\left|P(z)\right| \text{ for } |z|=1. \quad (12)$$

Inequality (12) reduces to Bernstein's Inequality (1) for $r=1$.

For polynomials $P\in P_n$ having no zero in $|z|<1$, we next prove the following interesting improvement of (8) which among other things include De-Bruijn's theorem (Inequality (5)) and a result of Boas and Rahman (Inequality (6)) as special cases.

Theorem 2. *If* $P\in P_n$ *and* $P(z)$ *does not vanish in* $|z|<1$, *then for every real or complex number* β *with* $|\beta|\le 1$, $R>r\ge 1$ *and* $p>0$

$$\left\|P(Rz)-\beta P(rz)\right\|_p \le \frac{\left\|\left(R^n-\beta r^n\right)z+(1-\beta)\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p . \quad (13)$$

The result is best possible and equality in (13) *holds for* $P(z)=az^n+b, |a|=|b|=1.$

For $\beta=0$, Theorem 2 reduces to Inequality (6). A variety of interesting results can be easily deduced from Theorem 2. Here we mention a few of these. The following corollary immediately follows from Theorem 2 by taking $\beta=1$.

Corollary 4. *If* $P\in P_n$ *and* $P(z)$ *does not vanish in* $|z|<1$, *then for* $R>r\ge 1$ *and* $p>0$,

$$\left\|P(Rz)-P(rz)\right\|_p \le \frac{\left(R^n-r^n\right)}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p . \quad (14)$$

The result is sharp and equality in (14) *holds for* $P(z)=az^n+b, |a|=|b|=1.$

Remark 4. For $r=1$, if we divide the two sides of (14) by $R-1$ and let $R\to 1$, we immediately get De-Bruijn's theorem (Inequality (5)) for each $p>0$.

Next we mention the following compact generalization of a theorem of Erdös and Lax (Inequality (5) for $p=\infty$) and a result of Ankeny and Rivlin (Inequality (5) for $p=\infty$) which immediately follows from Theorem 2 by letting $p\to\infty$ in (13).

Corollary 5. *If* $P\in P_n$ *and* $P(z)$ *does not vanish in* $|z|<1$, *then for every real or complex number* β *with* $|\beta|\le 1$ and $R>r\ge 1$,

$$\left|P(Rz)-\beta P(rz)\right| \le \frac{\left|R^n-\beta r^n\right|+\left|1-\beta\right|}{2}\max_{|z|=1}\left|P(z)\right| \quad (15)$$

for $|z|=1$.

The result is best possible and equality in (15) *holds for* $P(z)=az^n+b, |a|=|b|=1.$

Remark 5. For $\beta = 1$, if we divide the two sides of (15) by $R-r$ and let $R \to r$,we get

$$\left|P^{'}(rz)\right| \le \frac{n}{2} r^{n-1} \max_{|z|=1}\left|P(z)\right| \text{ for } |z| = 1. \qquad (16)$$

For $r = 1$, Inequality (16) was conjectured by Erdös and later verified by Lax[10]. If we take $\beta = 0$ in (15),we immediately get

$$\left\|P(Rz)_{\infty}\right\| \le \frac{R^n + 1}{2}\left\|P(z)\right\|_{\infty}, R > 1. \qquad (17)$$

Inequality (17) is due to Ankeny and Rivlin [1].

A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = uQ(z)$ for all $z \in C$ where $|u| = 1$ and $Q(z) = z^n \overline{P(1/\overline{z})}$. It is known[16, 17] that if $P \in P_n$ is self-inversive polynomial, then for every $p \ge 1$,

$$\left\|P^{'}(z)\right\|_p \le n\frac{\left\|P(z)\right\|_p}{\left\|1+z\right\|_p}, \qquad (18)$$

Finally, we present the following result which include some well-known results for self-inversive polynomials as special cases.

Theorem 3. *If* $P \in P_n$ *is self-inversive polynomial, then for every real or complex number* β *with* $|\beta| \le 1$, $R > r \ge 1$ *and* $p > 0$,

$$\left\|P(Rz) - \beta P(rz)\right\|_p \le \frac{\left\|\left(R^n - \beta r^n\right)z + (1-\beta)\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p. \qquad (19)$$

The result is best possible and equality in (19) *holds for* $P(z) = z^n + 1$.

Remark 6. Taking $\beta = 0$ in Theorem 3, it follows that *if* $P \in P_n$ *is self-inversive polynomial, then for* $R > 1$ *and* $p > 0$,

$$\left\|P(Rz)\right\|_p \le \frac{\left\|\left(R^n z + 1\right)\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p. \qquad (20)$$

The result is sharp.

Many interesting results can be deduced from Theorem 3 in exactly the same way as we have deduced from Theorem 2.

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1. If $P \in P_n$ *and* $P(z)$ *has all its zeros in* $|z| \le k$ where $k \le 1$, *then for every* $R \ge r \ge 1$ *and* $|z| = 1$,

$$\left|P(Rz)\right| \ge \left(\frac{R+k}{r+k}\right)^n \left|P(rz)\right|. \qquad (21)$$

Proof of Lemma 1. Since all the zeros of $P(z)$ lie in $|z| \le k$, we write

$$P(z) = C\prod_{j=1}^{n}\left(z - r_j e^{i\theta_j}\right)$$

where $r_j \le k$. Now for $0 \le \theta < 2\pi$, $R \ge r \ge 1$, we have

$$\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| = \left\{\frac{R^2 + r_j^2 - 2Rr_j Cos\left(\theta - \theta_j\right)}{r^2 + r_j^2 - 2rr_j Cos\left(\theta - \theta_j\right)}\right\}^{1/2}$$

$$\ge \left\{\frac{R + r_j}{r + r_j}\right\} \ge \left\{\frac{R+k}{r+k}\right\}, j = 1, 2, \cdots, n.$$

Hence

$$\left|\frac{P\left(Re^{i\theta}\right)}{P\left(re^{i\theta}\right)}\right| = \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|$$

$$\ge \prod_{j=1}^{n}\left(\frac{R+k}{r+k}\right) = \left(\frac{R+k}{r+k}\right)^n$$

for $0 \le \theta < 2\pi$. This implies for $|z| = 1$ and $R > r \ge 1$,

$$\left|P(Rz)\right| \ge \left(\frac{R+k}{r+k}\right)^n \left|P(rz)\right|,$$

which completes the proof of Lemma 1.

Lemma 2. If $P \in P_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every real or complex number* β *with* $|\beta| \le 1$, $R \ge r \ge 1$, *and* $|z| = 1$,

$$\left|P(Rz) - \beta P(rz)\right| \le \left|Q(Rz) - \beta P(rz)\right| \qquad (22)$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ *.The result is sharp and equality in* (22) *holds for* $P(z) = z^n + 1$.

Proof of Lemma 2. For the case $R = r$, the result follows by observing that $|P(z)| \le |Q(z)|$ for $|z| \ge 1$. Henceforth, we assume that $R > r$. Since the polynomial $P(z)$ has all its zeros in $|z| \ge 1$, therefore, for every real or complex number α with $|\alpha| > 1$, the polynomial $f(z) = P(z) - \alpha Q(z)$, where $Q(z) = z^n \overline{P(1/\overline{z})}$, has all its zeros in $|z| \le 1$. Applying Lemma 1 to the polynomial $f(z)$ with $k = 1$, we obtain for every $R > r \ge 1$ and $0 \le \theta < 2\pi$,

$$\left|f\left(Re^{i\theta}\right)\right| \ge \left(\frac{R+1}{r+1}\right)^n \left|f\left(re^{i\theta}\right)\right|. \qquad (23)$$

Since $f\left(Re^{i\theta}\right) \ne 0$ for every $R > r \ge 1, 0 \le \theta < 2\pi$ and $R + 1 > r + 1$, it follows from (23) that

$$\left|f\left(Re^{i\theta}\right)\right| > \left(\frac{r+1}{R+1}\right)^n \left|f\left(Re^{i\theta}\right)\right| \geq \left|f\left(re^{i\theta}\right)\right|$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. This gives

$$\left|f(rz)\right| < \left|f(Rz)\right|,$$

for $|z| = 1$ and $R > r \geq 1$.

Using Rouche's theorem and noting that all the zeros of $f(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, we conclude that the polynomial

$$\begin{aligned} T(z) &= f(Rz) - \beta f(rz) \\ &= \left\{P(Rz) - \beta P(rz)\right\} - \alpha\left\{Q(Rz) - \beta Q(rz)\right\} \end{aligned} \quad (24)$$

has all its zeros in $|z| < 1$ for every real or complex number β, α with $|\beta| \leq 1, |\alpha| > 1$ and $R > r \geq 1$. This implies

$$\left|P(Rz) - \beta P(rz)\right| \leq \left|Q(Rz) - \beta Q(rz)\right| \quad (25)$$

for $|z| \geq 1$ and $R > r \geq 1$. If Inequality (25) is not true, then exist a point $z = w$ with $|w| \geq 1$ such that

$$\left|P(Rw) - \beta P(rw)\right| > \left|Q(Rw) - \beta Q(rw)\right|.$$

But all the zeros of $Q(z)$ lie in $|z| \leq 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $Q(Rz) - \beta Q(rz)$ lie in $|z| < 1$. Hence $Q(Rw) - \beta Q(rw) \neq 0$ with $|w| \geq 1$. We take

$$\alpha = \frac{P(Rw) - \beta P(rw)}{Q(Rw) - \beta Q(rw)},$$

then α is a well defined real or complex number with $|\alpha| > 1$ and with this choice of α, from (24) we obtain $T(w) = 0$ where $|w| \geq 1$. This contradicts the fact that all the zeros of $T(z)$ lie in $|z| < 1$. Thus

$$\left|P(Rz) - \beta P(rz)\right| \leq \left|Q(Rz) - \beta Q(rz)\right|$$

for $|z| \geq 1$ and $R > r \geq 1$. This proves Lemma 2.

Next we describe a result of Arestov.

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ and $P(z) = \sum_{j=0}^{n} a_j z^j \in P_n$, we define

$$\Lambda_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator Λ_γ is said to be admissible if it preserves one of the following properties:

1) $P(z)$ has all its zeros in $\left\{z \in C : |z| \leq 1\right\}$,
2) $P(z)$ has all its zeros in $\left\{z \in C : |z| \geq 1\right\}$,

The result of Arestov may now be stated as follows.

Lemma 3. [8] Let $\phi(x) = \psi(\log x)$ where ψ is a convex nondecreasing function on *R*. Then for all $P \in P_n$ and each admissible operator Λ_γ,

$$\int_0^{2\pi} \phi\left(\left|\Lambda_\gamma P\left(e^{i\theta}\right)\right|\right) d\theta \leq \int_0^{2\pi} \phi\left(C(\gamma, n)\left|P\left(e^{i\theta}\right)\right|\right) d\theta$$

where $C(\gamma, n) = \text{Max}\left(|\gamma_0|, |\gamma_n|\right)$.

In particular, Lemma 3 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$. Therefore, we have

$$\left\{\int_0^{2\pi} \left(\left|\Lambda_\gamma P\left(e^{i\theta}\right)\right|^p\right) d\theta\right\}^{1/p} \leq \left(C(\gamma, n)\left\{\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{1/p}. \quad (26)$$

We use (26) to prove the following interesting result.

Lemma 4. If $P \in P_n$ *and* $P(z)$ *does not vanish in* $|z| < 1$, *then for every real or complex number* β *with* $|\beta| \leq 1, R > r \geq 1$, $p > 0$ *and* α real,

$$\begin{aligned} &\int_0^{2\pi} \left|\left(P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right)\right)\right. \\ &\left.+ e^{i\alpha}\left(R^n P\left(e^{i\theta}/R\right) - \bar{\beta} r^n P\left(e^{i\theta}/r\right)\right)\right|^p d\theta \quad (27) \\ &\leq \left|\left(R^n - \beta r^n\right) + e^{i\alpha}\left(1 - \bar{\beta}\right)\right|^p \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta. \end{aligned}$$

Proof of Lemma 4. Let $Q(z) = z^n \overline{P(1/\bar{z})}$. Since $P(z)$ does not vanish in $|z| < 1$, by Lemma 2, for every real or complex number β with $|\beta| \leq 1, R > r \geq 1$ and $|z| = 1$, we have

$$\begin{aligned} &\left|P(Rz) - \beta P(rz)\right| \\ &\leq \left|Q(Rz) - \beta Q(rz)\right| = \left|R^n P(z/R) - \bar{\beta} r^n P(z/r)\right| \end{aligned}$$

Now(as in the proof of Lemma 2), the polynomial

$$H(z) = Q(Rz) - \beta Q(rz) = R^n z^n \overline{P(1/R\bar{z})} - \beta r^n z^n \overline{P(1/r\bar{z})}$$

has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| \leq 1$ and $R > r$, it follows that the polynomial

$$z^n \overline{H(1/\bar{z})} = R^n P(z/R) - \bar{\beta} r^n P(z/r)$$

has all its zeros in $|z| > 1$. Hence the function

$$f(z) = \frac{P(Rz) - \beta P(rz)}{R^n P(z/R) - \bar{\beta} r^n P(z/r)}$$

is analytic in $|z| \leq 1$ and $|f(z)| \leq 1$ for $|z| = 1$. Since $f(z)$ is not a constant, it follows by the Maximum Modulus Principle that

$$|f(z)| < 1 \text{ for } |z| < 1,$$

or equivalently,

$$\left|P(Rz) - \beta P(rz)\right| < \left|R^n P(z/R) - \bar{\beta} r^n P(z/r)\right| \text{ for } |z| < 1. \quad (28)$$

A direct application of Rouche's theorem shows that

$$\Lambda_\gamma P(z) = \left(P(Rz) - \beta P(rz)\right) + e^{i\alpha}\left(R^n P(z/R) - \overline{\beta} r^n P(z/r)\right) = \left(\left(R^n - \beta r^n\right) + e^{i\alpha}\left(1-\overline{\beta}\right) a_n z^n + \cdots + \left(\left(1-\beta\right) + e^{i\alpha}\left(R^n - \overline{\beta} r^n\right)\right) a_0\right.$$

does not vanish in $|z|<1$ for every β with $|\beta|\le 1$, $R>r\ge 1$ and α real. Therefore, Λ_γ is admissibe operator. Applying (26) of Lemma 3, the desired result follows immediately for each $p>0$. This completes the proof of Lemma 4.

From lemma 4, we deduce the following more general lemma which is a result of independent interest with variety of application.

Lemma 5. If $P\in P_n$, *then for every real or complex number* β *with* $|\beta|\le 1, R>r\ge 1$, $p>0$ *and* α real,

$$\int_0^{2\pi}\left|\left(P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right)\right) + e^{i\alpha}\left(R^n P\left(e^{i\theta}/R\right) - \overline{\beta} r^n P\left(e^{i\theta}/r\right)\right)\right|^p d\theta \le \left|\left(R^n - \beta r^n\right) + e^{i\alpha}\left(1-\overline{\beta}\right)\right|^p \int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta. \tag{29}$$

The result is sharp and equality in (29) *holds for* $P(z)=\lambda z^n, \lambda\ne 0$

Proof of Lemma 5. Since $P(z)$ is a polynomial of degree at most n, we can write

$$P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k}\left(z-z_j\right)\prod_{j=k+1}^{n}\left(z-z_j\right), k\ge 1$$

where all the zeros of $P_1(z)$ lie in $|z|\ge 1$ and all the zeros of $P_2(z)$ lie in $|z|<1$. First we suppose that $P_1(z)$ has no zero on $|z|=1$ so that all the zeros of $P_1(z)$ lie in $|z|>1$. Let $Q_2(z) = z^{n-k}\overline{P_2\left(1/\overline{z}\right)}$, then all the zeros of $Q_2(z)$ lie in $|z|>1$ and $|Q_2(z)| = |P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$g(z) = P_1(z)Q_2(z) = \prod_{j=1}^{k}\left(z-z_j\right)\prod_{j=k+1}^{n}\left(1-z\overline{z_j}\right),$$

then all the zeros of $g(z)$ lie in $|z|>1$ and for $|z|=1$,

$$|g(z)| = |P_1(z)||Q_2(z)| = |P_1(z)||P_2(z)| = |P(z)|. \tag{30}$$

By the Maximum Modulus Principle, it follows that

$$|P(z)|\le |g(z)| \text{ for } |z|\le 1. \tag{31}$$

We claim that the polynomial $h(z) = P(z) + \lambda g(z)$ does not vanish in $|z|\le 1$ for every λ with $|\lambda|>1$. If this is not true, then $h(z_0)=0$ for some z_0 with $|z_0|\le 1$. This gives

$$|P(z_0)| = |\lambda||g(z_0)|.$$

Since $g(z_0)\ne 0$ and $|\lambda|>1$, it follows that

$$|P(z_0)| > |g(z_0)| \text{ with } |z_0|\le 1,$$

which clearly contradicts (31). Thus $h(z)$ does not vanish in $|z|\le 1$ for every λ with $|\lambda|>1$, so that all the zeros of $h(z)$ lie in $|z|\ge\rho$ for some $\rho>1$ and hence all the zeros of $h(\rho z)$ lie in $|z|\ge 1$. Applying (28) to the polynomial $h(\rho z)$, we get

$$|h(R\rho z) - \beta h(r\rho z)| < \left|R^n h(\rho z/R) - \overline{\beta} r^n h(\rho z/r)\right| \quad \text{for } |z|<1, R>r\ge 1.$$

Taking $z = e^{i\theta}/\rho, 0\le\theta<2\pi$, then $|z| = (1/\rho) < 1$ as $\rho>1$ and we get

$$\left|h\left(Re^{i\theta}\right) - \beta h\left(re^{i\theta}\right)\right| < \left|R^n h\left(e^{i\theta}/R\right) - \overline{\beta} r^n h\left(e^{i\theta}/r\right)\right|,$$

$0\le\theta<2\pi$, $R>r\ge 1$ and $|\beta|\le 1$. This implies

$$|h(Rz) - \beta h(rz)| < \left|R^n h(z/R) - \overline{\beta} r^n h(z/r)\right| \text{ for } |z|=1.$$

An application of Rouche's theorem shows that the polynomial

$$T(z) = \left(h(Rz) - \beta h(rz)\right) + e^{i\alpha}\left(R^n h(z/R) - \overline{\beta} r^n h(z/r)\right)$$

does not vanish in $|z|\le 1$ for every real or complex number β with $|\beta|\le 1$, $R>r\ge 1$ and α real. Replacing $h(z)$ by $P(z)+\lambda h(z)$, it follows that the polynomial

$$T(z) = \left\{P(Rz) - \beta P(rz) + e^{i\alpha}\left(R^n P(z/R) - \overline{\beta} r^n P(z/r)\right)\right\} + \lambda\left\{\left(g(Rz) - \beta g(rz)\right) + e^{i\alpha}\left(R^n g(z/R) - \overline{\beta} r^n g(z/r)\right)\right\} \tag{32}$$

does not vanish in $|z|\le 1$ for every β,λ with $|\beta|\le 1$ and $|\lambda|>1$. This implies

$$\left|\left(P(Rz) - \beta P(rz)\right) + e^{i\alpha}\left(R^n P(z/R) - \overline{\beta} r^n P(z/r)\right)\right| \le \left|\left(g(Rz) - \beta g(rz)\right) + e^{i\alpha}\left(R^n g(z/R) - \overline{\beta} r^n g(z/r)\right)\right| \tag{33}$$

for $|z|\le 1$, $|\beta|\le 1$, $R>r\ge 1$ and α real. If Inequality (33) is not true, then there is a point $z=z_0$ with $|z_0|\le 1$ such that

$$\left|\left(P(Rz_0) - \beta P(rz_0)\right) + e^{i\alpha}\left(R^n P(z_0/R) - \overline{\beta} r^n P(z_0/r)\right)\right| > \left|\left(g(Rz_0) - \beta g(rz_0)\right) + e^{i\alpha}\left(R^n g(z_0/R) - \overline{\beta} r^n g(z_0/r)\right)\right|.$$

Since all the zeros of polynomials $g(z)$ lie in $|z|>1$, it follows (as before) that all the zeros of polynomial $\left(g(Rz) - \beta g(rz)\right) + e^{i\alpha}\left(R^n g(z/R) - \overline{\beta} r^n g(z/r)\right)$ also lie in $|z|>1$ for every real or complex number β with $|\beta|\le 1$, $R>r\ge 1$ and α real. Hence

$$g(Rz_0)-\beta g(rz_0))+e^{i\alpha}\left(R^n g(z_0/R)-\bar{\beta}r^n g(z_0/r)\right)\neq 0$$
$$\text{with } |z_0|\le 1.$$

We take

$$\lambda = -\frac{\left(P(Rz_0)-\beta P(rz_0)\right)+e^{i\alpha}\left(R^n P(z_0/R)-\bar{\beta}r^n P(z_0/r)\right)}{\left(g(Rz_0)-\beta g(rz_0)\right)+e^{i\alpha}\left(R^n g(z_0/R)-\bar{\beta}r^n g(z_0/r)\right)}$$

so that λ is a well-defined real or complex number with $|\lambda|>1$ and with this choice of λ, from (32) we get $T(z_0)=0$ with $|z_0|\le 1$ This clearly is a contradiction to the fact that $T(z)$ does not vanish in $|z|\le 1$. Thus for every β with $|\beta|\le 1$, $R>r\ge 1$ and α real,

$$\left|\left(P(Rz)-\beta P(rz)\right)+e^{i\alpha}\left(R^n P(z/R)-\bar{\beta}r^n P(z/r)\right)\right|$$
$$\le\left|\left(g(Rz)-\beta g(rz)\right)+e^{i\alpha}\left(R^n g(z/R)-\bar{\beta}r^n g(z/r)\right)\right|$$

for $|z|\le 1$, which in particular gives for each $p>0$ and $0\le\theta<2\pi$,

$$\int_0^{2\pi}\left|\left(P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right)\right.$$
$$\left.+e^{i\alpha}\left(R^n P\left(e^{i\theta}/R\right)-\bar{\beta}r^n P\left(e^{i\theta}/r\right)\right)\right|^p d\theta$$
$$\le\int_0^{2\pi}\left|\left(g\left(Re^{i\theta}\right)-\beta g\left(re^{i\theta}\right)\right)\right.$$
$$\left.+e^{i\alpha}\left(R^n g\left(e^{i\theta}/R\right)-\bar{\beta}r^n g\left(e^{i\theta}/r\right)\right)\right|^p d\theta$$

Using lemma 4 and (30), it follows that for every β with $|\beta|\le 1$, $R>r$, $p>0$ and α real,

$$\int_0^{2\pi}\left|\left(P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right)\right.$$
$$\left.+e^{i\alpha}\left(R^n P\left(e^{i\theta}/R\right)-\bar{\beta}r^n P\left(e^{i\theta}/r\right)\right)\right|^p d\theta$$
$$\le\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar{\beta}\right)\right|^p\int_0^{2\pi}\left|g\left(e^{i\theta}\right)\right|^p d\theta$$
$$=\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar{\beta}\right)\right|^p\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta. \quad (34)$$

Now if $P_1(z)$ has a zero on $|z|=1$, then applying (34) to the polynomial $P^*(z)=P_1(tz)P_2(z)$ where $t<1$, we get for every β with $|\beta|\le 1$, $R>r\ge 1$, $p>0$ and α real,

$$\int_0^{2\pi}\left|\left(P^*\left(Re^{i\theta}\right)-\beta P^*\left(re^{i\theta}\right)\right)\right.$$
$$\left.+e^{i\alpha}\left(R^n P^*\left(e^{i\theta}/R\right)-\bar{\beta}r^n P^*\left(e^{i\theta}/r\right)\right)\right|^p d\theta \quad (35)$$
$$\le\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar{\beta}\right)\right|^p\int_0^{2\pi}\left|P^*\left(e^{i\theta}\right)\right|^p d\theta.$$

Letting $t\to 1$ in (35) and using continuity, the desired result follows immediately and this proves Lemma 5.

3. Proofs of the Theorems

Proof of Theorem 1. Since $P(z)$ is a polynomial of degree at most n, we can write

$$P(z)=P_1(z)P_2(z)=\prod_{j=1}^{k}\left(z-z_j\right)\prod_{j=k+1}^{n}\left(z-z_j\right), k\ge 1$$

where all the zeros of $P_1(z)$ lie in $|z|\le 1$ and all the zeros of $P_2(z)$ lie in $|z|>1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z|<1$. Let $Q_2(z)=z^{n-k}\overline{P_2(1/\bar{z})}$, then all the zeros of $Q_2(z)$ lie in $|z|<1$ and $|Q_2(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$F(z)=P_1(z)Q_2(z)=\prod_{j=1}^{k}\left(z-z_j\right)\prod_{j=k+1}^{n}\left(1-z\bar{z}_j\right),$$

then all the zeros of $F(z)$ lie in $|z|<1$ and for $|z|=1$,

$$|F(z)|=|P_1(z)||Q_2(z)|=|P_1(z)||P_2(z)|=|P(z)|. \quad (29)$$

By the Maximum Modulus Principle, it follows that

$$|P(z)|\le|F(z)| \text{ for } |z|\ge 1.$$

Since $F(z)\neq 0$ for $|z|\ge 1$ and $|\lambda|>1$, a direct application of Rouche's theorem shows that the polynomial $H(z)=P(z)+\lambda F(z)$ has all its zeros in $|z|<1$ for every λ with $|\lambda|>1$. Applying lemma 1 to the polynomial $H(z)$, we deduce (as before)

$$|H(rz)|<|H(Rz)|$$

for $|z|=1$ and $R>r\ge 1$.

Since all the zeros of $H(Rz)$ lie in $|z|<\frac{1}{R}\le 1,$ we conclude that for every β,λ with $|\beta|\le 1$ and $|\lambda|>1$, all the zeros of polynomial

$$G(z)=H(Rz)-\beta H(rz)$$
$$=\left(P(Rz)-\beta P(rz)\right)+\lambda\left(F(Rz)-\beta F(rz)\right)$$

lie in $|z|<1$. This implies (as in the case of Lemma 2)

$$|P(Rz)-\beta P(rz)|\le|F(Rz)-\beta F(rz)| \text{ for } |z|\ge 1$$
$$\text{and } R>r\ge 1,$$

which in particular gives for $R>r$ and $p>0$,

$$\int_0^{2\pi}\left|P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right|^p d\theta$$
$$\le\int_0^{2\pi}\left|F\left(Re^{i\theta}\right)-\beta F\left(re^{i\theta}\right)\right|^p d\theta \quad (30)$$

Again, since all the zeros of $F(z)$ lie in $|z|<1$, as before, $F(Rz)-\beta F(rz)$ has all its zeros in $|z|<1$ for every real or complex number β with $|\beta|\le 1$. Therefore, the operator Λ_γ defined by

$$\begin{aligned}\Lambda_\gamma F(z) &= F(Rz)-\beta F(rz)\\ &= \left(R^n-\beta r^n\right)b_n z^n+\cdots+(1-\beta)b_0\end{aligned}$$

is admissible. Hence by (26) of Lemma (3), for each $p>0$, we have

$$\begin{aligned}&\int_0^{2\pi}\left|F\left(Re^{i\theta}\right)-\beta F\left(re^{i\theta}\right)\right|^p d\theta\\ &\le \left|R^n-\beta r^n\right|^p\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^p d\theta.\end{aligned} \tag{31}$$

Combining Inequalities (37) and (38) and noting that $\left|F\left(e^{i\theta}\right)\right|=\left|P\left(e^{i\theta}\right)\right|$, we obtain for $R>r\ge 1$ and $p>0$

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right|^p d\theta\right\}^{1/p}\\ &\le \left|R^n-\beta r^n\right|\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right\}^{1/p}.\end{aligned} \tag{32}$$

In case $P_1(z)$ has a zero on $|z|=1$, the Inequality (39) follows by using similar argument as in the case of Lemma 5. This completes the proof of Theorem 1.

Proof of Theorem 2. By hypothesis $P\in P_n$ and $P(z)$ does not vanish in $|z|<1$, therefore, by Lemma 2 for every real or complex number β with $|\beta|\le 1$, $0\le\theta<2\pi$ and $R>r\ge 1$,

$$\begin{aligned}&\left|P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right|\\ &\le\left|R^nP\left(e^{i\theta}/R\right)-\bar\beta r^nP\left(e^{i\theta}/r\right)\right|\end{aligned} \tag{33}$$

Also, by Lemma 5,

$$\begin{aligned}&\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^p d\theta\\ &\le\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar\beta\right)\right|^p\int_0^{2\pi}\left|P\left(e^{i\theta}\right|^p d\theta\end{aligned} \tag{34}$$

where

$$F(\theta)=P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right) \text{ and}$$

$$G(\theta)=R^nP\left(e^{i\theta}/R\right)-\bar\beta r^nP\left(e^{i\theta}/r\right).$$

Integrating both sides of (41) with respect to α from 0 to 2π, we get for each $p>0$, $R>r\ge 1$ and α real,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^p d\alpha d\theta\\ &\le\left\{\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar\beta\right)\right|^p d\alpha\right\}\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right|^p d\theta\right\}\end{aligned} \tag{35}$$

Now for every real α, $t\ge 1$ and $p>0$, we have

$$\int_0^{2\pi}\left|t+e^{i\alpha}\right|^p d\alpha\ge\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha.$$

If $F(\theta)\ne 0$, we take $t=\left|G(\theta)\right|/\left|F(\theta)\right|$, then by (40) $t\ge 1$ and we get

$$\begin{aligned}&\int_0^{2\pi}\left|F(\theta)+e^{i\alpha}G(\theta)\right|^p d\alpha\\ &=\left|F(\theta)\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\frac{G(\theta)}{F(\theta)}\right|^p d\alpha\\ &=\left|F(\theta)\right|^p\int_0^{2\pi}\left|\frac{G(\theta)}{F(\theta)}+e^{i\alpha}\right|^p d\alpha\\ &=\left|F(\theta)\right|^p\int_0^{2\pi}\left|\left|\frac{G(\theta)}{F(\theta)}\right|+e^{i\alpha}\right|^p d\alpha\\ &\ge\left|F(\theta)\right|^p\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha.\end{aligned}$$

For $F(\theta)=0$, this inequality is trivially true. Using this in(42), we conclude that for every real or complex number β with $|\beta|\le 1$, $R>r\ge 1$ and α real,

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|1+e^{i\alpha}\right|^p d\alpha\right\}\left\{\int_0^{2\pi}\left|P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)\right|^p d\theta\right\}\\ &\le\left\{\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar\beta\right)\right|^p d\alpha\right\}\left\{\int_0^{2\pi}\left|P\left(e^{i\theta}\right|^p d\theta\right\}.\end{aligned} \tag{43}$$

Since

$$\begin{aligned}&\left\{\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)+e^{i\alpha}\left(1-\bar\beta\right)\right|^p d\alpha\right\}\\ &=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|+e^{i\alpha}\left|1-\bar\beta\right|\right|^p d\alpha\right\}\\ &=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|+e^{i\alpha}\left|1-\beta\right|\right|^p d\alpha\right\}\\ &=\left\{\int_0^{2\pi}\left|\left|R^n-\beta r^n\right|e^{i\alpha}+\left|1-\beta\right|\right|^p d\alpha\right\}\\ &=\left\{\int_0^{2\pi}\left|\left(R^n-\beta r^n\right)e^{i\alpha}+\left(1-\beta\right)\right|^p d\alpha\right\},\end{aligned} \tag{44}$$

the desired result follows immediately by combining (43) and (44). This completes the proof of Theorem 2.

Proof of Theorem 3. Since $P(z)$ is a self-inversive polynomial, we have $P(z)=uQ(z)$ for all $z\in C$ where $|u|=1$ and $Q(z)=z^n\overline{P}\left(1/\bar z\right)$. Therefore, for every real or complex number β and $R>r\ge 1$,

$$\left|P(Rz)-\beta P(rz)\right|=\left|Q(Rz)-\beta Q(rz)\right| \text{ for all } z\in C$$

so that

$$\left|G(\theta)/F(\theta)\right|=\left|\frac{P\left(Re^{i\theta}\right)-\beta P\left(re^{i\theta}\right)}{R^nP\left(e^{i\theta}/R\right)-\bar\beta r^nP\left(e^{i\theta}/r\right)}\right|=1.$$

Using this in (41) with $|\beta| \le 1$ and proceeding similarly as in the proof of Theorem 2, we get the desired result. This completes the proof of Theorem 3.

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