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Hyperbolic Fibonacci and Lucas Functions, "Golden" Fibonacci Goniometry, Bodnar's Geometry, and Hilbert's Fourth Problem

—Part III. An Original Solution of Hilbert's Fourth Problem

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Abstract

This article refers to the "*Mathematics of Harmony*" by Alexey Stakhov [1], a new interdisciplinary direction of modern science. The main goal of the article is to describe two modern scientific discoveries—*New Geometric Theory of Phyllotaxis* (*Bodnar's Geometry*) and *Hilbert's Fourth Problem* based on the *Hyperbolic Fibonacci and Lucas Functions* and "Golden" Fibonacci λ*-Goniometry* ($\lambda > 0$ is a given positive real number). Although these discoveries refer to different areas of science (mathematics and theoretical botany), however they are based on one and the same scientific ideas—The "*golden mean*," which had been introduced by Euclid in his *Elements*, and its generalization—The "*metallic means*," which have been studied recently by Argentinian mathematician Vera Spinadel. The article is a confirmation of interdisciplinary character of the "*Mathematics of Harmony*", which originates from Euclid's Elements.

Keywords: Euclid's Fifth Postulate, Lobachevski's Geometry, Hyperbolic Geometry, Phyllotaxis, Bodnar's Geometry, Hilbert's Fourth Problem, the "Golden" and "Metallic" Means, Binet Formulas, Hyperbolic Fibonacci and Lucas Functions, Gazale Formulas, "Golden" Fibonacci λ-Goniometry

1. Introduction

In Part III we study *Hilbert's Fourth Problem*, concerning to hyperbolic geometry, from new point of view-the "Golden" Fibonacci λ *-Goniometry* ($\lambda > 0$ is given positive real number). This goniometry is based on a new class of hyperbolic functions-*hyperbolic Fibonacci* and *Lucas* λ *-functions* [2,3], which are connected with the "metallic" means and Gazale formulas. The main result of this study is a creation of infinite set of the golden isometric λ-models of Lobachevski's plane that is directly relevant to Hilbert's Fourth Problem. Also we discuss a connection between Poincare's model of Lobachevski's plane on the unit disc and the golden λ-models of Lobachevski's plane. This study can be considered as an unexpected variant of Hilbert's Fourth Problem solution based on the "metallic means" [4], which are a generalization of the "golden mean" (Theorem II.11 of Euclid's Elements).

2. Euclid's Fifth Postulate and Lobachevski's Geometry

On February 23, 1826 on the meeting of the Mathematics and Physics Faculty of Kazan University the Russian mathematician Nikolai Lobachevski had proclaimed on the creation of new geometry named *imaginary geometry*. This geometry was based on the traditional Euclid's postulates, excepting Euclid's Fifth Postulate about parallels. New Fifth Postulate about parallels was formulated by Lobachevski as follows: "*At the plane through a point outside a given straight line, we can conduct two and only two straight lines parallel to this line, as well as an endless set of straight lines, which do not overlap with this line and are not parallel to this line, and the endless set of straight lines, intersecting the given straight line*." For the first time, a new geometry was published by Lobachevski in 1829 in the article About *the Foundations of Geometry* in the magazine *Kazan Bulletin*.

Independently on Lobachevski, the Hungarian mathematician Janos Bolyai came to such ideas. He published his work Appendix in 1832, that is, three years later Lobachevski. Also the prominent German mathematician Carl Friedrich Gauss came to the same ideas. After his death some unpublished sketches on the non-Euclidean geometry were found.

Lobachevski's geometry got a full recognition and wide distribution 12 years after his death, when it is became clear that scientific theory, built on the basis of a system of axioms, is considered to be fully completed only when the system of axioms meets three conditions: *independence, consistency* and *completeness*. Lobachevski's geometry satisfies these conditions. Finally this became clear in 1868 when the Italian mathematician Eugenio Beltrami in his memoirs *The Experience of the Non-Euclidean Geometry Interpretation* showed that in Euclidean space R^3 at pseudospherical surfaces the geometry of Lobachevski's plane arises, if we take geodesic lines as straight lines.

Later the German mathematician Felix Christian Klein and the French mathematician Henri Poincare proved a consistency of Non-Euclidean geometry, by means of the construction of corresponding models of Lobachevski's plane. The interpretation of Lobachevski's geometry on the surfaces of Euclidean space contributed to general recognition of Lobachevski's ideas.

The creation of *Riemannian geometry* by Georg Friedrich Riemann became the main outcome of such Non-Euclidean approach. The Riemannian geometry developed a mathematical doctrine about geometric space, a notion of differential of a distance between elements of diversity and a doctrine about curvature. The introduction of the *generalized Riemannian spaces*, whose particular cases are *Euclidean space* and *Lobachevski's space*, and the so-called *Riemannian geometry*, opened new ways in the development of geometry. They found their applications in physics (theory of relativity) and other branches of theoretical natural sciences.

Lobachevski's geometry also is called *hyperbolic geometry* because it is based on the *hyperbolic functions* (1.8) (see Part I) introduced in 18th century by the Italian mathematician Vincenzo Riccati.

The most famous classical *isometric* interpretations of Lobachevski's plane with the *Gaussian curvature* $K = -1$ are the following:

- Beltrami's interpretation on a disk;
- Beltrami's interpretation of hyperbolic geometry on pseudo-sphere;
- Euclidean model by Keli-Klein;
- Projective model by Keli-Klein;
- Poincare's interpretation at a half-plane;
- Poincare's interpretation inside a circle;
- Poincare's interpretation on a hyperboloid.

In particular, the classical model of Lobachevski's plane in *pseudo-spherical coordinates* $(u,v), 0 < u < +\infty$, $-\infty < v < +\infty$ with the Gaussian curvature $K = -1$ (Beltrami's interpretation of hyperbolic geometry on pseudo-sphere) has the following form:

$$(ds)^2 = (du)^2 + sh^2(u)(dv)^2, \tag{3.1}$$

where ds is an element of length and $sh(u)$ is hyperbolic sine.

Lobachevski's geometry has remarkable applications in many fields of modern natural sciences. This concerns not only applied aspects (*cosmology*, *electrodynamics*, *plasma theory*), but, first of all, it concerns the most fundamental sciences and their foundation—Mathematics (*number theory*, *theory of automorphic functions created by A. Poincare*, *the geometry of surfaces* and so on).

Since on the closed surfaces of negative Gaussian curvature, Lobachevski's geometry is fulfilled and Lobachevski's plane is universal covering for these surfaces, it is very fruitful to study various objects (dynamical systems with continuous and discrete time, layers, fabrics and so on), defined on these surfaces. By developing this idea, we can raise these objects to the level of universal covering, which is replenished by the absolute ("infinity"), and further we can study smooth topological properties of these objects with the help of the absolute.

Samuil Aranson studied this problem about four decades. The works [5-10] written by Samuil Aranson with co-authors give presentation about these results and research methods. Aranson's DrSci dissertation "*Global problems of qualitative theory of dynamic systems on surfaces*" (1990) is devoted to this themes.

3. Hilbert's Fourth Problem

In the lecture "*Mathematical Problems*" presented at the *Second International Congress of Mathematicians* (Paris, 1900), David Hilbert had formulated his famous 23 mathematical problems. These problems determined considerably the development of the 20th century mathematics. This lecture is a unique phenomenon in the mathematics history and in mathematical literature. The Russian translation of Hilbert's lecture and its comments are given in the work [11]. In particular, Hilbert's Fourth Problem asserts:

"*Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry*".

Note, Hilbert considered that *Lobachevski's Geometry*and *Riemannian geometry* are nearest to the *Euclidean geometry.* In [12] the history of the Hilbert's Fourth

Problem solution and some approaches to its solution are described. Also Hilbert's understanding of the Fourth Problem is discussed. It is clear that in mathematics, Hilbert's Fourth Problem was a fundamental problem in geometry. In the citation, taken from Hilbert's original, we found the following description of Hilbert' Fourth Problem: "*the problem is to find geometries whose axioms are closest to those of Euclidean geometry if the ordering and incidence axioms are retained, the congruence axioms is weakened, and the equivalent of the parallel postulate is omitted*."

In mathematical literature Hilbert's Fourth Problem is sometimes considered as formulated very vague what makes difficult its final solution [12]. In [13] American geometer Herbert Busemann analyzed the whole range of issues related to Hilbert's Fourth Problem and also concluded that the question related to this issue, unnecessarily broad. Note also the book [14] by Alexei Pogorelov devoted to partial solution to *Hilbert's Fourth Problem*. The book identifies all, up to isomorphism, implementations of the axioms of classical geometries (Euclid, Lobachevski and elliptical), if we delete the axiom of congruence and refill these systems with the axiom of "triangle inequality."

In spite of critical attitude of mathematicians to *Hilbert's Fourth Problem*, we should emphasize great importance of this problem for mathematics, particularly for geometry. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry* do not exhaust all possible variants of non-Euclidean geometries. *Hilbert's Fourth Problem* directs attention of researchers at finding new non-Euclidean geometries, which are the nearest geometries to the traditional Euclidean geometry.

In this connection, a discovery of new class of hyperbolic functions based on the "golden mean" and "metallic means" [2,3,15] and following from them new geometric theory of phyllotaxis (*Bodnar's geometry*) [16] have a principal importance for the development of geometry because it shows an existence of new non-Euclidean geometries in surrounding us world. Recently Alexey Stakhov gave a wide generalization of the symmetric hyperbolic Fibonacci and Lucas functions (1.9) and (1.10) (see Part I) and developed the *so*-called *hyperbolic Fibonacci and Lucas λ-functions* [3]. It is proved in [3] an existence of infinite variants of new hyperbolic functions, which can be a base for new non-Euclidean geometries.

The main purpose of Part III of the article is to develop this idea, that is, to create new non-Euclidean geometries based on the hyperbolic Fibonacci and Lucas λ-functions introduced in [3]. This study can be considered as unexpected and original solution of *Hilbert's Fourth Problem* based on the the "*metallic means*" [4]. The authors of this article announced this idea in [17]. In Part III of the article we give a detailed proof of this idea.

4. The "Golden" Fibonacci λ-Goniometry and Hilbert's Fourth Problem

4.1. "Golden" Metric λ-Forms of Lobachevski's Plane

In connection with *Hilbert's Fourth Problem* the authors of the present article Alexey Stakhov and Samuil Aranson suggested in [17] infinite set of *metric forms* of *Lobachevski's plane* in dependence on real parameter $\lambda > 0$. These metric forms are given in the coordinates (u, v), $0 < u < +\infty, -\infty < v < +\infty$; they have the *Gaussian curvature K*= –1 and can be represented in the form:

$$(ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \quad (3.2)$$

where $\Phi_\lambda = \dfrac{\lambda + \sqrt{4+\lambda^2}}{2}$ is the "*metallic mean*" and $sF_\lambda(u)$ is hyperbolic Fibonacci λ-sine. Let us name the forms (3.2) *metric λ-forms of Lobachevski's plane.*

In [17] we asserted (without proof) that for any real parameter $\lambda > 0$ the metric forms (3.2) are *isometric* on the base of diffeomorphisms

$$\begin{aligned}\bar{u} = \bar{u}(u,v) &= Arcch\left[\frac{\sqrt{4+\lambda^2}}{2}cF_\lambda(u)\right] \\ &= Arcsh\left[\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda(u)\right], \bar{v} = \bar{v}(u,v) = v\end{aligned} \quad (3.3)$$

to the classical metric forms of Lobachevski's plane (3.1) in semi-geodesic coordinates ($\bar{u}$, $\bar{v}$), $0 < \bar{u} < +\infty$, $-\infty < \bar{v} < +\infty$.

Since the forms of the kind (3.1) are isometric to all previously known classical metric forms of Lobach-evski's plane what is noted, for instance, in [18], then it follows from here that the forms (3.2) are isometric to all these classical forms.

Here we give direct proof of isometrics of the form (3.2) and (3.1).

Next we describe the basic geometric objects so, for instance, as geodesic lines and intersection angles, which are induced by the form (3.2).

In the concluding part for completeness of presentation it is shown also isometrics between *Poincare's model* of *Lobachevski's plane* on unit disc and the form (3.2). From here it is easy to deduce for the form (3.2) the formula for distance and also the formula for metrics movement.

The proof of isometrics of the forms (3.2) and (3.1) is fulfilled in tree steps.

Step 1. Let us prove that for the metric form (3.2) *Gaussian curvature* $K=-1$. For this purpose let us introduce the following designations:

$$A=\ln\left(\Phi_\lambda\right),\ B=\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda\left(u\right) \tag{3.4}$$

Here, according to (1.35) (see Part I), $sF_\lambda\left(u\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{\sqrt{4+\lambda^2}}$, therefore for the second correlation in (3.4) we have:

$$B=\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda\left(u\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2} \tag{3.5}$$

where, by virtue (1.24) (see Part I), $\Phi_\lambda=\frac{\lambda+\sqrt{4+\lambda^2}}{2}$. Then, the expressions (3.4) can be written in the form:

$$\begin{gathered}A=\ln\left(\Phi_\lambda\right)=\ln\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right),\\ B=\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda\left(u\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}.\end{gathered} \tag{3.6}$$

Therefore, the metric form (3.2) can be written in the form:

$$\left(ds\right)^2=A^2\left(du\right)^2+B^2\left(dv\right)^2. \tag{3.7}$$

Taking into consideration the expression (3.6) and the obvious conditions:

$$\lambda>0,\quad 0<u<+\infty,\quad -\infty<v<+\infty,$$

We can write:

$$A>0,\quad B>0. \tag{3.8}$$

It is known [18] that for the metric form of the kind (3.1) *Gaussian curvature K* is determined from the correlation:

$$K=K(u,v)=-\frac{1}{AB}\left[\left(\frac{A_v}{B}\right)_v+\left(\frac{B_u}{A}\right)_u\right]. \tag{3.9}$$

Here the symbols $(\ )_v$ and $(\ )_u$ mean partial derivatives on v and u.

By using definition (3.6), we can get the following expressions:

$$A_v=0;\ B_u=\frac{\Phi_\lambda^u+\Phi_\lambda^{-u}}{2}\ln\left(\Phi_\lambda\right);\ B_{uu}=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}\ln^2\left(\Phi_\lambda\right);$$

$$AB=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}\ln\left(\Phi_\lambda\right);\left(\frac{A_v}{B}\right)_v=0;$$

$$\left(\frac{B_u}{A}\right)_u=\frac{B_{uu}}{\ln\left(\Phi_\lambda\right)}=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}\ln\left(\Phi_\lambda\right);$$

$$K=-\frac{1}{AB}\left[\left(\frac{A_v}{B}\right)_v+\left(\frac{B_u}{A}\right)_u\right]$$

$$=-\left[\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}\ln\left(\Phi_\lambda\right)\right]^{-1}\left[\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}\ln\left(\Phi_\lambda\right)\right]=-1$$

Step 2. Let us prove that the transformations (3.3) can be written in the form:

$$\bar{u}=\left[\ln\left(\Phi_\lambda\right)\right]u,\ \bar{v}=v \tag{3.10}$$

Since

$$\frac{\sqrt{4+\lambda^2}}{2}cF_\lambda\left(u\right)=\frac{\Phi_\lambda^u+\Phi_\lambda^{-u}}{2}$$

and

$$\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda\left(u\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}, \tag{3.11}$$

then, by virtue (3.11), from the transformations (3.3) for the cases $\bar{u}>\bar{u}>0,u>0$ we get:

$$ch\left(u\right)=\frac{\Phi_\lambda^u+\Phi_\lambda^{-u}}{2}\text{ and }sh\left(u\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}. \tag{3.12}$$

Take the differential d of the first correlation in (3.12):

$$\begin{aligned}d\left[ch\left(u\right)\right]&=sh\left(\bar{u}\right)d\bar{u}\\ &=d\left[\frac{\Phi_\lambda^u+\Phi_\lambda^{-u}}{2}\right]\\ &=\left[\ln\left(\Phi_\lambda\right)\right]\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}du\end{aligned} \tag{3.13}$$

Since $sh\left(\bar{u}\right)=\frac{\Phi_\lambda^u-\Phi_\lambda^{-u}}{2}$, then after substituting this expression into (3.13) we get: $sh\left(\bar{u}\right)d\bar{u}=\left[\ln\left(\Phi_\lambda\right)\right]sh\left(\bar{u}\right)du$. Since $\bar{u}>0$, then after the reduction by $sh\left(\bar{u}\right)$ we come to differential equation:

$$\frac{d\bar{u}}{du}=\ln\left(\Phi_\lambda\right). \tag{3.14}$$

Hence we have:

$$\bar{u}=\left[\ln\left(\Phi_\lambda\right)\right]u+C, \tag{3.15}$$

where C = *const.* Since in (3.3) we have $\bar{u}=0$ at $u=0$, then in (3.15) we can assume $C=0$, and therefore we get from (3.15) that $\bar{u}=\left[\ln\left(\Phi_\lambda\right)\right]u$. Then at all

$$\begin{gathered}0<\bar{u}<+\infty,\quad \infty<\bar{v}<+\infty,\\ 0<u<+\infty,\quad \infty<v<+\infty\end{gathered} \tag{3.16}$$

instead (3.3) we can consider the transformations (3.10).

Step 3. Let us prove that the metric forms (3.1) and (3.2) are *isometric.* For the proof we use the transformations (3.10), which is analytical diffeomorphism at the values of variables given by (3.16).

With this purpose let us consider more general situation, when two isometric metric forms are given:

$$\begin{cases}(d\bar{s})^2 = \bar{E}(\bar{u},\bar{v})(d\bar{u})^2 + 2\bar{F}(\bar{u},\bar{v})d\bar{u}d\bar{v} + \bar{G}(\bar{u},\bar{v})(d\bar{v})^2 \\ (ds)^2 = E(u,v)(du)^2 + 2F(u,v)dudv + G(u,v)(dv)^2\end{cases} \tag{3.17}$$

where

$\bar{E} > 0,\ \bar{G} > 0,\ \bar{E}\bar{G} - (\bar{F})^2 > 0,\ E > 0,\ G > 0,\ EG - F^2 > 0$,

here isometrics is realized by using the diffeomorphism:

$$h : \bar{u} = \bar{u}(u,v),\ \bar{v} = \bar{v}(u,v). \tag{3.18}$$

Let us consider differential of $\bar{u}$:

$$d\bar{u} = d\left[\bar{u}(u,v)\right] = \bar{u}_u du + \bar{u}_v dv$$

and

$$d\bar{v} = d\left[\bar{v}(u,v)\right] = \bar{v}_u du + \bar{v}_v dv \tag{3.19}$$

Substitute (3.19) into (3.17) and note that according to our assumption the first and second metric forms in (3.17) are isometric. Taking into consideration that they have common linear element *ds*, we get the following identity:

$$\begin{aligned}(ds)^2 &= \bar{E}\times(\bar{u}_u du + \bar{u}_v dv)^2 + 2\bar{F}\times(\bar{u}_u du + \bar{u}_v dv) \\ &\quad\times(\bar{v}_u du + \bar{v}_v dv) + \bar{G}\times(\bar{v}_u du + \bar{v}_v dv)^2 \\ &= (ds)^2 = E\times(du)^2 + 2F\times dudv + G\times(dv)^2\end{aligned} \tag{3.20}$$

By equating in the left-hand and right-hand parts of the identity (3.20) the equal coefficients at $(du)^2$, $du\,dv$ and $(dv)^2$, we get the following correlations:

$$\begin{pmatrix}E\\F\\G\end{pmatrix} = \begin{pmatrix}(\bar{u}_u)^2 & 2\bar{u}_u\bar{v}_u & (\bar{v}_u)^2 \\ \bar{u}_u\bar{u}_v & \bar{u}_u\bar{u}_v + \bar{u}_v\bar{u}_u & \bar{v}_u\bar{v}_v \\ (\bar{u}_v)^2 & 2\bar{u}_v\bar{v}_v & (\bar{v}_v)^2\end{pmatrix}\begin{pmatrix}\bar{E}\\\bar{F}\\\bar{G}\end{pmatrix} \tag{3.21}$$

Also the back statement is correct, that is, if we have two metric forms

$$\begin{cases}(d\bar{s})^2 = \bar{E}(\bar{u},\bar{v})(d\bar{u})^2 + 2\bar{F}(\bar{u},\bar{v})d\bar{u}d\bar{v} + \bar{G}(\bar{u},\bar{v})(d\bar{v})^2 \\ (ds)^2 = E(u,v)(du)^2 + 2F(u,v)dudv + G(u,v)(dv)^2\end{cases} \tag{3.22}$$

and there is the diffeomorphism (3.18) so that the correlations (3.21) are fulfilled, then $(d\bar{s})^2 = (ds)^2$, that is, the metric forms (3.22) are isometric.

In our situation we have two metric forms (3.1) and (3.2); here, as is shown in correlation (3.6), the metric form (3.2) can be rewritten in the form (3.7), whose coefficients A^2 and B^2 are equal:

$$A^2 = \ln^2(\Phi_\lambda),\ B^2 = \left(\frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2.$$

Let us prove isometrics of the forms (3.1) and (3.7):

$$\begin{cases}(d\bar{s})^2 = (d\bar{u})^2 + sh^2(\bar{u})(d\bar{v})^2 \\ (ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \left(\dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2(dv)^2\end{cases} \tag{3.23}$$

This isometrics is proved by using the analytical diffeomorphism (3.10):

$$\bar{u} = \bar{u}(u,v) = \left[\ln(\Phi_\lambda)\right]u,\ \bar{v} = \bar{v}(u,v) = v,$$

which, as is shown (step 2), can be rewritten in the form (3.3).

Note that the area of parameters and variables change has the following range:

$$\begin{aligned}&\lambda > 0, (0 < u < +\infty, -\infty < v < +\infty), \\ &(0 < \bar{u} < +\infty, -\infty < \bar{v} < +\infty)\end{aligned} \tag{3.24}$$

where, in virtue of (1.24) (Part I), we have:

$$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}.$$

Then, in terms of the correlations (3.17) for the metric forms (3.23) we get the following expressions for the coefficients of these forms:

$$\begin{cases}\bar{E}(\bar{u},\bar{v}) = 1,\ \bar{F}(\bar{u},\bar{v}) = 0,\ \bar{G}(\bar{u},\bar{v}) = sh^2(\bar{u}), \\ E(u,v) = \ln^2(\Phi_\lambda),\ F(u,v) = 0,\ G(u,v) = \left(\dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2\end{cases} \tag{3.25}$$

From the transformations (3.10) we get the following derivatives:

$$\bar{u}_u = \ln(\Phi_\lambda),\ \bar{u}_v = 0,\ \bar{v}_u = 0,\ \bar{v}_v = 1 \tag{3.26}$$

Then, with regard to (3.25) and (3.26), the transformation (3.21) can be written in the form:

$$\begin{bmatrix}\ln^2(\Phi_\lambda) \\ 0 \\ \left(\dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2\end{bmatrix} = \begin{bmatrix}\ln^2(\Phi_\lambda) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 \\ 0 \\ sh^2(\bar{u})\end{bmatrix} \tag{3.27}$$

From here we get the following identities:

$$\begin{cases}\left[\ln(\Phi_\lambda)\right]^2 \equiv \left[\ln(\Phi_\lambda)\right]^2 \\ 0 \equiv 0 \\ \left(\dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2 \equiv sh^2(\bar{u})\end{cases} \tag{3.28}$$

The first two identities from (3.28) are obvious. The last identity from (3.28) follows from the second correlation in (3.12), which is proved in step 2 at $\bar{u} = \left[\ln\left(\Phi_\lambda\right)\right]u$, where $\lambda > 0$ and $u > 0$.

Thus, by using the transformations (3.10), we have proved that the metric forms (3.1) and (3.2) are isometric.

Hence, at all $\lambda > 0, (0 < u < +\infty, -\infty < v < +\infty)$, the *golden metric* λ*-forms of Lobachevski's plane of the kind* (3.2) are isometric to all previous known isometric between themselves metric forms of Lobachevski's plane.

4.2. Partial Cases of the Golden Metric λ-Forms of Lobachevski's Plane.

1) The *golden metric form* of Lobachevski's plane

For the case $\lambda = 1$ we have $\Phi_1 = \frac{1+\sqrt{5}}{2} \approx 1.61803$—the *golden mean*, and hence the form (3.2) is reduced to the following:

$$(ds)^2 = \ln^2\left(\Phi_1\right)(du)^2 + \frac{5}{4}\left[sFs(u)\right]^2(dv)^2 \quad (3.29)$$

where $\ln^2\left(\Phi_1\right) = \ln^2\left(\frac{1+\sqrt{5}}{2}\right) \approx 0.231565$ and

$sFs(u) = \frac{\Phi_1^u - \Phi_1^{-u}}{\sqrt{5}}$ is symmetric hyperbolic Fibonacci sine (see Part I).

Let us name the metric form (3.29) the *golden metric form of Lobachevski's plane.*

2) The *silver metric form* of Lobachevski's plane

For the case $\lambda = 2$ we have $\Phi_2 = 1+\sqrt{2} \approx 2.1421$—The *silver mean,* and hence the form (3.2) is reduced to the following:

$$(ds)^2 = \ln^2\left(\Phi_2\right)(du)^2 + 2\left[sF_2(u)\right]^2(dv)^2, \quad (3.30)$$

where $\ln^2\left(\Phi_2\right) \approx 0.776819$ and $sF_2(u) = \frac{\Phi_2^u - \Phi_2^{-u}}{2\sqrt{2}}$.

Let us name the metric form (3.30) the *silver metric form of Lobachevski's plane.*

3) The bronze metric form of Lobachevski's plane

For the case $\lambda = 3$ we have

$\Phi_3 = \frac{3+\sqrt{13}}{2} \approx 3.30278$—the *bronze mean,* and hence the form (3.2) is reduced to the following:

$$(ds)^2 = \ln^2\left(\Phi_3\right)(du)^2 + \frac{13}{4}\left[sF_3(u)\right]^2(dv)^2 \quad (3.31)$$

where $\ln^2\left(\Phi_3\right) \approx 1.42746$ and $sF_3(u) = \frac{\Phi_3^u - \Phi_3^{-u}}{\sqrt{13}}$.

Let us name the metric form (3.31) the *bronze metric form of Lobachevski's plane.*

4) The *cooper metric form* of Lobachevski's plane

For the case $\lambda = 4$ we have

$\Phi_4 = 2+\sqrt{5} \approx 4.23607$—The *cooper mean,* and hence the form (3.2) is reduced to the following:

$$(ds)^2 = \ln^2\left(\Phi_4\right)(du)^2 + 5\left[sF_4(u)\right]^2(dv)^2, \quad (3.32)$$

where $\ln^2\left(\Phi_4\right) \approx 2.08408$ and $sF_4(u) = \frac{\Phi_4^u - \Phi_4^{-u}}{2\sqrt{5}}$.

Let us name the metric form (3.32) the *cooper metric form of Lobachevski's plane.*

5) The *classical metric form* of Lobachevski's plane in semi-geodesic coordinates. For the case $\lambda = \lambda_e = 2sh(1)$ ≈ 2.350402. we have $\Phi_{\lambda_e} = e \approx 2.7182$ -*Napier number,* and hence the form (3.2) is reduced to the expression (3.1), which gives *classical metric form of Lobachevski's plane* in semi-geodesic coordinates (u, v), where $0 < u < +\infty$, $-\infty < v < +\infty$.

4.3. Geodesic Lines of the Golden Metric λ-Forms of Lobachevski's Plane and Other Geometric Objects.

Geodesic lines and angles between these geodesic lines are one of basic geometric concepts of inner geometry. If metric form is given, then geodesic lines are determined as extremums of functional of curve length.

We proved above (see step 3) that the golden metric λ-forms of Lobachevski's plane of the kind (3.2) coincide with the metric forms of (3.7). For convenience, we introduce new designations for these forms:

$$(ds)^2 = A^2(du)^2 + \left[B(u)\right]^2(dv)^2, \quad (3.33)$$

where

$$A = \ln\left(\Phi_\lambda\right) > 0,\ B(u) = \frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2} > 0,$$
$$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2},\ \lambda > 0,\ 0 < u < +\infty,\ -\infty < v < +\infty \quad (3.34)$$

It is easy to prove (see also the formula (3.12)), that for the conditions (3.34) we have:

$$sh(Au) = B(u) = \frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2} > 0,$$
$$ch(Au) = C(u) = \frac{\Phi_\lambda^u + \Phi_\lambda^{-u}}{2} > 0 \quad (3.35)$$

Let us consider three-dimensional Minkowski's space $L^3 = (X, Y, Z)$ with Minkowski's metrics

$$\left(dl\right)^2 = dX^2 - dY^2 - dZ^2, \quad (3.36)$$

where dl is linear element of the space L^3. Now let us consider the upper half M^2 of the two-sheet hyperboloid:

$$X^2 - Y^2 - Z^2 = 1,\ X > 0. \quad (3.37)$$

The surface M^2 is given in implicit form. We can give the surface M^2, if we fulfill the following special parameterization:

$$\begin{cases} X = X\left(u,v\right) = ch\left(Au\right) = C\left(u\right) \\ Y = Y\left(u,v\right) = sh\left(Au\right)\cos\left(v\right) = B\left(u\right)\cos\left(v\right) \\ Z = Z\left(u,v\right) = sh\left(Au\right)\sin\left(v\right) = B\left(u\right)\sin\left(v\right) \end{cases} \quad (3.38)$$

By direct testing, we can verify that

$$\left[X\left(u,v\right)\right]^2 - \left[Y\left(u,v\right)\right]^2 - \left[Z\left(u,v\right)\right]^2 \equiv 1,\ X\left(u,v\right) > 0. \quad (3.39)$$

Let us substitute (3.38) into the correlation (3.36); then on the upper half M^2 of two-sheet hyperboloid we get the metric form:

$$\left(dl\right)^2 = -\left\{A^2\left(du\right)^2 + \left[B\left(u\right)\right]^2\left(dv\right)^2\right\}, \quad (3.40)$$

where A and $B(u)$ has a form (3.34). From here we get the *golden metric* λ*-form of Lobachevski's plane* of the kind (3.33):

$$\left(ds\right)^2 = -\left(dl\right)^2 = A^2\left(du\right)^2 + \left[B\left(u\right)\right]^2\left(dv\right)^2$$

Let us consider in the space $L^3 = \left(X,Y,Z\right)$ the following planes

$$aX + bY + cZ = 0,\left(a^2 + b^2 + c^2 > 0\right), \quad (3.41)$$

which pass through the coordinate origin $O\left(0,0,0\right)$ and intersect the upper half of the two-sheet hyperboloid (3.37), if the coefficients of the equation (3.41) satisfy to the following restriction:

$$-a^2 + b^2 + c^2 > 0. \quad (3.42)$$

Then the intersection lines of the planes (3.41) with the surface (3.37) are *geodesic lines* on the surface (3.37) in the metrics (3.36) (see [17]). This is analogous to the case, when at the unit sphere $S^2: X^2 + Y^2 + Z^2 = 1$ the intersection lines of the planes $aX + bY + cZ = 0$ (where $a^2 + b^2 + c^2 > 0$) with this sphere are *geodesic lines* in the metrics of constant *Gaussian curvature* $K = 1$.

If we substitute (3.38) into (3.41), then in the golden λ-metrics (3.33) in the coordinates $\left(u,v\right)$ we get the following equation of geodesic lines in the following implicit form:

$$ach\left(Au\right) + bsh\left(Au\right)\cos\left(v\right) + csh\left(Au\right)\sin\left(v\right) = 0, \quad (3.43)$$

where $\lambda > 0, (0 < u < +\infty, -\infty < v < +\infty)$,

$$A = \ln\left(\Phi_\lambda\right) > 0,\ \Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2} > 1.$$

Note that coefficients a, b, c in (3.43) satisfy to the restrictions:

$$a^2 + b^2 + c^2 > 0 \quad , \quad -a^2 + b^2 + c^2 > 0. \quad (3.44)$$

Let us rewrite (3.43) in the form:

$$F\left(u,v\right) = a\frac{\Phi_\lambda^u + \Phi_\lambda^{-u}}{\Phi_\lambda^u - \Phi_\lambda^{-u}} + b\cos\left(v\right) + c\sin\left(v\right) = 0. \quad (3.45)$$

Let $\left(u_0, v_0\right)$ be coordinates of the intersection point of two geodesic lines given the equations:

$$\begin{cases} F_1\left(u,v\right) = a_1\dfrac{\Phi_\lambda^u + \Phi_\lambda^{-u}}{\Phi_\lambda^u - \Phi_\lambda^{-u}} + b_1\cos\left(v\right) + c_1\sin\left(v\right) = 0 \\ F_2\left(u,v\right) = a_2\dfrac{\Phi_\lambda^u + \Phi_\lambda^{-u}}{\Phi_\lambda^u - \Phi_\lambda^{-u}} + b_2\cos\left(v\right) + c_2\sin\left(v\right) = 0 \end{cases}$$

The angle α *of intersection of two geodesic lines* (counted counter-clockwise), according to the formulas of differential geometry, can be found from the correlation:

$$tg\left(\alpha\right) = \frac{\sqrt{EG - F^2}\left(\dfrac{\partial F_1}{du}\dfrac{\partial F_2}{dv} - \dfrac{\partial F_1}{dv}\dfrac{\partial F_2}{du}\right)}{E\dfrac{\partial F_1}{du}\dfrac{\partial F_1}{dv} + F\left(\dfrac{\partial F_1}{du}\dfrac{\partial F_2}{dv} - \dfrac{\partial F_1}{dv}\dfrac{\partial F_2}{du}\right) + G\dfrac{\partial F_2}{du}\dfrac{\partial F_2}{dv}} \quad (3.46)$$

where the right-hand parts in (3.46) are taken in the point $\left(u_0, v_0\right)$ and $E = E\left(u,v\right)$, $F = F\left(u,v\right)$, $G = G\left(u,v\right)$ are coefficients of the metric form

$$\left(ds\right)^2 = E\left(dv\right)^2 + 2Fdudv + G\left(dv\right)^2.$$

Note that the formulas for $\sin\left(\alpha\right)$ and $\cos\left(\alpha\right)$ are written by analogy.

In our situation, according to (3.2), (3.4), (3.33), (3.34), we have:

$$E = \ln^2\left(\Phi_\lambda\right),\ F = 0,$$

$$G = \left(\frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2}\right)^2 = \left(\frac{\sqrt{4+\lambda^2}}{2}sF_\lambda\left(u\right)\right)^2 = sh^2\left[u\ln\left(\Phi_\lambda\right)\right]. \quad (3.47)$$

Partial derivatives on u and v of the function $F\left(u,v\right)$ of the kind (3.45) have the following forms:

$$\frac{\partial F}{\partial u} = a\frac{\ln\left(\Phi_\lambda\right)}{ch^2\left[u\ln\left(\Phi_\lambda\right)\right]},\ \frac{\partial F}{\partial v} = -b\sin\left(v\right) + c\cos\left(v\right) \quad (3.48)$$

Further by analogy on Lobachevski's plane, provided by the golden metric λ-forms (3.2), at any real number $\lambda > 0$ we can find corresponding formulas for distances between two points, transformations of movement, and all other mathematical objects, inherent in this remarkable geometry.

The authors of the present article do not pursue a goal to write out all corresponding formulas and geometric constructions, which are connected with the golden metric λ-forms of Lobachevski's plane because this problem is a subject of separate study.

In this connection, it would be very fruitful to unite in further the developed by the authors *golden metric λ-forms of Lobachevski's plane* of the kind (3.2), realized at any real number λ on the half-plane $\Pi^+ = (u,v)\left(0 < u < +\infty, -\infty < v < +\infty\right)$ and having *Gaussian curvature* $K = -1$, with the well-known studied and convenient for applications classical model of *Lobachevski's plane*, suggested in 1882 by Great French mathematician, physicist, and astronomer Henry Poincare on a disc

$$D^2 : x^2 + y^2 < 1, \tag{3.49}$$

completed by the *absolute* $E : x^2 + y^2 = 1$, which plays a role of a carrier of the infinitely distant points of *Lobachevski's plane*.

4.4. Poincare's Model of Lobachevski's Plane on the Unit Disc

Let us remind the basic facts of Lobachevski's geometry for Poincare's realization on a disc (3.49). Information is taken from [18].

Poincare's metric form of *Gaussian curvature* $K = -1$ has the following form:

$$(ds)^2 = \frac{4\left[(dx)^2 + (dy)^2\right]}{\left(1 - x^2 - y^2\right)^2}. \tag{3.50}$$

Geodesic planes for Poincare's model are or circle arcs, which are orthogonal to *absolute* (if these geodesic lines do not contain the coordinate origin $O(0.0)$) or segments of right lines (if these geodesic lines pass through the coordinate origin).

In general case the geodesic lines equation in Poincare's model has the following form:

$$F(x,y) = a\left(1 + x^2 + y^2\right) - 2bx - 2cy = 0,\ x^2 + y^2 < 1, \tag{3.51}$$

where $a^2 + b^2 + c^2 > 0, -a^2 + b^2 + c^2 > 0.$

The angle α of intersection of two geodesic lines, which is counted counter-clockwise, is determined from the correlations:

$$tg(\alpha) = \frac{\dfrac{\partial F_1}{\partial x}\dfrac{\partial F_2}{\partial y} - \dfrac{\partial F_1}{\partial y}\dfrac{\partial F_2}{\partial x}}{\dfrac{\partial F_1}{\partial x}\dfrac{\partial F_1}{\partial y} + \dfrac{\partial F_2}{\partial x}\dfrac{\partial F_2}{\partial y}} \tag{3.52}$$

where the right-hand parts in (3.52) is taken in the point (x_0, y_0) being a common point of the geodesic lines intersection:

$$\begin{cases} F_1(x,y) = a_1\left(1 + x^2 + y^2\right) - 2b_1x - 2c_1y = 0 \\ F_2(x,y) = a_2\left(1 + x^2 + y^2\right) - 2b_2x - 2c_2y = 0 \end{cases} \tag{3.53}$$

Thus, in the metrics (3.50) the angles are measured in Euclidean sense.

Let $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ be arbitrary points of Lobachevski's plane, which is realized in the form of a circle (3.49) with the metrics (3.50).

We use complex numbers further. We designate the point $A(x,y)$ by $z = x + iy$, where $i = \sqrt{-1}$ is imaginary unit. A module of the complex number z is equal to $|z| = \sqrt{x^2 + y^2}$. Let $\bar{z} = x - iy$ be a complex number conjugate to the complex number $z = x + iy$.

For this case the points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ in complex notation can be represented as follows: $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. It is well-known that a *distance* $\rho(A_1, A_2)$ between two points $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$ in complex notation has the following form:

$$\rho(A_1, A_2) = \ln\left(\frac{1 + \left|\dfrac{z_1 - z_2}{z_1 - \bar{z}_2}\right|}{1 - \left|\dfrac{z_1 - z_2}{z_1 - \bar{z}_2}\right|}\right) \tag{3.54}$$

In complex notation the metrics (3.50) has the following form:

$$(ds)^2 = \frac{4}{\left(1 - |z|^2\right)^2} dz d\bar{z}, |z| < 1. \tag{3.55}$$

The movement of the metrics (3.55) of Lobachevski's plane is written as follows:

$$z' = f(z) = \frac{Az + \bar{B}}{Bz + \bar{A}},\ |A|^2 - |B|^2 = 1, \tag{3.56}$$

where $z = x + iy$ and $z' = x' + iy'$. Note that at the movements (3.56) the distances between points and angles between geodesic lines are kept.

4.5. Connection Between Poincare's Model of Lobachevski's Plane in the Unit Disc and the Golden λ-Models of Lobachevski's Plane

It is proved in [18] that the upper half M^2 of the two-sheet hyperboloid (3.37) assumes a parameterization of the following kind:

$$\begin{aligned} X &= X(x,y) = \frac{2}{1-x^2-y^2} - 1, \\ Y &= Y(x,y) = \frac{2x}{1-x^2-y^2}, \\ Z &= Z(x,y) = \frac{2y}{1-x^2-y^2}, \end{aligned} \quad (3.57)$$

where $x^2 + y^2 < 1$.

Direct calculation shows that that if we substitute (3.57) into the correlation (3.36), then we get a kind of the metrics (3.50) on M^2 in coordinates (x, y), that is, the metrics of *Poincare's model of Lobachevski's plane* in the unit disc $x^2 + y^2 < 1$ of the kind

$$(ds)^2 = -(dl)^2 = \frac{4\left[(dx)^2 + (dy)^2\right]}{\left(1-x^2-y^2\right)^2},$$

where *dl* is a linear element of the kind (3.36).

Thus, the transformations (3.57) result in Poincare's model on the unit disc; we have described its basic properties in Section 3.4 of this Part III.

In order to pass from Poincare's model of Lobachevski's plane in the unit disc $x^2 + y^2 < 1$ to the golden λ-models of Lobachevski's plane in the half-plane $0 < u < +\infty, -\infty < v < +\infty$, we introduce another parameterization of the upper half M^2 of the two-sheet hyperboloid (3.37), connected with the previous parameterization (3.57) by the following correlations:

$$\begin{cases} X = X(x,y) = \dfrac{2}{1-x^2-y^2} - 1 = ch(Au) \\ Y = Y(x,y) = \dfrac{2x}{1-x^2-y^2} = sh(Au)\cos(v), \\ Z = Z(x,y) = \dfrac{2x}{1-x^2-y^2} = sh(Au)\sin(v) \end{cases} \quad (3.58)$$

where

$$A = \ln(\Phi_\lambda) > 0, \Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2}, \lambda > 0 \quad (3.59)$$

We have proved in Section 3.3 that if we consider the upper half of M^2 of the two-sheet hyperboloid (3.37) in the form (3.38):

$$X = ch(Au), Y = sh(Au)\cos(v), Z = sh(Au)\sin(v),$$

then we directly came to the golden λ-forms of Lobachevski's plane of the kind (3.2) or, in another notation, of the kind (3.33):

$$(ds)^2 = A^2(du)^2 + \left[B(u)\right]^2(dv)^2,$$

where

$$B(u) = \frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2} > 0, 0 < u < +\infty, -\infty < v < +\infty. \quad (3.60)$$

According to the formulas (3.35), at the condition $0 < u < +\infty$ the following correlations are valid:

$$sh(Au) = \frac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2} > 0, \; ch(Au) = \frac{\Phi_\lambda^u + \Phi_\lambda^{-u}}{2} > 0. \quad (3.61)$$

Then, from the correlations (3.58) and (3.61) we get directly the following connection between parameters (x, y) and (u, v):

$$\begin{cases} x = x(u,v) = \dfrac{sh(Au)}{1+ch(Au)}\cos(v) = \dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2+\Phi_\lambda^u + \Phi_\lambda^{-u}}\cos(v) \\ y = y(u,v) = \dfrac{sh(Au)}{1+ch(Au)}\sin(v) = \dfrac{\Phi_\lambda^u - \Phi_\lambda^{-u}}{2+\Phi_\lambda^u + \Phi_\lambda^{-u}}\sin(v) \end{cases} \quad (3.62)$$

Note that at any $\lambda > 0, 0 < u < +\infty$ the transformations (3.62) are diffeomorphisms, because their jacobian is not equal to 0, and they establish connection between the golden λ-models of Lobachevski's plane in the coordinates $0 < u < +\infty, -\infty < v < +\infty$ and the classical Poincare's model of Lobachevski's plane in the unit disc $x^2 + y^2 < 1$.

Most in all, the transformations (3.62) establish an isometry between Poincare's metric form (3.50) and the golden metric λ-forms of the kind (3.2) or, in another notation, of the kind (3.33).

By using the transformations (3.62) and the formulas (3.54) and (3.56), for the golden λ-models of Lobachevski's plane in the coordinates $0 < u < +\infty$, $-\infty < v < +\infty$, for every real $\lambda > 0$ we can got the formula for the *distance* between two arbitrary points (u_1, v_1) and (u_2, v_2) and the formula of *metrics movement* (3.2), that is, for the metrics (3.33).

Thus, the main result of Part III of the article is an unexpected variant of Hilbert's Fourth Problem solution based on the the "metallic means" [4], which is a generalization of the "golden mean" (Theorem II.11 of Euclid's *Elements*).

5. General Conclusion to the Article

5.1. Euclid's Fifth Postulate, Hyperbolic Geometry, and Hilbert's Fourth Problem

A study of *Euclid's Fifth Postulate* led in 19th century to *Lobachevski's geometry*, which can refer to one of the greatest mathematical discoveries of 19th century. *Lobachevski's hyperbolic geometry* can be considered as a break-through of hyperbolic ideas into mathematics and theoretical physics. Interest in hyperbolic geometry, which was studied during 19th century by many outstanding mathematicians Bolyai, Gauss, Beltrami, Klein, Poincare, Riemann, increased in the end of 19th century so much that Great mathematician David Hilbert in his famous lecture *Mathematical Problems* (Second International Congress of Mathematicians, Paris, 1900) [11] was forced to include problems of *hyperbolic geometry* to the list of the most important 23 Mathematical Problems.

We are talking about *Hilbert's Fourth Problem*, which sounds: "*Whether is possible from the other fruitful point of view to construct geometries, which with the same right can be considered the nearest geometries to the traditional Euclidean geometry.*"

During 20th century many attempts to solve this problem were undertaken. Finally, mathematicians came to conclusion that *Hilbert's Fourth Problem* was formulated very vague what makes difficult its final solution [12, 13].

5.2. Euclid's Golden Section and the Mathematics of Harmony

A problem of the *Golden Section* was formulated by Euclid as a problem of *division of line segments in extreme and mean ratio* (Theorem II.11). This problem was introduced by Euclid with the purpose to create a full geometric theory of *Platonic Solids* (the Book XIII), expressed in Plato's cosmology the harmony of Universe.

During two last centuries the interest in the Golden Section and Platonic Solids increased rapidly what led to many scientific discoveries (quasi-crystals, fullerenes, "golden" genomatrices and so on). Besides, the development of this direction led to the creation of the **Mathematics of Harmony**-a new interdisciplinary theory and the "golden" paradigm of modern science [1].

5.3. Hyperbolic Fibonacci and Lucas Functions and Bodnar's Geometry

Hyperbolic Fibonacci and Lucas functions based on Euclid's Golden Section united together two great problems formulated by Euclid—*Euclid's Fifth Postulate*, which led to *Lobachevski's hyperbolic geometry,* and *Euclid's Golden Section problem* (Teorem II.11), which led to the "*Mathematics of Harmony*" [1]. The hyperbolic Fibonacci and Lucas functions [2,15] led to *Bodnar's geometry* [16], which discovered for us a new "hyperbolic world"-*the world of phyllotaxis.*

5.4. The "Golden" Fibonacci Goniometry and Hilbert's Fourth Problem

The "golden" Fibonacci goniometry based on Spinadel's *metallic means,* which are a generalization of the classical Euclid's *Golden Section,* led to obtaining an original solution of *Hilbert's Fourth Problem,* which considered until now as formulated very vague, what makes difficult its final solution [12,13].

The "golden" Fibonacci goniometry [3] generates a theoretically infinite number of new hyperbolic functions, in particular, *hyperbolic Fibonacci and Lucas functions* [2,15], which underlie *Bodnar's geometry* [16]. The "golden" Fibonacci goniometry extends considerably the sphere of *hyperbolic researches* and attracts an attention of theoretical natural sciences to the question of a search of *new hyperbolic worlds of Nature,* based on the *hyperbolic Fibonacci and Lucas λ-functions* ($\lambda > 0$ is a given real number) [3].

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