Applied Mathematics, 2010, 1, 416-424
doi:10.4236/am.2010.15055 Published Online November 2010 (http://www.SciRP.org/journal/am)

Groups Having Elements Conjugate to Their Squares and Connections with Dynamical Systems

Geoffrey R. Goodson
Department of Mathematics, Towson University, Towson, USA
E-mail: *ggoodson@towson.edu*

Received July 15, 2010; *revised September* 23, 2010; *accepted September* 26, 2010

Abstract

In recent years, dynamical systems which are conjugate to their squares have been studied in ergodic theory. In this paper we study the consequences of groups having elements which are conjugate to their squares and consider examples arising from topological dynamics and more general dynamical systems.

Keywords: Conjugacy to Square, Homeomorphism on a Compact Space

1. Introduction

Our intention is to introduce the reader to a number of topics in dynamical systems related by the group theoretic notion of conjugacy of an element to its square. In recent years there has been some interest in dynamical systems that are conjugate to their composition squares (see [1,2] and [3]). Although these ideas appeared in an ergodic theory setting, they can just as well be considered in the areas of topological and chaotic dynamics, and this is our point of view. This work arose during the teaching of a Senior Seminar (capstone) course given on chaotic dynamics. Some of the topics were introduced with the aim of having students give presentations in related areas using articles from various journals. The intention was to motivate students to learn more about chaotic and topological dynamics, ergodic theory and other aspects of dynamical systems (see [4] and [5]). We start in Section 2 with some group theoretic consequences of having elements conjugate to their square. The rest of the paper is concerned with examples. These include homeomorphisms of the interval $[0,1]$, the circle S^1, and the 3-adic adding machine. We also take a look at transitive homeomorphisms having topological discrete spectrum and which are conjugate to their squares.

2. General Groups with Elements Conjugate to Their Squares

Let G be a group and $a \in G$. We consider the consequences of a having the property that it is conjugate to its square, *i.e.*, there exists $g \in G$ such that

$$ga = a^2 g.$$

Denote by $C(a)$ the *centralizer* of a:

$$C(a) = \{g \in G : ag = ga\},$$

and set

$$S(a) = \{g \in G : ga = a^2 g\}.$$

If $g \in S(a)$, we see that $ga^n = a^{2n} g$ for $n \in \mathbb{Z}$ and $g^n a = a^{2^n} g^n$ for all $n \in \mathbb{Z}^+$. It follows that if G is abelian, $a = e$, the identity of G.

Fix $k \in S(a)$ and define a map $\Phi : C(a) \to C(a^2)$ by $\Phi(s) = ksk^{-1}$, for $s \in C(a)$. We can see that Φ is well defined since

$$\Phi(s)a^2 = ksk^{-1}a^2 = ksak^{-1} = kask^{-1} = a^2 ksk^{-1} = a^2\Phi(s),$$

so that $\Phi(s) \in C(a^2)$. Clearly Φ is one--to--one, and it is a homomorphism since

$$\Phi(ts) = ktsk^{-1} = ktk^{-1}ksk^{-1} = \Phi(t)\Phi(s).$$

Φ is onto, for if $r \in C(a^2)$, set $s = k^{-1}rk$, then $\Phi(s) = r$ and $s \in C(a)$, which shows that Φ is a group isomorphism.

Since $C(a) \subseteq C(a^2)$, there is an $s \in C(a)$ with $\Phi(s) = a$, *i.e.*, $ksk^{-1} = a$, or $s = k^{-1}ak$, so that

$$s^2 = k^{-1}akk^{-1}ak = k^{-1}ka = a.$$

This shows that s is a square root of a which is conjugate to a. We summarize this, and give some additional properties of Φ as follows:

Proposition 1 *Suppose that* $a \in G$ *satisfies* $ka = a^2 k$ *for some* $k \in G$. *Define a map*

Copyright © 2010 SciRes.

$$\Phi : C(a) \to C(a^2) \quad by \quad \Phi(s) = ksk^{-1}, \quad s \in C(a).$$

Then

(a) Φ *is a well defined group isomorphism.*

(b) $a = k^{-1}ak$ *is a square root of* a *in* $C(a)$.

(c) $C(a) = C(a^2)$ *if and only if there is exactly one square root of* a *in* $C(a)$*, which is conjugate to* a.

(d) *If* $C(a)$ *is abelian, then* Φ *is independent of the conjugation* k*,* $C(a) = C(a^2)$*, so* Φ *is a group automorphism.*

(e) *If* $C(a) = \{a^n : n \in \mathbb{Z}\}$*, then* $a^m = e$ *for some* $m > 1$ *odd.*

Proof. (c) First assume $C(a) = C(a^2)$ and $s_1 = k_1^{-1}ak_1$ and $s_2 = k_2^{-1}ak_2$ are two square roots of a conjugate to a. Then $a = k_1^{-1}a^2k_1 = k_2^{-1}a^2k_2$. It follows that $k_2k_1^{-1}a^2 = a^2k_2k_1^{-1}$ and $k_2k_1^{-1} \in C(a^2) = C(a)$. In particular

$$k_2k_1^{-1}a = ak_2k_1^{-1} \Rightarrow k_1^{-1}ak_1 = k_2^{-1}ak_2 \Rightarrow s_1 = s_2$$

(a may have other square roots either in $C(a)$ or not, which are not conjugate to a).

Conversely suppose that $b \in C(a^2) \setminus C(a)$. We have $ka = a^2k$, so set $k' = bk$, then

$$k'a = bka = ba^2k = a^2bk = a^2k',$$

since $b \in C(a^2)$. Therefore $(k')^{-1}ak'$ is a square root of a conjugate to a. If $(k')^{-1}ak' = k^{-1}ak$, then $ak'k^{-1} = k'k^{-1}a$ or $ab = ba$ so $b \in C(a)$, a contradiction. It follows that $k^{-1}ak$ and $(k')^{-1}ak'$ are distinct square roots and the result follows.

(d) If $k_ia = a^2k_i$, $i = 1,2$, we see that $k_2^{-1}k_1 \in C(a)$, and since $C(a)$ is abelian

$$\left(k_2^{-1}k_1\right)s = s\left(k_2^{-1}k_1\right) \text{ for all } s \in C(a),$$

or $k_1sk_1^{-1} = k_2sk_2^{-1}$, which says that Φ is independent of the conjugation k.

Write $s = a^{\frac{1}{2}}$, then for $r \in C(a)$

$$\Phi(r)a = krk^{-1}a = kra^{\frac{1}{2}}k^{-1} = ka^{\frac{1}{2}}ra^{-1} = akrk^{-1} = a\Phi(r),$$

so that $\Phi(r) \in C(a)$ and since $C(a) \subseteq C(a^2)$, it follows that $C(a) = C(a^2)$. In particular, Φ is a group automorphism.

(e) Note that $k^{-1}ak \in C(a)$, so that $k^{-1}ak = a^n$ for some $n \in \mathbb{Z}$, and this implies $a^{2n-1} = e$. (The same proof can be used to show that if $C(a)$ is an infinite, finitely generated abelian group, then a cannot be conjugate to its square).

Remark. If G is a topological group, then the homomorphism $\Phi : C(a) \to C(a^2)$ is continuous.

Examples. 1. Set $G = \{\langle a,k \rangle : ka = a^2k\}$, a countably infinite, finitely generated non-abelian group.

Let $Q_2 =$ the subgroup of G generated by all conjugations gag^{-1}, $g \in G$. Q_2 is the *conjugacy class* of a, and every member of Q_2 is conjugate to its square. Elements of G can be written as products of the form $k^m a^q k^n$, $m,n,q \in \mathbb{Z}$, so that $gag^{-1} = (k^m a^q k^n)a(k^{-n}a^{-q}k^{-m})$ will be of the form $k^r a^p k^{-r}$, a member of $C(a)$ (consider the cases where $n \geq 0$ and $n < 0$ separately).

Q_2 is abelian for suppose that $q \geq m$ and consider

$$(k^m a^p k^{-m})(k^q a^r k^{-q})$$
$$= k^m a^p k^{q-m} a^r k^{-q}$$
$$= k^m a^p a^{r2^{q-m}} k^{q-m}k^{-q} = k^m a^{p+r2^{q-m}} k^{-m},$$

and

$$(k^q a^r k^{-q})(k^m a^p k^{-m})$$
$$= k^q a^r k^{m-q} a^p k^{-m}$$
$$= k^q k^{m-q} a^{r2^{q-m}} a^p k^{-m} = k^m a^{p+r2^{q-m}} k^{-m}.$$

In addition, Q_2 has the following properties (based on [1,6]):

Proposition 2 *Let* Q_2 *be the subgroup of* G *generated by the conjugations* gag^{-1}*,* $g \in G$.

(a) Q_2 *is an abelian subgroup of* $C(a)$*, consisting of elements of the form* $k^m a^n k^{-m}$*,* $m,n \in \mathbb{Z}$.

(b) *The map* $\phi : Q_2 \to \mathbb{R}$ *defined by* $\phi(k^m a^n k^{-m}) = n2^m$ *is an isomorphism onto the subgroup* H *of* $\mathbb{R}$*,* $H = \{n2^m : n,m \in \mathbb{Z}\}$ *of dyadic rationals.*

(c) *The commutator subgroup of* G *is* Q_2 *and* $G/Q_2 \cong \mathbb{Z}$.

(d) $G \cong H \times_2 \mathbb{Z}$ *where* $H \times_2 \mathbb{Z}$ *denotes the semi-direct product of* H *and* $\mathbb{Z}$ *where the multiplication is given by*

$$(h,n)(r,m) = (h + 2^n r, n+m), \quad h,r \in H, \; n,m \in \mathbb{Z}.$$

Proof. (b) ϕ is well defined for if $(k^m a^p k^{-m}) = (k^q a^r k^{-q})$, $q \geq m$, then $a^p = k^{q-m} a^r k^{m-q} = a^{r2^{q-m}} k^{q-m} k^{-q+m} = a^{r2^{q-m}}$, so that $p = r$ and $m = q$.

H is a subgroup of $\mathbb{R}$ with respect to addition and $\phi : Q_2 \to H$ is a homomorphism since if $q \geq m$, then

$$\varphi((k^m a^p k^{-m})(k^q a^r k^{-q}))$$
$$= \varphi(k^m a^{p+r2^{q-m}} k^{-m})$$
$$= p2^m + r2^q = \varphi(k^m a^p k^{-m}) + \varphi(k^q a^r k^{-q}).$$

Clearly ϕ is one-to-one and onto H.

(c) If $g \in k^iQ_2 \cap k^jQ_2$, then $k^{i-j} \in Q_2 \Rightarrow i = j$. In addition, if $g \in G$ then g is of the form

$g = k^m a^q k^n = k^{m+n}(k^{-n}a^q k^n) \in k^{m+n}Q_2$. This shows that $G = \cup_{i\in\mathbb{Z}} k^i Q_2$ as a disjoint union. In addition, If $g_1, g_2 \in G$ we can write $g_1 = k^i q_1$, $g_2 = k^j q_2$ for some $i, j \in \mathbb{Z}$ and $q_1, q_2 \in Q_2$, and we can check that $g_1 g_2 \in k^{i+j} Q_2$. It follows that $G/Q_2 \cong \mathbb{Z}$.

Let G' be the commutator subgroup of G. If $g_1, g_2 \in G$, then we can write $g_1 = k^i q_1$, $g_2 = k^j q_2$ for some $i, j \in \mathbb{Z}$ and $q_1, q_2 \in Q_2$. Then $g_1 g_2 = k^{i+j} g$ and in a similar way $g_1^{-1} g_2^{-1} = k^{-i-j} h$, for some $g, h \in Q_2$, so that $g_1 g_2 g_1^{-1} g_2^{-1} \in Q_2$, and $G' \subseteq Q_2$. Also $Q_2 \subseteq G'$ since the commutator

$$[k, ak^{-1}] = kak^{-1}k^{-1}ka^{-1}$$
$$= kak^{-1}a^{-1} = a^2 kk^{-1}a^{-1} = a^2 a^{-1} = a,$$

which gives

$$gag^{-1} = g[k, ak^{-1}]g^{-1} = [gkg^{-1}, (gag^{-1})(gk^{-1}g^{-1})] \in G'$$

(d) Set $G_1 = H \times_2 \mathbb{Z}$ and define $\psi : G_1 \to G$ by $\psi(h, n) = a^h k^n$ where we interpret $a^{1/2} = k^{-1}ak$ and similarly for other fractional powers. We see that ψ is well defined and onto. It is easy to check that ψ is one-to-one, and

$$\psi[(h,n)(r,m)] = \psi(h + 2^n r, n+m) = a^{h+2^n r} k^{n+m},$$

$$\psi(h,n)\psi(r,m) = a^h k^n a^r k^m = a^h a^{2^n r} k^n k^m = a^{h+2^n r} k^{n+m},$$

so ψ is an isomorphism.

2 If we set $G_n = \{\langle a,k\rangle : ka = a^2 k, k^n = e\}$, then necessarily $a^m = e$ for $m = 2^n - 1$, so G_n will be finite, with order a multiple of m. The group $H_n = \{\langle a,k\rangle : ka = a^2 k, a^n = e\}$, for some n odd, is not finite.

3 Let us consider an example related to the horocycle flow in ergodic theory (see [7]). Set

$G = SL(2,\mathbb{R}) = 2\times 2$ matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$,

and $g_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in G$, then for $t \neq 0$, the centralizer of g_t is the abelian group:

$$C(g_t) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 = 1 \right\}.$$

In addition, we can check that g_t is conjugate to its square $g_t^2 = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}$, via elements of G of the form $K = \begin{pmatrix} c & d \\ 0 & c/2 \end{pmatrix}$, where $c^2 = 2$.

The automorphism $\Phi : C(g_t) \to C(g_t)$ is defined by

$$\Phi \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 2b \\ 0 & a \end{pmatrix},$$

which is of course independent of the conjugation K (because $C(g_t)$ is abelian).

This type of example can be generalized to $\mathrm{SL}(n,\mathbb{R})$,

e.g., when $n = 3$, in place of g_t, taking $\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$, again conjugate to its square with abelian centralizer.

Remark. Actions of the groups in Examples (1) and (2) above have importance in measurable dynamics (see [1] and [6] where the existence of weakly mixing rank-one transformations conjugate to their square is demonstrated. This answered an open question in ergodic theory, see [2,3]).

3. Groups of Homeomorphisms

Denote by $\mathcal{H}[0,1]$ the set of homeomorphisms of the unit interval $[0,1]$, a group when given the operation of composition of functions. There are two possibilities for $f \in \mathcal{H}[0,1]$: either f is *orientation preserving* (then f is continuous, increasing, $f(0) = 0$ and $f(1) = 1$, possibly with other fixed points, but no period 2-points or points of greater period), or f is *orientation reversing* (so f is continuous, decreasing, $f(0) = 1$, $f(1) = 0$ with a unique fixed point and no additional period 2-points or points of a greater period). We shall show that for $f \in \mathcal{H}[0,1]$ orientation preserving, f is conjugate to its square $f^2(x) = f(f(x))$, and we shall study properties of the conjugating map. The result below shows that any orientation preserving homeomorphism of $[0,1]$ has a square root (in fact infinitely many). We illustrate a standard method of showing conjugacy using the notion of *fundamental domain* (see [8,9] and [10] for related results).

Proposition 3 *Let $f \in \mathcal{H}[0,1]$ be orientation preserving, with fixed points* $0 = \boldsymbol{c}_1 < \boldsymbol{c}_2 < \cdots < \boldsymbol{c}_{n-1} < \boldsymbol{c}_n = 1$. *Then*

(a) *f is conjugate to its square f^2, i.e., there is a map $\phi \in \mathcal{H}[0,1]$ satisfying $\phi \circ f = f^2 \circ \phi$.*

(*b*) *If $\phi \in \mathcal{H}[0,1]$ with $\phi \circ f = f^2 \circ \phi$, then either $\phi(c_i) = c_i$, $\boldsymbol{i} = 1, \cdots, \boldsymbol{n}$, or $n = 2$ and $\varphi(0) = 1$, $\varphi(1) = 0$.*

(*c*) *The conjugation ϕ in (b) must have fixed points d_i with $c_i < d_i < c_{i+1}$ for $\boldsymbol{i} = 1, 2, \cdots, \boldsymbol{n} - 1$.*

(*d*) *The centralizers $C(f)$ and $C(f^2)$ are not equal, and f has infinitely many distinct square roots ($g \in \mathcal{H}[0,1]$ with $g^2 = f$), each conjugate to f.*

Proof. (a) To keep the proof simple, we prove this for

Copyright © 2010 SciRes.

the case where there are only two fixed points, 0 and 1. The general case is similar.

Either $f(x) > x$ for all $0 < x < 1$ or $f(x) < x$ for all $0 < x < 1$ (since otherwise f will have additional fixed points). Suppose the latter holds (the former is treated similarly). In this case, if $x \in (0,1)$ then $\lim_{n\to\infty} f^n(x) = 0$, so $x = 0$ is an attracting fixed point and $\lim_{n\to\infty} f^{-n}(x) = 1$, or $x = 1$ is a repelling fixed point.

Fix $a,b \in (0,1)$ and define a map ϕ linearly on the interval $I = (f(a), a]$ onto the interval $J = (f^2(b), b]$ (so that $\phi(a) = b$, $\phi(f(a)) = f^2(b)$). These are called *fundamental domains* for f and f^2. Roughly speaking, if we know the values $\phi(x)$ takes on I, then the equation $\phi(f(x)) = f^2(\phi(x))$ determines the values of $\phi(x)$ on the rest of $[0,1]$.

Extend ϕ to all of

$$(0,1) = \cup_{n=-\infty}^{\infty} f^n(I)$$

(a disjoint union since $f(x) < x$ and f is strictly increasing and continuous) as follows: Given any $x \in (0,1)$, there is exactly one $n \in \mathbb{Z}$ with

$$x \in f^n(I), \quad \text{so that} \quad f^{-n}(x) \in I,$$

and we set $\varphi(x) = f^{2n}(\varphi(f^{-n}(x))).$

We can check that $\phi(f^n(a)) = f^{2n}(b)$ for all $n \in \mathbb{Z}$ and that ϕ is well defined. If we let $\phi(0) = 0$, $\phi(1) = 1$, then ϕ is a homeomorphism with the property that $\phi \circ f(x) = f^2 \circ \phi(x)$ for all $x \in [0,1]$.

(b) For $\boldsymbol{i} = 1,\cdots,\boldsymbol{n}$, $\phi(f(c_i)) = f^2(\phi(c_i))$, so that $f^2(\phi(c_i)) = \phi(c_i)$, but f has no period 2-points, so $f(\phi(c_i)) = \phi(c_i)$.

It follows that $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \cdots, \boldsymbol{c}_{\boldsymbol{n}}\} = \{\varphi(\boldsymbol{c}_1), \varphi(\boldsymbol{c}_2), \cdots, \varphi(\boldsymbol{c}_{\boldsymbol{n}})\}$, so that $\phi(c_i) = c_j$ for some $1 \le j \le n$.

Since $\phi \in \mathcal{H}[0,1]$, ϕ cannot have periodic points that are non-fixed in $(0,1)$, we must have $\phi(c_i) = c_i$ for $1 < i < n$. If in addition $\phi(0) = 0, \phi(1) = 1$, there is nothing more to prove, so suppose that $\phi(0) = 1, \phi(1) = 0$. In this case there is a unique fixed point, so either $n = 2$ or $n = 3$. It follows from (c) (below) that $n = 3$ cannot happen as this would imply the existence of other fixed points. This concludes the proof of (b).

(c) Again we treat the case where f has only two fixed points as the general case is similar. Let $f, \phi \in \mathcal{H}[0,1]$ satisfy $f(0) = 0$, $f(1) = 1$ and $\phi(f(x)) = f^2(\phi(x))$, $f(x) < x$ for $x \in [0,1]$. Then for any $0 < x < 1$, $f^n(x) \to 0$ as $n \to \infty$.

Suppose that there exists $0 < a < b < 1$ with $\phi(a) = b$ (if not there must exist $0 < b < a < 1$ with $\phi(a) = b$ and we proceed in an analogous way). Set $g(x) = \phi(x) - x$, then $g(a) = \phi(a) - a = b - a > 0$.

Now $\phi(f(a)) = f^2(\phi(a)) = f^2(b)$, and generally $\phi(f^n(a)) = f^{2n}(b)$. We can find $k \in \mathbb{Z}^+$ with $f^k(b) < a$, so that $f^{2k}(b) = f^k(f^k(b)) < f^k(a)$. Consequently,

$$g(f^k(a)) = \phi(f^k(a)) - f^k(a) = f^{2k}(b) - f^k(a) < 0.$$

Applying the Intermediate Value Theorem, we see that there exists $x^* \in [f^k(a), a]$ with $g(x^*) = 0$, or $\phi(x^*) = x^*$.

(d) Part (a) shows that f has a square root $\phi^{-1} f \phi$. An argument similar to that in (a) can be used to show that every square root of f is conjugate to f. Suppose now that there exists $h \in C(f^2) \setminus C(f)$. Set $\psi = h \cdot \phi$, then by Proposition 1(d) we see that $\phi^{-1} f \phi$ and $\psi^{-1} f \psi$ are distinct square roots. It therefore suffices to show that $C(f^2) \setminus C(f)$ is an infinite set.

The method of part (a) can be used to construct $h \in C(f^2)$ with the property that $h(a) = b$ where a and b are chosen arbitrarily in (0,1) (again, for simplicity, assume that the only fixed points f has are 0 and 1 and that $f(x) < x$ for all $x \in (0,1)$). Set $h(f^2(a)) = f^2(b)$ and extend $h: I \to J$ by linearity, where $I = (f^2(a), a]$, $J = (f^2(b), b]$ and continue the definition of h as in part (a). Because of the way h is defined on I we cannot have $h(f(a)) = f(h(a)) = f(b)$ (unless $a = b$), so that $h \notin C(f)$. By varying $a, b \in (0,1)$ we can define infinitely many distinct $h \in C(f^2) \setminus C(f)$, thus giving infinitely many distinct square roots of f.

Our hypotheses exclude $f(x) = x$, $x \in [0,1]$ from consideration, but the identity map has many square roots in $\mathcal{H}[0,1]$. These are the *involutions* of $\mathcal{H}[0,1]$ and are easily constructed. Kuczma [10] has given a general treatment of the square roots of members of $\mathcal{H}[0,1]$. We have used different methods with an aim of keeping the technicalities to a minimum. For example, if $f(x) = x^2$ for $x \in [0,1]$, then f has infinitely many square roots in $\mathcal{H}[0,1]$ (besides $g(x) = x^{\sqrt{2}}$).

Clearly orientation reversing homeomorphisms of $[0,1]$ cannot be conjugate to their square. This is also seen from the following proposition. Denote by $\mathrm{Fix}(T)$ the set of fixed points of a homeomorphism T.

Proposition 4 *Let S and T be homeomorphisms of a compact metric space X. If $ST = T^2 S$ and T has finitely many fixed points, then*

$$Fix(T) = Fix(T^2) = S(Fix(T)).$$

In particular, T has no period 2-points.

Proof. Let $A = Fix(T)$, $B = Fix(T^2)$, and let $\# A$ denote the number of elements of A. Clearly $A \subseteq B$.

If $q \in B$ then

$$S^{-1}T^2(q) = TS^{-1}(q) \Rightarrow S^{-1}(q) \in A \Rightarrow q \in SA$$

so $A \subseteq B \subseteq SA$. Now A is a finite set, so that B is also a finite set and we must have $A = B = SA$, since S is a homeomorphism.

4. Groups with the Weak Closure Property

Suppose now that G is a complete metric topological group. We say that $a \in G$ has the *weak closure property* if the closure of $\{a^n : n \in \mathbb{Z}\}$ in G is equal to $C(a)$.

In this case there is a dichotomy: the centralizer $C(a)$ of a is either trivial ($C(a) = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$), or there is a sub--sequence k_n of integers such that $a^{k_n} \to e$ as $n \to \infty$, and the centralizer $C(a)$ is uncountable (see [11]).

Now if $C(a)$ is trivial, with a^2 conjugate to a, then we have seen that a must be of finite order, so we shall assume throughout that $C(a)$ is uncountable.

Let $\mathrm{WC}(a)$ denote the closure of the powers a^n, $n \in \mathbb{Z}$. It is clear that if a has the weak closure property, then $C(a)$ is an abelian group. As before $\Phi : C(a) \to C(a^2)$ is the isomorphism of Proposition 1.

Proposition 5 *Suppose that* a *has the weak closure property, and* $k \in G$ *satisfies* $ka = a^2k$ ($a^n \neq e$ *for all* $n \neq 0$). *Then*

(a) Φ *can be represented as an automorphism* $\Phi : C(a) \to C(a)$, *defined by* $\Phi(s) = s^2$.

(b) *Every member of* $C(a)$ *is conjugate to its square via* k *and has a unique square root in* $C(a)$.

(c) $C(a)$ *does not contain elements of even order.*

Proof. (a) If $s \in WC(a)$, then $s = \lim_{i\to\infty} a^{n_i}$, for some sequence n_i, so

$$\Phi(s) = \lim_{i\to\infty} \Phi(a^{n_i}) = \lim_{i\to\infty} a^{2n_i} = s^2,$$

therefore $\Phi(s) \in WC(a^2)$. We have shown that

$$\Phi(WC(a)) \subseteq WC(a^2).$$

Now it is clear that $WC(a^2) \subseteq WC(a)$ and $C(a) \subseteq C(a^2)$. Furthermore, we are assuming that a has the weak closure property, so $WC(a) = C(a)$. Hence

$$WC(a^2) \subseteq WC(a)$$
$$= C(a) \subseteq C(a^2) = \Phi(C(a)) = \Phi(WC(a)) \subseteq WC(a^2),$$

so we must have equality throughout. It follows that a^2 has the weak closure property and $C(a^2) = C(a)$. Furthermore, $\Phi(s) = ksk^{-1} = s^2$ defines an isomorphism from $C(a)$ to $C(a^2)$, which may be regarded as an automorphism of the group $C(a)$.

(b) If $s \in C(a)$, then $\Phi(s) = ksk^{-1} = s^2$, so s is conjugate to its square. Also, there exists a unique $r \in C(a)$, such that $\Phi(r) = r^2 = s$, *i.e.*, every member of $C(a)$ has a unique square root in $C(a)$ which is conjugate to its square.

(c) Let $r \in C(a) \setminus \{e\}$ be of finite even order, $r^{2m} = e$ say, then $kr^m k^{-1} = (r^2)^m = e \Rightarrow r^m = e$, a contradiction.

Most of our results are equally valid for groups with elements a conjugate to a^3 or a^4 etc. The following proposition is an exception to this claim:

Proposition 6 *If* a *has the weak closure property, then all conjugations between* a *and* a^2 *are conjugate.*

Proof. Suppose that s_1 and s_2 are two conjugations between a and a^2, then $s_2^{-1}s_1 \in C(a)$. Then since a has the weak closure property

$$s_2^{-1}s_1 = \lim_{i\to\infty} a^{n_i}$$

for some subsequence n_i. We have $s_1 a^{n_i} = a^{2n_i} s_1$, for all $i \geq 1$, so we deduce, on taking limits, that

$$s_1(s_2^{-1}s_1) = (s_2^{-1}s_1)^2 s_1,$$

or $s_2(s_1 s_2^{-1}) = (s_1 s_2^{-1}) s_1$. This says that s_1 and s_2 are conjugate ($rs_1 = s_2 r$), via the conjugation $r = s_1 s_2^{-1}$.

5. The Topological Discrete Spectrum Theorem

The main examples having the weak closure property that we consider are those with *(topological) discrete spectrum*. Let $T : X \to X$ be a homeomorphism of a compact metric space X. Denote by $C(X)$ the Banach space of complex valued continuous functions $f : X \to \mathbb{C}$ with the supremum norm. A non-zero function $f \in C(X)$ is an *eigenfunction* of T with corresponding *eigenvalue* $\lambda \in \mathbb{C}$ if $f(Tx) = \lambda f(x)$ for $x \in X$.

Suppose now that T is a *transitive map*: there is a point $x_0 \in X$ whose orbit $O(x_0) = \{T^n x_0 : n \in \mathbb{Z}\}$ is dense in X. If there is $g \in C(X)$ with $g(Tx) = g(x)$ for all $x \in X$ (an invariant function), then g must be constant, because it is a continuous function constant on the orbit of x_0 (a dense set).

Let $f \in C(X)$ with $f(Tx) = \lambda f(x)$ for all $x \in X$. Then

$$|f(Tx)| = |\lambda| |f(x)| \Rightarrow \sup_{x\in X} |f(Tx)|$$
$$= |\lambda| \sup_{x\in X} |f(x)| \Rightarrow \sup_{x\in X} |f(x)| = |\lambda| \sup_{x\in X} |f(x)|,$$

so $|\lambda| = 1$ and $|f(x)| =$ constant on X.

It can also be seen that the eigenspace corresponding to any one eigenvalue is one-dimensional, and any finite set of eigenfunctions having distinct eigenvalues is a

Copyright © 2010 SciRes.

linearly independent set. Furthermore, the set of all eigenvalues of T is a countable subgroup of the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ (see Walters [6]).

A homeomorphism $T : X \to X$ on a compact metric space X is said to have *(topological) discrete spectrum* if the eigenfunctions of T span $C(X)$ (*i.e.*, the smallest closed linear subspace containing the eigen-functions is $C(X)$). One of the main problems of topological dynamics is the question of when two dynamical systems (homeomorphisms on compact metric spaces) are conjugate. This is answered quite satisfactorily in the case of maps having discrete spectrum:

The *Discrete Spectrum Theorem* of Halmos and von Neumann says:

Let $S, T : X \to X$ be transitive homeomorphisms defined on a compact metric space X and having discrete spectrum, then S and T are conjugate if and only if they have the same eigenvalue group Λ.

In addition, a transitive homeomorphism T having discrete spectrum is conjugate to a rotation $R : G \to G$, $R(g) = ag, g \in G$ on some compact abelian group G ($a \in G$ fixed with the property that $\{a^n : n \in \mathbb{Z}\}$ is dense in G): see [5,12].

If G is a compact abelian group and $\mathcal{H}(G) =$ the set of self-homeomorphisms of G, define $T \in \mathcal{H}(G)$ by $T(g) = bg$ for some $b \in G$, a rotation on G. If we set $S(g) = g^2$ then we see that $ST = T^2 S$ since

$$ST(g) = S(bg) = (bg)^2 = b^2 g^2,$$

$$\text{and} \quad T^2 S(g) = T^2(g^2) = b^2 g^2.$$

T and T^2 need not be conjugate, but will be conjugate if S is a group automorphism. For example, they are conjugate when

$G = \mathbb{Z}_q = \{e^{2\pi i k/q} : k = 0, 1, \ldots, q-1\}$, (q odd), $T : G \to G$, $T(g) = bg$ where $b = e^{2\pi i/q}$ and $S(g) = g^2$.

However, if we set $G = S^1$, the circle group, and define a transitive map with discrete spectrum $T : S^1 \to S^1$ by $T(z) = az$ where $a \in S^1$ is fixed with $a^n \neq 1$ for all $n \in \mathbb{Z}$, then $S(z) = z^2$ is onto but not one--to--one and $ST = T^2 S$ (T^2 is said to be a *factor* of T in this case). Clearly T and T^2 are not conjugate since T has eigenvalue group $\Lambda = \{a^n : n \in \mathbb{Z}\}$ and T^2 has eigenvalue group $\Lambda^2 = \{a^{2n} : n \in \mathbb{Z}\}$. The eigenfunctions of T are the *continuous characters* of S^1. These are the continuous homomorphisms $\chi : S^1 \to S^1$, and are of the form $\chi_n(z) = z^n$ for each $n \in \mathbb{Z}$. T is an example of what is called an *irrational rotation of the circle* and is conjugate to the map

$T_\alpha : [0,1) \to [0,1)$, $T_\alpha(x) = x + \alpha \pmod 1$ where $a = e^{2\pi i \alpha}$.

We now show that transitive rotations have the weak closure property. In order to do this we need to specify the topology on $\mathcal{H}(G)$. It is known that on any compact metrizable group G there is a metric ρ which is rotation invariant: $\rho(gx, gy) = \rho(x, y) = \rho(xg, yg)$ for all $x, y, g \in G$. Now we define a metric d on $\mathcal{H}(G)$ giving the *compact-open* topology:

$$d(S,T) = \sup_{x \in G} \rho(Sx, Tx) + \sup_{x \in G} \rho(S^{-1}x, T^{-1}x), \ S, T \in \mathcal{H}(G).$$

With this topology, $\mathcal{H}(G)$ is complete metric topological group.

Lemma 1 *Let G be a compact metric abelian group. If $R : G \to G$ is the rotation $R(g) = ag$ where $\{a^n : n \in \mathbb{Z}\}$ is dense in G, then R has the weak closure property. In addition, $C(R)$, the centralizer of R, is the set of all rotations on G.*

Proof. For each $b \in G$, define a rotation $T_b : G \to G$ by $T_b(g) = bg$, then there is a sequence of positive integers n_k such that $a^{n_k} \to b$ in G, as $k \to \infty$. It follows that $R^{n_k} \to T_b$ in $\mathcal{H}(G)$, as $k \to \infty$.

It therefore suffices to show that the centralizer of R is given by

$$C(R) = \{T_b : b \in G\}.$$

Clearly $\{T_b : b \in G\} \subseteq C(R)$, so suppose that $T \in C(R)$ then

$$TR = RT \Rightarrow T(ag) = aT(g) \Rightarrow T(a^n g) = a^n T(g),$$
$$g \in G.$$

Let $x \in G$ be arbitrary and choose n_k so that $a^{n_k} \to x$, and set $b = T(e)$, then $T(a^{n_k}) = a^{n_k} T(e)$ for $\boldsymbol{k} = 1, 2, \cdots$, implies that $T(x) = bx$ for all $x \in G$, and so $T \in \{T_b : b \in G\}$.

Proposition 7 *Let $T : X \to X$ be transitive with discrete spectrum and suppose that $S : X \to X$ is a homeomorphism satisfying $ST = T^2 S$. Then T can be represented as a rotation $R : G \to G$, $R(g) = ag$ on a compact abelian group G, and S can be represented as $S : G \to G$, $S(g) = bg^2$ for some $a, b \in G$.*

Proof. The Discrete Spectrum Theorem tells us that T can be represented as a rotation $R(g) = ag$ on a compact abelian group G, so that we can assume

$$SR(g) = R^2 S(g)$$

$$\text{and} \quad SR^n(g) = R^{2n} S(g) \Rightarrow S(a^n g) = a^{2n} S(g),$$

$$\text{for all} \quad g \in G,$$

Let $x \in G$ and choose a sequence $a^{n_k} \to x$ as $k \to \infty$, and set $b = S(e)$, then we must have $S(x) = bx^2$ for all $x \in G$.

In the following proof we talk about the *character* *group* $\widehat{G}$ of a (locally) compact abelian group G. This is the set of all continuous characters $\chi : G \to S^1$ (*i.e.*, continuous homomorphisms $\chi : G \to S^1$). If G is a compact group, $\widehat{G}$ is a countable group (called the *dual group* of G). The Pontryagin Duality Theorem says that the dual of $\widehat{G}$ (the second dual of G) is topologically isomorphic to G (both a homeomorphism and a group isomorphism: see [13]).

Theorem 1 *Let T be transitive with discrete spectrum and eigenvalue group Λ. T is conjugate to T^2 if and only if the map $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$ is a group automorphism.*

Proof. T is conjugate to a rotation $R : G \to G$, $R(g) = ag$ for some compact abelian group G and some $a \in G$, where R has the weak closure property. It follows that the map $\Phi : C(R) \to C(R)$, $\Phi(s) = s^2$ is a group automorphism. Because $C(R) = \{T_b : b \in G\}$, (where $T_b(g) = bg$), this can be written as $\Phi(T_b) = T_b^2 = T_{b^2}$. We deduce that the map

$$\psi : G \to G, \ \ \psi(g) = g^2,$$

is a group automorphism.

The eigenfunctions of R are the continuous characters $\chi : G \to S^1$ since $\chi(R(g)) = \chi(ag) = \chi(a)\chi(g)$, and the group of eigenvalues of R is $\Lambda = \{\chi(a) : \chi \in \widehat{G}\}$, where $\widehat{G}$ is the group of characters of G. We then see that the map

$$\widehat{\psi} : \Lambda \to \Lambda, \ \ \widehat{\psi}(\chi)(a) = \chi(\psi(a)) = \chi(a^2) = \chi^2(a),$$

is an automorphism which can be identified with $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$ which is therefore also an automorphism.

Conversely, if $T : G \to G$ has eigenvalues Λ, a countable subgroup of S^1, for which the map $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$ is a group automorphism, then $\phi(\Lambda) = \Lambda^2$, so R and R^2 have the same eigenvalues, and are conjugate by the Discrete Spectrum Theorem.

Examples. 1. Set $S_i^1 = S^1$, $i \geq 0$ and $\Omega = \Pi_{i=0}^{\infty} S_i^1$, a compact group when given the product topology and usual group operation. Let G be the subgroup of Ω defined by

$$G = \{(z_0, z_1, z_2, \cdots) \in \Omega : z_0 = z_1^2, z_1 = z_2^2, \cdots, z_n = z_{n+1}^2, \cdots\}.$$

(G is actually the *inverse limit* of the sequence $\boldsymbol{S}^1 \leftarrow \boldsymbol{S}^1 \leftarrow \cdots$, where all the arrows denote the power two homomorphism. This is written $G = \lim_{\leftarrow} S^1$).

Fix $z_0 = e^{i\theta}$ where $z_0^n \neq 1$ for all $n \in \mathbb{Z}$, and set $\omega_0 = (z_0, z_1, z_2, \cdots)$ where $z_n = z_{n+1}^2$, for $\boldsymbol{n} = 0, 1, 2 \cdots$.

Define a group rotation $T : G \to G$ by $T(\omega) = \omega_0 \cdot \omega$ for $\omega \in G$. It can be shown that T is a transitive homeomorphism having discrete spectrum, the eigenvalue group being $H = \{e^{in\theta/2^m} : m, n \in \mathbb{Z}\}$, and it is easy to check that $S(\omega) = \omega^2$ is an automorphism which conjugates T to T^2.

If we define $\phi : H \to H$, by $\phi(\lambda) = \lambda^2$, then ϕ is a group automorphism and we see that the hypotheses of Theorem 1 are satisfied.

Actually, it is a consequence of the Discrete Spectrum Theorem that for any countable subgroup Λ of the circle S^1, there is a transitive homeomorphism $T : G \to G$ ($G = \widehat{\Lambda}$ a compact abelian group) having discrete spectrum, and which has Λ as its eigenvalue group. It follows that if Λ is any countable subgroup of S^1 for which $\phi : \Lambda \to \Lambda$, $\phi(\lambda) = \lambda^2$, is an automorphism, then there is a transitive homeomorphism having discrete spectrum and eigenvalue group Λ, which is conjugate to its square.

2. Let $G =$ the group of *3-adic integers*, and $T : G \to G$ the *adding machine*. We can think of G as $G = \Pi_{n=1}^{\infty} \mathbb{Z}_3$ where $\mathbb{Z}_3 = \{0,1,2\}$, the group of integers modulo 3. The group operation on G can be defined as "carry to the right", so for example

$$(2,1,0,2,1,\cdots) + (1,2,1,0,1,\cdots) = (0,1,1,2,2,\cdots).$$

The adding machine is then defined by

$$T(g) = g + 1,$$

where $1 = (1,0,0,\cdots)$ and $g = (g_1, g_2, \cdots) \in G$. T is transitive and since it is a group rotation, it has discrete spectrum. The eigenvalues are the 3^n th roots of unity.

The map $S(g) = 2g$ is a group automorphism of G which conjugates T to T^2. S is not transitive as sets of the form $\{(0, g_2, g_3, \cdots) \in G : g_i \in \mathbb{R}_3\}$ are S invariant. However, the only fixed point of S is the identity of the group G. Since T has the weak closure property, all other conjugations are conjugate to S.

3. One of the earliest results in dynamical systems is due to Poincaré concerning homeomorphisms $f : S^1 \to S^1$ of the unit circle in the complex plane. Before stating his result we shall give some properties of *circle homeomorphisms* (see [8] or [4] for example).

Let $F : \mathbb{R} \to \mathbb{R}$ be a strictly increasing continuous function satisfying

$$F(x+1) = F(x) + 1, \text{ for all } x \in \mathbb{R}$$

(respectively strictly decreasing with $\boldsymbol{F}(\boldsymbol{x}+1) = \boldsymbol{F}(\boldsymbol{x}) - 1$ for all $x \in \mathbb{R}$). F determines a homeomorphism of the circle $f : S^1 \to S^1$ defined by

$$f(e^{2\pi i x}) = e^{2\pi i F(x)}.$$

In addition, every homeomorphism of S^1 arises in

Copyright © 2010 SciRes.

this way. The homeomorphism is said to be *orientation preserving* if F is strictly increasing and *orientation reversing* if F is strictly decreasing. The map F is said to be a *lift* of f. For example $f(e^{2\pi ix}) = e^{2\pi ix^2}$ is a circle homeomorphism with a single fixed point. $F(x) = x^2$, $0 \le x < 1$ and extended to all of $\mathbb{R}$ so that $F(x+1) = F(x)+1$, is a lift of f. Circle homeomorphisms and their *rotation numbers* have the following properties:

1) If f is orientation preserving with lift F, then for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{F^n(x)}{n} = \rho(f)$$

exists and is independent of x. Since the lift F is not unique (any two differ by an integer), by choosing $0 \le F(0) < 1$, we may assume that $0 \le \rho(f) < 1$ is unique and we call $\rho(f)$ the rotation number of f. It satisfies $\rho(f^n) = n\rho(f)$ (mod 1) for $n \in \mathbb{Z}$, and $\rho(hfh^{-1}) = \rho(f)$ for any other orientation preserving circle homeomorphism h.

2) If $\rho(f) = 0$, then f has a fixed point. If $\rho(f)$ is rational, then f has a periodic point and $\rho(f)$ is irrational if and only if f has no periodic points. There are circle homeomorphisms having points of any given period.

3) Poincaré showed: *A transitive circle homeomorphism having irrational rotation number is conjugate to an irrational rotation of the circle.*

4) An orientation preserving circle homeomorphism having points of period n can have points of no other period.

5) An orientation reversing circle homeomorphism has exactly 2 fixed points and can have any number of 2-cycles, but cannot have points of period greater than 2.

We have seen that irrational rotations of the circle cannot be conjugate to their squares, so the same is true for transitive circle homeomorphisms having irrational rotation number. The situation is different for circle homeomorphisms having fixed points.

Theorem 2 *Let* $f: S^1 \to S^1$ *be an orientation preserving circle homeomorphism. Then*

(a) f is conjugate to f^2 if and only if f has at least one fixed point.

(b) If f has a single fixed point c and $h \circ f = f^2 \circ h$ for some circle homeomorphism h, then $h(c) = c$, and h has at least one other fixed point. If f has fixed points Fix $(f) = \{c_1, c_2, \cdots, c_n\}$, then h permutes the set Fix (f). If n is prime these points are either fixed or constitute an n-cycle.

Proof. (a) Suppose f has a fixed point, then all other periodic points are fixed. Use these to partition the circle into subintervals and give an argument similar to that in Proposition 3(a) (see [9] or [14]).

Conversely if $h \circ f = f^2 \circ h$, we may assume h is orientation preserving (otherwise look at h^2), so the above properties imply $\rho(f) = \rho(h^{-1} f^2 h) = \rho(f^2) = 2\rho(f)$, so that $\rho(f) = 0$, and f must have a fixed point.

(b) If $f(c) = c$ then $h(f(c)) = f^2(h(c))$, so $f(h(c)) = h(c)$ as f cannot have any period 2-points. Since c is unique, we must have $h(c) = c$. An argument similar to that in Proposition 3(c) can be used to show that h must have an additional fixed point.

If f has n fixed points $\{c_1, c_2, \ldots,, c_n\}$ then as above, $f(h(c_i)) = h(c_i)$ for $i = 1, 2, \ldots, n$. We deduce that h is a permutation of Fix (f).

If n is prime and the points are not periodic, then since a circle homeomorphism can only have points of one period, it must be an n-cycle for h.

4. Let $X = [0,1]$, $\mathcal{B} =$ the Borel measurable subsets of $[0,1]$, and μ a Borel measure on X. Denote by Aut (X) the group of all invertible measure preserving transformations of $T: X \to X$ (T will be one-to-one and onto, but possibly only after a set of measure zero is omitted). These satisfy $T(A), T^{-1}(A) \in \mathcal{B}$, $\mu(T(A)) = \mu(A) = \mu(T^{-1}(A))$ for all $A \in \mathcal{B}$. Aut (X) is a Polish group (but not a topological group). The 3-adic adding machine can be realized as a member of Aut (X) for $\mu =$ Lebesgue measure on $[0,1]$ in the following way.

We define T as a *rank-one* (*rational discrete spectrum*) transformation whose eigenvalues are the 3^n th roots of unity, and constructed as follows:

Starting with the unit interval $[0,1)$, subdivide into 3 equal subintervals and stack, placing $[1/3, 2/3)$ above $[0, 1/3)$ and $[2/3, 1)$ on top. Now define T by linearly mapping the bottom interval to the middle interval and mapping the middle interval to the top interval, but leaving T undefined on the top level. Continue this process inductively, so that at the n th stage, T is defined on the levels of a column consisting of 3^n equal subintervals. Again subdivide the column into 3 equal subcolumns and stack as before to extend the definition of T. Ultimately, T is defined almost everywhere on $[0,1]$. Denote by $B_i(n)$ the i th level of the n th column ($0 \le i \le 3^n - 1$, $n \ge 0$), then S can now be defined inductively by mapping the level B_i to the level B_{2i} (working modulo 3^n for $n \ge 1$, where $B_0 = [0,1)$). By following the orbit of $x \in B_i(n)$, $0 \le i < 3^n - 1$, we can see that $ST = T^2 S$ where $S \in Aut(X)$.

5. Set $X = [0,1]^{\mathbb{Z}} = \Pi_{i=-\infty}^{\infty}[0,1]_i$ (where $[0,1]_i = [0,1]$),

Copyright © 2010 SciRes.

the *Hilbert cube*. Suppose we have a map $T:[0,1]\to[0,1]$ conjugate to its square (for example $T(x)=x^2$) with $kT=T^2k$. Set $T^{1/2}=k^{-1}Tk$ and define $\tilde{T}:X\to X$ by

$$\tilde{T}(\cdots,x_{-2},x_{-1},\overset{*}{x_0},x_1,x_2,\cdots)$$
$$=(\cdots,T^{1/4}(x_{-2}),T^{1/2}(x_{-1}),\overset{*}{T(x_0)},T^2(x_1),T^4(x_2),\cdots),$$

then if $S:X\to X$ is the left shift map:

$$\boldsymbol{S}(\cdots,\boldsymbol{x}_{-1},\overset{*}{\boldsymbol{x}_0},\boldsymbol{x}_1,\cdots)=(\cdots,\boldsymbol{x}_0,\overset{*}{\boldsymbol{x}_1},\boldsymbol{x}_2,\cdots),$$

(where the * represents the 0th coordinate), then we can check that $S\tilde{T}=\tilde{T}^2S$. In this case $\tilde{T}$ is a homeomorphism having uncountably many fixed points (e.g., if $T(x)=x^2$), but no period 2-points. S has uncountably many periodic points of every order.

6. Concluding Remarks

1) In dynamical systems theory one studies the actions of groups on sets of homeomorphisms or on sets of measure preserving transformations. In the study of a single transformation we are looking at actions of the countable group $\mathbb{Z}$. The examples of this paper may be thought of as actions of the group $G=\{\langle a,k\rangle : ka=a^2k\}$, the countable non-abelian group that was studied in Section 2. Let X be a compact space and $\mathcal{H}(X)$ its group of self-homeomorphisms. We can define a representation of G (an action) as a continuous homomorphism $V:G\to\mathcal{H}(X)$, $V(g)=V_g$. Suppose we set $V_a=T$ and $V_k=S$, then $V_{ka}=V_kV_a=ST$ and $V_{a^2k}=V_{a^2}V_k=T^2S$, so we see that actions of G on $\mathcal{H}(X)$ are determined by a pair of homeomorphisms S, T satisfying $ST=T^2S$.

2) All the examples we have considered have zero topological entropy (except for the last example whose entropy is infinite). This is because conjugate homeomorphisms have the same topological entropy, and if $h(T)$ denotes the entropy of a homeomorphism T, then $h(T^2)=2h(T)$ (see [4]).

3) Other examples of interest are the homeomorphisms of the Cantor set C, in particular for proving category/density type results. Every invertible measure preserving transformation may be modeled as a homeomorphism of C. The adding machine can be seen directly to be such an example.

4) It is natural to talk about conjugacy between a continuous transformation f defined on some metric space X and its square f^2. The logistic map $f_\mu(x)=\mu x(1-x)$, $x\in[0,1]$ cannot be conjugate to f_μ^2 for $\mu>3$, because such maps have period 2-points. However, conjugacies between continuous maps and their squares on higher dimensional spaces may lead to interesting dynamics.

7. References

[1] O. N. Ageev, "Spectral Rigidity of Group Actions: Applications to the Case $\mathrm{gr}(t,s:ts=s^2t)$," *Proceedings of the American Mathematical Society*, Vol. 134, No. 5, 2005, pp. 1331-1338.

[2] G. R. Goodson, "Ergodic Dynamical Systems Conjugate to Their Composition Squares," *Journal of Acta Mathematica Universitatis Comenianae*, Vol. 71, No. 7, 2002, pp. 201-210.

[3] G. R. Goodson, "Spectral Properties of Ergodic Dynamical Systems Conjugate to Their Composition Squares," *Colloquium Mathematicum*, Vol. 107, 2007, pp. 99-118.

[4] B. Hasselblatt and A. Katok, "A First Course in Dynamical Systems," Cambridge University Press, Cambridge, 2003.

[5] P. Walters, "An Introduction to Ergodic Theory," Springer Verlag, Berlin, 1982.

[6] A. I. Danilenko, "Weakly Mixing Rank-One Transformations Conjugate to Their Squares," *Studia Mathematica,* Vol. 187, No. 1, 2008, pp. 75-93.

[7] B. Marcus, "Unique Ergodicity of the Horocycle Flow: Variable Negative Curvature Case," *Israel Journal of Mathematics*, Vol. 21, No. 2-3, 1975, pp. 133-144.

[8] R. L. Devaney, "Chaotic Dynamical Systems," Benjamin Cummings Publishing Company, California, 1986.

[9] N. J. Fine and G. E. Schweigert, "On the Group of Homeomorphisms of an Arc," *Annals of Mathematics*, Vol. 62, No. 2, 1955, pp. 237-253.

[10] M. Kuczma, "On the Functional Equation $\varphi^n(x)=g(x)$," *Annales Polonici Mathematici*, Vol. 11, 1961, pp. 161-175.

[11] J. L. King. "The Commutant is the Weak Closure of the Powers, for Rank-One Transformations," *Ergodic Theory and Dynamical Systems*, Vol. 6, No. 3, 1983, pp. 363-384.

[12] P. R. Halmos, "Ergodic Theory," Chelsea Publishing Company, Vermont, 1956.

[13] E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis," 2nd Edition, Springer-Verlag, Berlin, 1979.

[14] N. Gill, A. G. O'Farell and I. Short, "Reversibility in the Group of Homeomorphisms of the Circle," *Bulletin of the London Mathematical Society*, Vol. 41, No. 5, 2009, pp. 885-897.

Copyright © 2010 SciRes.