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New Periodic Solitary Wave Solutions for a Variable-Coefficient Gardner Equation from Fluid Dynamics and Plasma Physics

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Abstract

The Gardner equation with a variable-coefficient from fluid dynamics and plasma physics is investigated. Different kinds of solutions including breather-type soliton and two soliton solutions are obtained using bilinear method and extended homoclinic test approach. The proposed method can also be applied to solve other types of higher dimensional integrable and non-integrable systems.

1. Introduction

In nonlinear science, many important phenomena in various fields can be describe by the nonlinear evolution equations. Seeking exact solutions of nonlinear partial differential equations is of great significance as it appears that these (NLPDEs) are mathematical models of complex physics phenomena arising in physics, mechanics, biology, chemistry and engineers. In order to help engineers and physicists to better understand the mechanism that governs these physical models or to better provide knowledge to the physical problem and possible applications, a vast variety of the powerful and direct methods have been derived. Various powerful methods for obtaining explicit travelling solitary wave solutions to nonlinear equations have proposed such as [1-8].

One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods to look for exact solutions for nonlinear partial differential equations. A search of directly seeking for exactly solutions of nonlinear equations has been more interest in recent years because of the availability of symbloic computation Mathematica or Maple. These computer systems allow us to perform some complicated and tedious algebraic and differential calculations on a computer.

Much attention has been paid to the variable coeffi-

cients nonlinear equation which can describe many nonlinear phenomena more realistically than their constant coefficient ones [9]. The Gardner equation, or extended KdV equation can describe various interesting physics phenomens, such as the internal waves in a stratified Ocean [10], the long wave propagation in an inhomogeneous two-layer shallow liquid [11] and ion acoustic waves in plasma with negative ion [12], we consider a generalized variable-coefficient Gardner equation [13]

$$\phi_t + f(t)\phi_{xxx} + g(t)\phi\phi_x + h(t)\phi^2\phi_x + r(t)\phi_x + \tau(t)\phi = 0, \quad (1)$$

where $\phi(x,t)$ is a function of x and t. The coefficients $f(t)$, $g(t)$, $h(t)$, $r(t)$ and $\tau(t)$ are differential functions of t. Equaion (1) is not completely integrable in the sense of the inverse scattering scheme it contains some important special cases:

In case of $h(t) = 0$, $r(t) = 0$ and $\tau(t) = 0$, Equation (1) reduces to

$$\phi_t + f(t)\phi_{xxx} + g(t)\phi\phi_x = 0, \quad (2)$$

and

$$f(t) = g(t)\left[a + b\int f(t)dt \right]. \quad (3)$$

Equation (2) possesses the Painleve property [14,15]. The Bäcklund transformation, Lax pair, similarity reduction and special analytic solution of Equation (2) have been obtained [16-20].

For $g(t) = -6a(t)$, $h(t) = -6r$ and $r(t) = \tau(t) = 0$,

Equation (1) reduces to

$$\phi_t - 6a(t)\phi\phi_x - 6r\phi^2\phi_x + f(t)\phi_{xxx} = 0, \qquad (4)$$

which describe strong and weak interactions of different mode internal solitary waves, etc. When $f(t) = r$, $g(t) = 6\alpha$, $h(t) = 6\beta$, $r(t) = \tau(t) = 0$, Equation (1) becomes the constant-coefficient Gardner equation

$$\phi_t + 6\alpha\phi\phi_x + 6\beta\phi^2\phi_x + r\phi_{xxx} = 0, \qquad (5)$$

where r, β and α are constants. It is widely applied to physics and quantum fields, such as solid state physics, plasma physics, fluid dynamics and quantum field theory.

When $g(t) = 6$, $f(t) = 1$, $h(t) = r(t) = 0$, Equation (1) reduces to constant coefficient KdV equation

$$\phi_t + 6\phi\phi_x + \phi_{xxx} + \tau(t)\phi = 0, \qquad (6)$$

which possesses the Painleve property. If $\tau(t) = 0$ or $\tau(t) = \dfrac{1}{2(t - t_0)}$, it corresponds to the well known cylindrical KdV equation.

The structure of this paper is organized as follows; In Section 2, with symbolic computation, the bilinear form of Equation (1) are obtained. In order to illustrate the proposed method, we consider for a variable-coefficient Gardner equation from fluid dynamics and plasma physics and new periodic wave solutions are obtained which included periodic two solitary solution, doubly periodic solitary solution. Finally, conclusion and discussion are given in Section 3.

2. Bilinear Form of the Gardner Equation with Variable Coefficients

Making use the dependent variable transformation as

$$\phi(x,t) = k(t)\frac{\partial}{\partial x}w(x,t), \qquad (7)$$

into Equation (1) and integrating once with respect to x, admits to [13]

$$k'(t)w + k(t)w_t + f(t)k(t)w_{xxx} + \frac{1}{2}g(t)k^2(t)w_x^2$$
$$+ \frac{1}{3}h(t)k^3(t)w_x^3 + r(t)k(t)w_x + \tau(t)k(t)w = 0 \qquad (8)$$

with the integration constant to zero. Then introducing the transformation

$$w(x,t) = \arctan\left[\frac{v(x,t)}{u(x,t)}\right], \qquad (9)$$

where $u(x,t)$ and $v(x,t)$ are differential functions of x and t into (8) yields [13]

$$[k'(t) + \tau(t)k(t)]\arctan\left[\frac{v(x,t)}{u(x,t)}\right] + k(t)\frac{D_t v.u}{v^2 + u^2}$$
$$+ f(t)k(t)\left[\frac{D_x^3 v.u}{v^2 + u^2} - 3\frac{D_x v.u[D_x^2(u.u + v.v)]}{(v^2 + u^2)^2}\right.$$
$$\left. - 8\left(\frac{D_x v.u}{v^2 + u^2}\right)^3\right] + \frac{1}{2}g(t)k^2(t)\left(\frac{D_x v.u}{v^2 + u^2}\right)^2 \qquad (10)$$
$$+ \frac{1}{3}h(t)k^3(t)\left(\frac{D_x v.u}{v^2 + u^2}\right)^3 + r(t)k(t)\left(\frac{D_x v.u}{v^2 + u^2}\right) = 0,$$

where the prime denotes the derivative with respect to t, and D_t, D_x, D_x^2 and D_x^3 are the bilinear derivative operators [7] defined as

$$D_x^m D_t^n f(x,t).g(x,t)$$
$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n [f(x,t)g(x',t')]_{x'=x,t'=t}. \qquad (11)$$

Decopling Equation (10), we obtain [13]

$$k'(t) + \tau(t)k(t) = 0, \qquad (12)$$

$$-8f(t)k(t) + \frac{1}{3}h(t)k^3(t) = 0, \qquad (13)$$

$$\left[D_t + f(t)D_x^3 + r(t)D_x\right]v.u = 0, \qquad (14)$$

$$3f(t)D_x^2(u.u + v.v] = \frac{1}{2}g(t)k(t)D_x v.u, \qquad (15)$$

Via Equations (12) and (13), we have the following relations

$$k(t) = c_0 e^{-\int \tau(t)dt}, f(t) = \frac{1}{24}h(t)k^2(t) \qquad (16)$$

where c_0 is a nonzero arbitrary constant. That is to say, through the dependent variable transformation

$$\phi(x,t) = c_0 e^{-\int \tau(t)dt}\left[\arctan\frac{v(x,t)}{u(x,t)}\right]_x. \qquad (17)$$

Equation (1) is transformed into its bilinear form, *i.e.*, Equations (14) and (15) under constraint (16).To solve the reduced Equations (14) and (15) using the extended homoclinic test function [21-29], we suppose a solution of Equations (14) and (15) as follows

$$v(x,t) = e^{a_1 x + b_1 t} + p_1 \cos[a_2 x + b_2 t] + q_1 e^{-a_1 x - b_1 t}, \qquad (18)$$

and

$$u(x,t) = e^{a_1 x + b_1 t} + p_2 \cos[a_2 x + b_2 t] + q_2 e^{-a_1 x - b_1 t}, \qquad (19)$$

where p_i, q_i, a_i, b_i $(i = 1, 2)$ are parameters to be determined later.

Substituting Equations (18) and (19) into Equations (14) and (15), and equating all coefficients of $[e^{j(a_1 x + b_1 t)}$

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($j = -1, 0, 1$), $\cos(a_2 x + b_2 t)$, $\sin(a_2 x + b_2 t)$] to zero, we get the set of algebraic equation for p_i, q_i, a_i, b_i ($i = 1, 2$). Solve the set of algebraic equations with the aid of Maple, we have many solutions, in which the following solutions are

Case (1):

$$b_2 = 0, \quad q_2 = 0, \quad q_1 = 0, \quad b_1 = -f(t)a_1^3 - r(t)a_1,$$

$$r(t) = r(t), \quad p_2 = p_2, \quad p_1 = p_1, \quad a_2 = 0, \quad a_1 = a_1, \quad (20)$$

$$g(t) = \frac{12 f(t) a_1 (p_2 + p_1)}{k(t)(p_1 - p_2)}$$

Case (2):

$$b_2 = b_2, \quad q_2 = q_2, \quad q_1 = q_1, \quad b_1 = -4 f(t)a_1^3 - r(t)a_1,$$

$$r(t) = r(t), \quad p_2 = 0, \quad p_1 = 0, \quad a_2 = a_2, \quad a_1 = a_1, \quad (21)$$

$$g(t) = \frac{4 f(t) a_1 (q_2 + q_1)}{k(t)(q_1 - q_2)}$$

Case (3):

$$p_2 = 0, \quad p_1 = 0, \quad q_2 = q_2, \quad q_1 = q_1,$$

$$b_2 = b_2, \quad b_1 = b_1 a_1 = a_1, \quad a_2 = 0, \quad (22)$$

$$r(t) = -\frac{4 f(t) a_1^3 + b_1}{a_1}, \quad g(t) = -\frac{24 f(t) a_1 (q_2 + q_1)}{k(t)(q_1 - q_2)}$$

Case (4):

$$g(t) = 0, \quad p_1 = p_1, \quad b_2 = b_2, \quad a_1 = a_1, \quad a_2 = a_2, \quad q_2 = \frac{a_2^2 p_1^2}{4 a_1^2},$$

$$q_1 = \frac{a_2^2 p_1^2}{4 a_1^2}, \quad b_1 = \frac{a_1 \left(b_2 + 2 f(t) a_2^3 + 2 f(t) a_2 a_1^2 \right)}{a_2},$$

$$r(t) = -\frac{b_2 - f(t) a_2^3 + 3 f(t) a_1^2 a_2}{a_2}, \quad p_2 = -p_1$$

$$(23)$$

Case (5):

$$b_2 = a_2 (4 f(t) a_2^2 - r(t)), \quad a_2 = a_2, \quad p_1 = p_1, \quad p_2 = p_2,$$

$$r(t) = r(t), \quad b_1 = i a_2 (4 f(t) a_2^2 - r(t)), \quad a_1 = i a_2,$$

$$g(t) = \frac{24 i f(t) a_2 (p_2 + p_1)}{k(t)(p_1 - p_2)},$$

$$q_1 = \frac{1}{8} p_1^2 + \frac{1}{4} p_1 p_2 - \frac{1}{8} p_2^2, \quad q_2 = -\frac{1}{8} p_1^2 + \frac{1}{4} p_1 p_2 + \frac{1}{8} p_2^2$$

$$(24)$$

Using Equation (20), Equations (18) and (19) can be written as

$$v(x,t) = e^{a_1 x + [-f(t)a_1^3 - r(t)a_1]t} + p_1, \qquad (25)$$

and

$$u(x,t) = e^{a_1 x + [-f(t)a_1^3 - r(t)a_1]t} + p_2. \qquad (26)$$

Inserting Equations (25) and (26) into Equation (17), admits to the new solitary wave solution of Equation (1) as

$$\phi(x,t) = c_0 e^{-\int \tau(t)dt} \left[\arctan \frac{v(x,t)}{u(x,t)} \right]_x \qquad (27)$$

With the aid of Equation (21), Equations (18) and (19) yields

$$v(x,t) = e^{a_1 x + [-4 f(t)a_1^3 - r(t)a_1]t} + q_1 e^{-a_1 x - [-4 f(t)a_1^3 - r(t)a_1]t}, \quad (28)$$

and

$$u(x,t) = e^{a_1 x + [-4 f(t)a_1^3 - r(t)a_1]t} + q_2 e^{-a_1 x - [-4 f(t)a_1^3 - r(t)a_1]t}, \quad (29)$$

Knowing Equations (28) and (29) with Equation (17), we have the solitary wave solution of Equation (1) as

$$\phi(x,t) = c_0 e^{-\int \tau(t)dt} \left[\arctan \frac{v(x,t)}{u(x,t)} \right]_x \qquad (30)$$

In view of case (3), Equations (18) and (19) reads

$$v(x,t) = e^{a_1 x + b_1 t} + + q_1 e^{-a_1 x - b_1 t}, \qquad (31)$$

and

$$u(x,t) = e^{a_1 x + b_1 t} + q_2 e^{-a_1 x - b_1 t}. \qquad (32)$$

Inserting Equations (31) and (32) into Equation (17), admits to the new solitary wave solution of Equation (1) as

$$\phi(x,t) = c_0 e^{-\int \tau(t)dt} \left[\arctan \frac{v(x,t)}{u(x,t)} \right]_x \qquad (33)$$

Via Equation (23) with Equations (18) and (19), we have

$$v(x,t) = e^{a_1 x + \left[\frac{a_1(b_2 + 2 f(t)a_2^3 + 2 f(t)a_2 a_1^2)}{a_2} \right]t} + \left[\frac{a_2^2 p_1^2}{4 a_1^2} \right] \cos[a_2 x + b_2 t]$$

$$+ \left[\frac{a_2^2 p_1^2}{4 a_1^2} \right] e^{-a_1 x - \left[\frac{a_1(b_2 + 2 f(t)a_2^3 + 2 f(t)a_2 a_1^2)}{a_2} \right]t},$$

$$(34)$$

and

$$u(x,t) = e^{a_1 x + \left[\frac{a_1(b_2 + 2 f(t)a_2^3 + 2 f(t)a_2 a_1^2)}{a_2} \right]t} + \left[\frac{a_2^2 p_1^2}{4 a_1^2} \right] \cos[a_2 x + b_2 t]$$

$$+ \left[\frac{a_2^2 p_1^2}{4 a_1^2} \right] e^{-a_1 x - \left[\frac{a_1(b_2 + 2 f(t)a_2^3 + 2 f(t)a_2 a_1^2)}{a_2} \right]t},$$

$$(35)$$

Using Equations (34) and (35), admits to the new solitary wave solution of Equation (1) as

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$$\phi(x,t) = c_0 e^{-\int r(t)dt} \left[\arctan \frac{v(x,t)}{u(x,t)} \right]_x \qquad (36)$$

According to case (5), Equations (18) and (19) becomes

$$v(x,t) = e^{ia_2 x + \left[ia_2(4f(t)a_2^2 - r(t)) \right] t} + p_1 \cos[a_2 x$$
$$+ [a_2(4f(t)a_2^2 - r(t))]t] + q_1 e^{-ia_2 x - \left[ia_2(4f(t)a_2^2 - r(t)) \right] t}, \qquad (37)$$

and

$$u(x,t) = e^{ia_2 x + \left[ia_2(4f(t)a_2^2 - r(t)) \right] t} + p_2 \cos[a_2 x$$
$$+ [a_2(4f(t)a_2^2 - r(t))]t] + q_2 e^{-ia_2 x - \left[ia_2(4f(t)a_2^2 - r(t)) \right] t}. \qquad (38)$$

By means of Equations (36) and (37) with Equation (17) we have a new solitary wave solutions as

$$\phi(x,t) = c_0 e^{-\int r(t)dt} \left[\arctan \frac{v(x,t)}{u(x,t)} \right]_x \qquad (39)$$

3. Conclusions

In this paper, with the aid of two methods, namely, bi-linear form and the extended homoclinic test approach, we obtain breather-type soliton and two soliton solutions for the Gardner equation with a variable-coefficient from fluid dynamics and plasma physics. The results reported here show that the extended homoclinic test approach is very effective in finding exact solitary wave solutions for nonlinear evolution equations with variable coefficients.

Finally, it is worthwhile to mention that, the proposed method is reliable and effective can also be applied to solve other types of higher dimensional integrable and non-integrable systems.

4. Acknowledgements

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