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A General Hermitian Nonnegative-Definite Solution to the Matrix Equation *AXB* = *C*

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Abstract

We derive necessary and sufficient conditions for the existence of a Hermitian nonnegative-definite solution to the matrix equation $AXB = C$. Moreover, we derive a representation of a general Hermitian nonnegative-definite solution. We then apply our solution to two examples, including a comparison of our solution to a proposed solution by Zhang in [1] using an example problem given from [1]. Our solution demonstrates that the proposed general solution from Zhang in [1] is incorrect. We also give a second example in which we derive the general covariance structure so that two matrix quadratic forms are independent.

Keywords

Matrix Equation $AXB = C$, Generalized Inverse Matrices, Parallel Summable Matrices, Symmetrization Device

1. Introduction

Let $A \in \mathbb{C}_{m\times n}\left(\mathbb{R}_{m\times n}\right)$ represent a matrix in the $m \times n$ vector space of complex (real) matrices $\mathbb{C}_{m\times n}\left(\mathbb{R}_{m\times n}\right)$, and let $A^* \in \mathbb{C}_{n\times m}\left(A' \in \mathbb{R}_{n\times m}\right)$ denote the conjugate transpose (transpose) of $A \in \mathbb{C}_{m\times n}\left(\mathbb{R}_{m\times n}\right)$. We frequently encounter linear matrix equations of the form

$$AXB = C \tag{1}$$

with $A \in \mathbb{C}_{m\times n}$, $B \in \mathbb{C}_{n\times k}$, and $C \in \mathbb{C}_{m\times k}$. Using the Moore-Penrose inverse, Penrose in [2] was the first to provide conditions for the existence and representation of the general solution to (1). Since then, numerous authors have derived representations of a general solution to (1) under varying restrictions on A, B, and C and on the type of solution X. Existence conditions and alternative expressions for the general solution have been studied by Dogaru in [3] and Chu in [4]. Also, Rosen in [5] has provided a representation of the general solution for (1) when $C = 0$.

Hermitian solutions to (1) have been considered by numerous authors as well such as by Khatri in [6], Wang, Yan, and Dai in [7], and Cvetković-Ilić in [8]. Additionally, Wang and Yang in [9] and Cvetković-Ilić and Dragana in [10] have found necessary and sufficient conditions for the existence of a real nonnegative-definite (Re-n.n.d.) solution and a representation of a general Re-n.n.d. solution to (1). Also, Zhang has proposed representations of the general Hermitian n.n.d. solutions to (1) in [1].

In this paper, we derive necessary and sufficient conditions for the existence of a Hermitian n.n.d. solution and a new representation of the general Hermitian n.n.d. solution to (1). Moreover, our representation is invariant with respect to the generalized inverse (g-inverse) involved, unlike the solution from Khatri in [6]. We then apply our solution to an example problem posed by Zhang in [1] and obtain a simpler solution that contradicts the proposed solution from Zhang in [1]. Furthermore, while Zhang employs an algorithmic method in [1], we obtain a closed-form solution. We also provide an example application where we employ our general Hermitian n.n.d. solution to demonstrate that two matrix quadratic forms are stochastically independent.

2. Notation and Definitions

In this section, we establish some notation to be used throughout the remainder of the paper. We use I_n to represent the $n \times n$ identity matrix and use I to denote the identity matrix if the order of the matrix is apparent. We use $\mathcal{C}(A)$ to denote the column space (range space) and $\mathcal{R}(A)$ to denote the row space of $A \in \mathbb{C}_{m\times n}$. The rank of $A \in \mathbb{C}_{m\times n}\left(\mathbb{C}_{m\times n}\right)$ is represented by $\operatorname{rank}(A)$. We let $\mathbb{C}_n^{\geq}\left(\mathbb{R}_n^{\geq}\right)$ denote the cone of all $n \times n$ Hermitian (symmetric) n.n.d. matrices in $\mathbb{C}_n\left(\mathbb{R}_n\right)$, where $\mathbb{C}_n\left(\mathbb{R}_n\right)$ is the set of all $n \times n$ complex (real) matrices. Given a matrix $A \in \mathbb{C}_{m\times n}\left(\mathbb{R}_{m\times n}\right)$, a g-inverse $A^{-} \in \mathbb{C}_{n\times m}\left(\mathbb{R}_{n\times m}\right)$ of A is a matrix that satisfies the property $AA^{-}A = A$. Finally, we let $\mathbb{C}_n^H$ denote the set of complex Hermitian $n \times n$ matrices.

3. Mathematical Preliminaries

This section contains the fundamental mathematical results that will be used in this paper. We provide a definition of parallel summable matrices and introduce five lemmas that are essential to our main results.

Definition. Let $A, B \in \mathbb{C}_{m\times n}$. A pair of matrices $A, B \in \mathbb{C}_{m\times n}$ is defined to be parallel summable if $A(A+B)^{-}B$ is invariant under the choice of the g-inverse $(A+B)^{-}$. That is, if

$$\mathcal{C}(A) \subset \mathcal{C}(A+B) \text{ and } \mathcal{R}(A) \subset \mathcal{R}(A+B),$$

or, equivalently,

$$\mathcal{C}(B) \subset \mathcal{C}(A+B) \text{ and } \mathcal{R}(B) \subset \mathcal{R}(A+B),$$

then the parallel sum of A and B is

$$A \mp B \equiv A(A+B)^{-} B.$$

We provide useful results for parallel summable matrices that are included in the next two lemmas. The first two lemmas are from Rao in [11].

Lemma 3.1. ([11], Lemma 2.2.4) Let $A \in \mathbb{R}_{m\times n}$, $B \in \mathbb{R}_{p\times n}$, and $C \in \mathbb{R}_{m\times q}$. If $\mathcal{R}(B) \subset \mathcal{R}(A)$ and $\mathcal{C}(C) \subset \mathcal{C}(A)$, then $BA^{-}C$ is invariant to the choice of the g-inverse A^{-}.

Lemma 3.2. ([11], Theorem 10.1.8) For a pair of parallel summable matrices $A, B \in \mathbb{C}_n^H$, we have $A \mp B = B \mp A$.

The following lemma comes from Khatri and Mitra in [12] and is used in the proof of the main result of this paper.

Lemma 3.3. Let $A \in \mathbb{C}_{m\times n}$, $B \in \mathbb{C}_{n\times k}$, and $C \in \mathbb{C}_{m\times k}$ such that $AXB = C$ is consistent. Then, X is a representation of a general solution for

$$AXB = C \tag{2}$$

if and only if X is a representation of the general solution for

$$A^{*}AXBB^{*} = A^{*}CB^{*}. \tag{3}$$

The following lemma verifies that, under certain conditions, a quadratic form is invariant under the choice of the g-inverse. Moreover, we verify that the quadratic form is n.n.d.

Lemma 3.4. Let $A, B \in \mathbb{C}_n^{\geq}$ and $C \in \mathbb{C}_n$. Also, let

$$T \equiv B(A+B)^{-} C(A+B)^{-} A \in \mathbb{C}_n^{\geq} \tag{4}$$

with

$$E \equiv (A+B)^{-} B, \tag{5}$$

$$F \equiv C(A+B)^{-} A, \tag{6}$$

$$K \equiv A(A+B)^{-}, \tag{7}$$

and

$$L \equiv B(A+B)^{-} C. \tag{8}$$

If $\operatorname{rank}(T) = \operatorname{rank}(F^{*}) = \operatorname{rank}(L)$, then $(F+L^{*})T^{-}(F^{*}+L) \in \mathbb{C}_n^{\geq}$ and is invariant to the choice of T^{-}.

Proof. First, because $T \in \mathbb{C}_n^{\geq}$, there exists a $T^{-} \in \mathbb{C}_n^{\geq}$. Also, we have that $LK^{*} = T = E^{*}F$ and $\operatorname{rank}(T) = \operatorname{rank}(F^{*}) = \operatorname{rank}(L)$. By Lemma 4.2.2 and Theorem 4.4.6 from Harville in [13], we have $\mathcal{C}(T) = \mathcal{C}(L) = \mathcal{C}(F^{*})$ and $\mathcal{R}(T) = \mathcal{R}(L^{*}) = \mathcal{R}(F)$. Also, from Lemma 4.5.10 of Harville in [13], we see that $\mathcal{R}(F+L^{*}) \subset \mathcal{R}(T)$ and $\mathcal{C}(L+F^{*}) \subset \mathcal{C}(T)$. Thus, by Lemma 3.1, the lemma holds.

The following lemma can be found in Theorem 1 from Albert in [14].

Lemma 3.5. Let $R \equiv \begin{bmatrix} Q & N \\ N^{*} & M \end{bmatrix}$, where $Q \in \mathbb{C}_n^H$, $N \in \mathbb{C}_{n\times p}$, and $M \in \mathbb{C}_p^H$. Then, $R \in \mathbb{C}_{n+p}^{\geq}$ if and only if $Q \in \mathbb{C}_n^{\geq}$, $QQ^{+}N = N$, and $M - N^{*}Q^{+}N \in \mathbb{C}_p^{\geq}$.

We use the following lemma in the proof of the second example. The lemma is well-known, and, therefore, is stated without proof.

Lemma 3.6. If $V \in \mathbb{R}_n^{\geq}$ is a n.n.d. matrix and $U \in \mathbb{R}_{n\times m}$ and $W \in \mathbb{R}_{m\times p}$ are matrices such that $UW \in \mathbb{R}_{n\times p}$, then $U'VUW = 0$ is equivalent to $VUW = 0$.

4. A General Hermitian N.N.D. Solution to AXB = C

In [6], Khatri provided existence conditions and have proposed a representation of the Hermitian n.n.d. solution to

$$AXB = C, \tag{9}$$

where $A, B \in \mathbb{C}_n^{\geq}$ and $C \in \mathbb{C}_n$. However, as noted by Baksalary in [15], his results are dependent on the choice of the g-inverse and, hence, do not represent a general Hermitian n.n.d. solution to (9).

In their efforts to derive a solution, Khatri and Mitra in [12] have employed an innovative technique that converts (9) to an equation in which the coefficient matrices are equal. We call this technique "symmetrization" because it effectively transforms (9) from a matrix bilinear form in A and B to the matrix equation form

$$(A+B)X(A+B)^{*} = D,$$

where $D \in \mathbb{C}_n^{\geq}$. We employ this symmetrization device in the proof of our main result.

The following theorem provides necessary and sufficient conditions for the existence of and a representation of the general Hermitian n.n.d. solution to (9) that is invariant to the choice of g-inverse. We remark that the general Hermitian n.n.d. solution given below in (11) is based on a result following Theorem 1 of Groß in [16].

Theorem. Let $A, B \in \mathbb{C}_n^{\geq}$ and $C \in \mathbb{C}_n$ such that (9) is consistent. Then, (9) has a Hermitian n.n.d. solution if and only if T is defined as in (4) and

$$\begin{aligned} \operatorname{rank}(T) &= \operatorname{rank}\left[A(A+B)^{-}C^{*}\right] \\ &= \operatorname{rank}\left[B(A+B)^{-}C\right]. \end{aligned} \tag{10}$$

A representation of the general Hermitian n.n.d. solution is

$$\begin{aligned} X &= (A+B)^{=}\left[C+C^{*}+Y+Z\right]\left[(A+B)^{=}\right]^{*} \\ &\quad + \left[I-(A+B)^{-}(A+B)\right]U\left[I-(A+B)^{-}(A+B)\right]^{*} \end{aligned} \tag{11}$$

where $(A+B)^{=}$ represents the class of g-inverses of $(A+B)$ given by

$$\begin{aligned} (A+B)^{=} &\equiv (A+B)^{-} + \left[I-(A+B)^{-}(A+B)\right]W \\ &\quad \times\left[\left(C+C^{*}+Y+Z\right)^{1/2}\right]^{-}, \end{aligned} \tag{12}$$

such that $Y, Z \in \mathbb{C}_n^{\geq}$ are arbitrary solutions of

$$Y(A+B)^{-}B = C(A+B)^{-}A \tag{13}$$

and

$$A(A+B)^{-}Z = B(A+B)^{-}C, \tag{14}$$

respectively. Also, $(C+C^{*}+Y+Z)\in\mathbb{C}_{n}^{\geq}$, $\left[(C+C^{*}+Y+Z)^{1/2}\right]^{-}\in\mathbb{C}_{n}$ is arbitrary but fixed, and $W\in\mathbb{C}_{n}$ and $U\in\mathbb{C}_{n}^{\geq}$ are free to vary. We remark that the form of the specialized g-inverse in (12) comes from Theorem 1 of Groß in [16].

Proof. First, assume $X_0\in\mathbb{C}_{n}^{\geq}$ is a solution to (9). Then,

$$\begin{aligned} T &= B(A+B)^{-}C(A+B)^{-}A \\ &= B(A+B)^{-}AX_0B(A+B)^{-}A, \end{aligned}$$

so that $T\in\mathbb{C}_{n}^{\geq}$. Next, let $X_0\equiv GG^{*}$, where $G\in\mathbb{C}_{n\times r}$ with $r=\operatorname{rank}(X_0)$. Then, using Lemma 3.2, we have

$$\begin{aligned} \operatorname{rank}(T) &= \operatorname{rank}\left(\left(B(A+B)^{-}AG\right)\left(B(A+B)^{-}AG\right)^{*}\right) \\ &= \operatorname{rank}\left(G^{*}A(A+B)^{-}B\right) \\ &= \operatorname{rank}\left(G^{*}B(A+B)^{-}A\right) \\ &= \operatorname{rank}\left(BX_0B(A+B)^{-}A\right) \\ &= \operatorname{rank}\left(BX_0A(A+B)^{-}B\right) \\ &= \operatorname{rank}\left(B(A+B)^{-}C\right). \end{aligned}$$

Similarly, we have that $\operatorname{rank}(T)=\operatorname{rank}\left(A(A+B)^{-}C^{*}\right)$.

Next, assume (4) and (10) hold. Following Khatri and Mitra in [6], we first write (13) and (14) as $YE=F$ and $KZ=L$, respectively, where E,F,K, and L are defined in (5)-(8), respectively. One can check that $LK^{*}=T=E^{*}F$ and $\operatorname{rank}(T)=\operatorname{rank}(F^{*})=\operatorname{rank}(L)$. From Lemma 3.1, we have that $FT^{-}F^{*}$ and $L^{*}T^{-}L$ are invariant with respect to T^{-}. Thus, by Theorem 2.2 of Khatri and Mitra in [6] and the fact that there exists a $T^{-}\in\mathbb{C}_{n}^{\geq}$, general Hermitian n.n.d. solutions to (13) and (14) are

$$Y = FT^{-}F^{*} + \left(I-EE^{-}\right)^{*}V_1\left(I-EE^{-}\right) \tag{15}$$

and

$$Z = L^{*}T^{-}L + \left(I-K^{-}K\right)V_2\left(I-K^{-}K\right)^{*}, \tag{16}$$

respectively, where $V_1,V_2\in\mathbb{C}_{n}^{\geq}$ are arbitrary. Also, because $\mathcal{C}(T)=\mathcal{C}(L)$ and $\mathcal{R}(T)=\mathcal{R}(F)$, we have $E^{*}\left(C-FT^{-}L\right)=0$ and $\left(C-FT^{-}L\right)K^{*}=0$, and, hence,

$$\left(I-EE^{-}\right)^{*}\left(C-FT^{-}L\right)\left(I-K^{-}K\right)^{*} = \left(C-FT^{-}L\right) \tag{17}$$

and

$$\left(I-K^{-}K\right)\left(C^{*}-L^{*}T^{-}F^{*}\right)\left(I-EE^{-}\right) = \left(C^{*}-L^{*}T^{-}F^{*}\right). \tag{18}$$

Using Equations (15) and (16), we have that

$$Y + C^* + C + Z = \left[FT^- F^* + \left(I - EE^-\right)^* V_1 \left(I - EE^-\right)\right] + C^* + C \\ + \left[L^* T^- L + \left(I - K^- K\right) V_2 \left(I - K^- K\right)^*\right]. \tag{19}$$

Adding $L^* T^- F^* - L^* T^- F^* + FT^- L - FT^- L$ to the right-hand side of (19) and letting $V_1 = I$ and $V_2 = P^* P$, we have that

$$\left(Y + C + C^* + Z\right) = \left(F^* + L\right) T^- \left(F + L^*\right) \\ + \left[\left(I - EE^-\right)^* : \left(I - K^- K\right)\right] J \begin{bmatrix} \left(I - EE^-\right) \\ \left(I - K^- K\right)^* \end{bmatrix}, \tag{20}$$

where $P \equiv C - FT^- L$ and

$$J \equiv \begin{bmatrix} I & P \\ P^* & P^* P \end{bmatrix}.$$

By Lemma 3.4, $\left(F + L^*\right) T^- \left(F^* + L\right) \in \mathbb{C}_n^{\geq}$. Also, using (17) and (18), we get that $J \in \mathbb{C}_{2n \times 2n}^{\geq}$ by Lemma 3.5. Thus, the right-hand side of equation (20) is Hermitian n.n.d., and, therefore, $\left(C + C^* + Y + Z\right) \in \mathbb{C}_n^{\geq}$. Because $\mathcal{C}\left(C + C^* + Y + Z\right) \subset \mathcal{C}\left(A + B\right)$, we have from Theorem 1 of Groß (2000) that

$$\left(A + B\right) X \left(A + B\right) = \left(C + C^* + Y + Z\right) \tag{21}$$

has a Hermitian n.n.d. solution.

Next, let $X_0 \in \mathbb{C}_n^{\geq}$ be given by (11). Then,

$$\begin{aligned} AX_0 B &= A\left(A + B\right)^= \left[C + C^* + Y + Z\right]\left[\left(A + B\right)^=\right]^* B \\ &\quad + A\left[I - \left(A + B\right)^- \left(A + B\right)\right] U \left[I - \left(A + B\right)^- \left(A + B\right)\right]^* B \\ &= A\left(A + B\right)^- \left[C + C^* + Y + Z\right]\left(A + B\right)^{-*} B \\ &= A\left(A + B\right)^- C\left(A + B\right)^- B + B\left(A + B\right)^- C\left(A + B\right)^- A \\ &\quad + A\left(A + B\right)^- C\left(A + B\right)^- A + B\left(A + B\right)^- C\left(A + B\right)^- B \\ &= \left(A + B\right)\left(A + B\right)^- C\left(A + B\right)^- \left(A + B\right) \\ &= C. \end{aligned}$$

Thus, if (21) has a Hermitian n.n.d. solution, then (9) has a Hermitian n.n.d. solution and, moreover, every Hermitian n.n.d. solution to (21) is a Hermitian n.n.d. solution to (9).

Now, let $X_0 \in \mathbb{C}_n^{\geq}$ be a solution to (9). Also, let $Y = AX_0 A$, $Z = BX_0 B$, and recall that $Y, Z \in \mathbb{C}_n^{\geq}$. Then, X_0 is a solution to (21). Thus, (11) is a general Hermitian n.n.d. solution to (9).

In our theorem, we derived a general Hermitian n.n.d. solution to (9) for the case where $A, B \in \mathbb{C}_n^{\geq}$. We next present the main result of the paper. We consider the general case by relaxing the n.n.d. and equal dimension constraints on the coefficient matrices A and B.

Corollary 1. Let $A \in \mathbb{C}_{n \times m}$, $B \in \mathbb{C}_{m \times p}$, and $C \in \mathbb{C}_{n \times p}$ such that

$$AXB = C \tag{22}$$

is consistent. Then, (22) has a Hermitian n.n.d. solution if and only if

$$T = BB^*H^-A^*CB^*H^-A^*A \in \mathbb{C}_m^{\geq},$$

and

$$\begin{aligned}\operatorname{rank}(T) &= \operatorname{rank}\left[A^*AH^-BC^*A\right]\\ &= \operatorname{rank}\left[BB^*H^-A^*CB^*\right],\end{aligned}$$

where $H = \left(A^*A + BB^*\right)$. A representation of the general Hermitian n.n.d. solution to (22) is given by

$$\begin{aligned}X &= H^{=}\left[A^*CB^* + BC^*A + Y + Z\right]\left[H^{=}\right]^*\\ &\quad + \left[I - H^-H\right]U\left[I - H^-H\right]^*\end{aligned} \tag{23}$$

such that $H^{=} = \left(A^*A + BB^*\right)^{=}$ represents the class of g-inverses of H given by

$$H^{=} \equiv H^- + \left[I - H^-H\right]S\left[\left(A^*CB^* + BC^*A + Y + Z\right)^{1/2}\right]^-, \tag{24}$$

where $Y, Z \in \mathbb{C}_m^{\geq}$ are arbitrary solutions of

$$YH^-BB^* = A^*CB^*H^-A^*A \tag{25}$$

and

$$A^*AH^-Z = BB^*H^-A^*CB^*, \tag{26}$$

respectively, such that $\left(A^*CB^* + BC^*A + Y + Z\right) \in \mathbb{C}_m^{\geq}$, $S \in \mathbb{C}_m$ and $U \in \mathbb{C}_m^{\geq}$ are free to vary, and $\left[\left(A^*CB^* + BC^*A + Y + Z\right)^{1/2}\right]^-$ is arbitrary but fixed.

Proof. The corollary follows from Lemma 3.3 and the theorem.

5. Two Examples

We now provide two example applications of our main results in Section 4, which were performed using R version 3.2.4.

5.1. Example 1

We utilize an example from Zhang in [1] to illustrate the computational ease and accuracy of our solution. Let

$$A \equiv \begin{bmatrix} 0 & 5 & 9 & 3\\ 2 & 2 & 1 & 3\end{bmatrix}, B \equiv \begin{bmatrix} 1 & 2 & 0\\ -1 & 2 & 1\\ 6 & 2 & -2\\ 3 & 1 & -1\end{bmatrix}, \text{ and } C \equiv \begin{bmatrix} 1 & -1 & 2\\ 1 & 0 & 2\end{bmatrix}, \tag{27}$$

so that $n = 2$, $m = 4$, and $p = 3$. The goal is to determine all Hermitian n.n.d. solutions to

$$AXB = C, \tag{28}$$

where A, B, and C are given in (27).

We first give the general Hermitian n.n.d. solution from Zhang in [1], which is of the form

$$X = \Gamma^{-1}\begin{bmatrix} 1.1078 & 1.6457 & -0.0034 & 0.0177 \\ 1.6457 & a & -0.0023 & -0.1233 \\ -0.0034 & -0.0023 & b & c \\ 0.0177 & -0.1233 & \bar{c} & d \end{bmatrix}\left(\Gamma^*\right)^{-1}, \tag{29}$$

where

$$\Gamma = \begin{bmatrix} 0.0964 & 0.4979 & 0.7709 & 0.3854 \\ 0.5903 & 0.1559 & -0.4868 & 0.6247 \\ -0.1667 & 0.3152 & -1.3266 & -0.6633 \\ -1.4985 & -1.4020 & -1.7901 & -0.8950 \end{bmatrix},$$

and a, b, c, and d are parameters satisfying $a > 2.4448$, $b_1 > 0$, and $d_1 > \bar{c}b_1^+c_1$, where $\begin{bmatrix} b_1 & c_1 \\ \bar{c}_1 & d_1 \end{bmatrix} \in \mathbb{C}_2^{\geq}$ with

$$\begin{cases} b_1 = b - 10^{-3}\times 0.0106 - \dfrac{0.0000}{a-2.4448} \\ c_1 = c + 10^{-3}\times 0.0546 + \dfrac{0.0004}{a-2.4448} \\ d_1 = d - 10^{-3}\times 0.2824 - \dfrac{0.0004}{a-2.4448} \end{cases}.$$

Next, we present our general Hermitian n.n.d. solution to (28). Using Corollary 4.1, we have

$$T = \begin{bmatrix} 0.00005 & 0.00027 & 0.00042 & 0.00021 \\ 0.00027 & 0.00141 & 0.00219 & 0.00109 \\ 0.00042 & 0.00219 & 0.00339 & 0.00170 \\ 0.00021 & 0.00109 & 0.00170 & 0.00085 \end{bmatrix} \in \mathbb{C}_4^{\geq},$$

where $\operatorname{rank}(T) = \operatorname{rank}\left(BB^*H^-A^*CB^*\right) = \operatorname{rank}\left(A^*AH^-BC^*A\right) = 1$. Therefore, a Hermitian n.n.d. solution to (28) exists. Note that

$$H = \begin{bmatrix} 9 & 7 & 12 & 11 \\ 7 & 35 & 43 & 19 \\ 12 & 43 & 126 & 52 \\ 11 & 19 & 52 & 29 \end{bmatrix},$$

where $\operatorname{rank}(H) = 4$. Next, we employ (25) and (26) to obtain

$$Y = \begin{bmatrix} 242.2352 & 898.8824 & 1303.083 & 757.3411 \\ 898.8824 & 3335.5588 & 4835.459 & 2810.3295 \\ 1303.0827 & 4835.4587 & 7009.818 & 4074.0496 \\ 757.3411 & 2810.3295 & 4074.050 & 2367.8047 \end{bmatrix},$$

and

$$Z = \begin{bmatrix} 0.0009 & 0.0009 & -0.0028 & -0.0014 \\ 0.0009 & 0.0009 & -0.0028 & -0.0014 \\ -0.0028 & -0.0028 & 0.0084 & 0.0042 \\ -0.0014 & -0.0014 & 0.0042 & 0.00201 \end{bmatrix}.$$

We remark that $\left(I - H^-H\right) = 0$ in (23), and, thus, from (23), we have that

$$X = \begin{bmatrix} 4.9620 & -3.4333 & 2.2782 & -7.4459 \\ -3.4333 & 2.3751 & -1.6240 & 5.2468 \\ 2.2782 & -1.6230 & 1.0759 & -3.4987 \\ -7.4459 & 5.2468 & -3.4988 & 11.3735 \end{bmatrix} \tag{30}$$

is the unique solution to (28) because $\operatorname{rank}(H) = 4$.

The solution given in (30) contradicts the general Hermitian n.n.d. solution given in (29). We remark that our general solution is closed form and is not obtained algorithmically as that from Zhang in [1].

5.2. Example 2

Next, consider the random matrix $X \sim N_{n,p}(M, \Psi \otimes \Sigma)$, where $\Psi \otimes \Sigma \in \mathbb{R}_{np}^{\geq}$. Several authors have studied the independence of matrix normal-based quadratic forms $X'A_iX$, $i = 1, \cdots, k$, $2 \leq \min(n, p)$. Numerous results can be found in work by Mathai and Provost in [17] and Gupta and Nagar in [18].

In the following corollary, we derive a representation of the general covariance structure of the form $V = \Psi \otimes \Sigma$ of a normal random matrix such that the two matrix quadratic forms $X'AX$ and $X'BX$ are independent when the coefficient matrices $A, B \in \mathbb{R}_n^{\geq}$.

Corollary 2. Let $X \sim N_{n,p}(M, \Psi \otimes \Sigma)$ with $M = 0$, $\Psi \in \mathbb{R}_n^{\geq}$, $\Sigma \in \mathbb{R}_p^{\geq}$, and $A, B \in \mathbb{R}_n^{\geq}$. Then, the two quadratic forms $X'AX$ and $X'BX$ are stochastically independent if and only if

$$\begin{aligned} \Psi &= [A+B]^{=}[Y+Z]\left[(A+B)'\right] \\ &\quad + \left[I - (A+B)^{-}(A+B)\right]U\left[I - (A+B)^{-}(A+B)\right]' \end{aligned} \tag{31}$$

where $[A+B]^{=}$ represents the class of generalized inverses defined by (12), $U \in \mathbb{R}_n^{\geq}$ is free to vary, and $Y, Z \in \mathbb{R}_n^{\geq}$ are arbitrary solutions of

$$Y(A+B)^{-}B = 0$$

and

$$A(A+B)^{-}Z = 0.$$

Proof. By Theorem 6.6b.1 from Mathai and Provost in [17], $X'AX$ and $X'BX$ are stochastically independent if and only if $\Psi A \Psi B \Psi = 0$, $M'A\Psi B\Psi = 0$, $M'B\Psi A\Psi = 0$, and $M'A\Psi BM = 0$. However, a direct application of Lemma 3.6 reduces these conditions to the single equation $A\Psi B = 0$. Thus, by the theorem in Section 4, $X'AX$ and $X'BX$ are stochastically independent if and only if (31) holds.

6. Discussion

In this paper, we derive necessary and sufficient conditions for the existence of a Hermitian n.n.d. solution and a new general Hermitian n.n.d. solution to the matrix equation $AXB = C$. Unlike the proposed n.n.d. solution by Khatri and Mitra in [12], our general representation of X is invariant with respect to the choice of g-inverse. Moreover, using an example from Zhang in [1], we demonstrate that our closed-form general Hermitian n.n.d. solution contradicts the proposed general Hermitian n.n.d. solution from Zhang in [1]. Finally, we apply our main result to obtain the general form of a matrix-normal random matrix with covariance matrix $V = \Psi \otimes \Sigma$ such that two matrix quadratic forms are independent.

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