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Group Inverse of 2 × 2 Block Matrices over Minkowski Space $\mathcal{M}$

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Abstract

Necessary and sufficient conditions for the existence of the group inverse of the block matrix $\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}$ in Minkowski Space are studied, where $P^{\sim}, Q^{\sim}$ are both square and $rank(Q^{\sim}) \geq rank(P^{\sim})$. The representation of this group inverse and some related additive results are also given.

Keywords

Block Matrix, Group Inverse, Minkowski Adjoint, Minkowski Space

1. Introduction

Let F be a skew field and $F^{n\times n}(\mathcal{C})$ be the set of all matrices over F. For $A \in F^{n\times n}(\mathcal{C})$, the matrix $X \in F^{n\times n}(\mathcal{C})$ is said to be the group inverse of A, if

$$AXA = A, \quad XAX = X, \quad AX = XA.$$

and is denoted by $X = A^{\#}$, and is unique by [1].

The generalized inverse of block matrix has important applications in statistical probability, mathematical programming, game theory, control theory etc. and for references see [2] [3] [4]. The research on the existence and the representation of the group inverse for block matrices in Euclidean space has been done in wide range. For the literature of the group inverse of block matrix in Euclidean space, see [5]-[11].

In [12] the existence of anti-reflexive with respect to the generalized reflection anti-symmetric matrix $P^{\sim}$ and solution of the matrix equation $AXB = C$ in Minkowski space $\mathcal{M}$ is given. In [13] necessary and sufficient condition for the existence of Re-nnd solution has been established of the matrix equation $AXA^{\sim} = C$ where

$A \in C^{n\times n}$ and $C \in C^{n\times n}$. In [14] partitioned matrix $M^{\sim}$ in Minkowski space $\mathcal{M}$ was taken of the form $M^{\sim} = \begin{bmatrix} A^{\sim} & -C^{\sim}G_1 \\ -G_1B^{\sim} & D^{*} \end{bmatrix}$ to yield a formula for the inverse of $M^{\sim}$ in terms of the Schur complement of D^{*}.

In this paper P^{*} and $P^{\sim}$ denote the conjugate transpose and Minkowski adjoint of a matrix P respectively. I_n denotes the identity matrix of order $n \times n$. Minkowski Space $\mathcal{M}$ is an indefinite inner product space in which the metric matrix associated with the indefinite inner product is denoted by G and is defined as

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix} \text{ satisfying } G^2 = I_n \text{ and } G^{*} = G.$$

G is called the Minkowski metric matrix. In case $u \in \mathbb{C}^n$, indexed as $u = (u_0, u_1, \cdots, u_{n-1})$, G is called the Minkowski metric tensor and is defined as $Gu = (u_0, -u_1, \cdots, -u_{n-1})$ [12]. For any $P \in C^{n\times n}$, the Minkowski adjoint of P denoted by $P^{\sim}$ is defined as $P^{\sim} = GP^{*}G$ where P^{*} is the usual Hermitian adjoint and G the Minkowski metric matrix of order n. We establish the necessary and sufficient condition for the existence and the representation of the group inverse of a block matrix $\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}$ or $\begin{pmatrix} P^{\sim} & Q^{\sim} \\ P^{\sim} & 0 \end{pmatrix}$ in Minkowski space, where $P^{\sim}, Q^{\sim} \in K^{n\times n}$, $rank(Q^{\sim}) \geq rank(P^{\sim})$. We also give a sufficient condition for $(P^{\sim}Q^{\sim})$ to be similar to $(Q^{\sim}P^{\sim})$.

2. Lemmas

Lemma 1. *Let* $P, Q \in F^{n\times n}(\mathcal{C})$. *If*

$$rank(P) = r, rank(Q) = rank(PQ) = rank(QP),$$

then there are unitary matrices $A, B \in F^{n\times n}(\mathcal{C})$ *such that*

$$P^{\sim} = B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}, \quad Q^{\sim} = (A^{*})^{\sim}\begin{pmatrix} Q_1^{*} & -Q_1^{*}X \\ -YQ_1^{*} & YQ_1^{*}X \end{pmatrix}(B^{*})^{\sim}$$

where $Q_1^{*} \in F^{r\times r}, X \in F^{r\times(n-r)}$ and $Y \in F^{(n-r)\times r}$.

Proof. Since $rank(P) = r$, there are two unitary matrices $A, B \in F^{n\times n}(\mathcal{C})$ such that

$$P = A\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}B, \qquad Q = B^{*}\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}A^{*}$$

where

$$Q_1 \in F^{r\times r}, Q_2 \in F^{r\times(n-r)}, Q_3 \in F^{(n-r)\times r}, Q_4 \in F^{(n-r)\times(n-r)}.$$

Now

$$P^{*} = B^{*}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{*} \Rightarrow P^{\sim} = GB^{*}G^2\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}G^2A^{*}G = B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}$$

and

$$Q = B^* \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} A^*$$

$$Q^* = \left(A^*\right)^* \begin{pmatrix} Q_1^* & Q_3^* \\ Q_2^* & Q_4^* \end{pmatrix} \left(B^*\right)^*$$

$$Q^\sim = G\left(A^*\right)^* G^2 \begin{pmatrix} Q_1^* & Q_3^* \\ Q_2^* & Q_4^* \end{pmatrix} G^2 (B^*)^* G = \left(A^*\right)^\sim \begin{pmatrix} Q_1^* & -Q_3^* \\ -Q_2^* & Q_4^* \end{pmatrix} \left(B^*\right)^\sim$$

From $rank(Q) = rank(PQ)$ we have

$$Q_2^* = YQ_1^*, \qquad Q_4^* = YQ_3^*, \qquad Y \in F^{(n-r)\times r}$$

and from $rank(Q) = rank(QP)$, we get

$$Q_3^* = Q_1^* X, \qquad Q_4^* = Q_2^* X = YQ_1^* X, \qquad X \in F^{r\times(n-r)}$$

So,

$$Q^\sim = \left(A^*\right)^\sim \begin{pmatrix} Q_1^* & -Q_1^* X \\ -YQ_1^* & YQ_1^* X \end{pmatrix} \left(B^*\right)^\sim$$

Lemma 2. *Let*

$$P \in F^{r\times r}, Q \in F^{(n-r)\times r}, M = \begin{pmatrix} P^\sim & 0 \\ Q^\sim & 0 \end{pmatrix} \in F^{n\times n}(\mathcal{C}).$$

Then the group inverse of M exists in $\mathcal{M}$ *if and only if the group inverse of* $P^\sim$ *exists in* $\mathcal{M}$ *and* $rank\left(P^\sim\right) = rank\begin{pmatrix} P^\sim \\ Q^\sim \end{pmatrix}$. *If the group inverse of* M^* *exists in M, then*

$$M^\# = \begin{pmatrix} \left(P^\sim\right)^\# & 0 \\ Q^\sim\left(\left(P^\sim\right)^\#\right)^2 & 0 \end{pmatrix}$$

Proof. Since $M = \begin{pmatrix} P^\sim & 0 \\ Q^\sim & 0 \end{pmatrix}$, suppose group inverse of $P^\sim$ exists in $\mathcal{M}$ and $rank\left(P^\sim\right) = rank\begin{pmatrix} P^\sim \\ Q^\sim \end{pmatrix}$. Now

$$rank(M) = rank\begin{pmatrix} P^\sim & 0 \\ Q^\sim & 0 \end{pmatrix} = rank\begin{pmatrix} P^\sim \\ Q^\sim \end{pmatrix} = rank\left(P^\sim\right).$$

But $rank\left(P^\sim\right) = rank\left(P^\sim\right)^2$ because $\left(P^\sim\right)^\#$ exists $\Rightarrow rank(M) = rank\left(M^2\right)$. Therefore $\left(M^\sim\right)^\#$ exists in $\mathcal{M}$.

Conversely, suppose the group inverse of M exists in $\mathcal{M}$, then it satisfies the following conditions: 1) $MM^\# M = M$, 2) $M^\# MM^\# = M^\#$ and 3) $MM^\# = M^\# M$. Also

$$rank(M) = rank\begin{pmatrix} P^\sim & 0 \\ Q^\sim & 0 \end{pmatrix} = rank\begin{pmatrix} P^\sim \\ Q^\sim \end{pmatrix} \Rightarrow rank\left(P^\sim\right) = rank\begin{pmatrix} P^\sim \\ Q^\sim \end{pmatrix}.$$

Let $M^{\#} = X = \begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}$ then,

1)

$$MXM = \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} P^{\sim}\left(P^{\sim}\right)^{\#}P^{\sim} & 0 \\ Q^{\sim}\left(P^{\sim}\right)^{\#}P^{\sim} & 0 \end{pmatrix} = \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}$$

$$MXM = M$$

2)

$$XMX = \begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \left(P^{\sim}\right)^{\#}P^{\sim}\left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2}P^{\sim}\left(P^{\sim}\right)^{\#} & 0 \end{pmatrix} = \begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}$$

$$XMX = X$$

3)

$$MX = \begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix}\begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix} = \begin{pmatrix} P^{\sim}\left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(P^{\sim}\right)^{\#} & 0 \end{pmatrix}$$

$$XM = \begin{pmatrix} \left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(\left(P^{\sim}\right)^{\#}\right)^{2} & 0 \end{pmatrix}\begin{pmatrix} P^{\sim} & 0 \\ Q^{\sim} & 0 \end{pmatrix} = \begin{pmatrix} P^{\sim}\left(P^{\sim}\right)^{\#} & 0 \\ Q^{\sim}\left(P^{\sim}\right)^{\#} & 0 \end{pmatrix}$$

$$XM = MX.$$

Lemma 3. *Let* $P \in F^{r\times r}, Q \in F^{r\times(n-r)}$, *and* $M = \begin{pmatrix} P^{\sim} & Q^{\sim} \\ 0 & 0 \end{pmatrix} \in F^{n\times n}(\mathcal{C})$. *Then the group inverse of M exists in* $\mathcal{M}$ *if and only if the group inverse of* $P^{\sim}$ *exists in* $\mathcal{M}$ *and* $rank\left(P^{\sim}\right) = rank\left(P^{\sim} \quad Q^{\sim}\right)$. *If the group inverse of M exists in* $\mathcal{M}$, *then,*

$$M^{\#} = \begin{pmatrix} \left(P^{\sim}\right)^{\#} & \left(\left(P^{\sim}\right)^{\#}\right)^{2}Q^{\sim} \\ 0 & 0 \end{pmatrix}$$

Proof. The proof is same as Lemma 2.

Lemma 4. *Let* $P, Q \in F^{n\times n}(\mathcal{C})$. *If*

$$rank\left(P^{\sim}\right) = rank\left(Q^{\sim}\right) = rank\left(P^{\sim}Q^{\sim}\right) = rank\left(Q^{\sim}P^{\sim}\right)$$

then the following conclusions hold:

1) $P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}=P^{\sim}$

2) $P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim}=P^{\sim}$

3) $Q^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim}=Q^{\sim}$

4) $Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}=Q^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}$

5) $P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}=\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}$

Proof. Suppose $rank\left(P\right)=r$, then by Lemma 1 we have

$$P^{\sim}=B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim},\qquad B^{\sim}=\left(A^{*}\right)^{\sim}\begin{pmatrix} Q_1^{*} & -Q_1^{*}X \\ -YQ_1^{*} & YQ_1^{*}X \end{pmatrix}\left(B^{*}\right)^{\sim}$$

where $Q_1^{*}\in F^{r\times r}, X\in F^{r\times(n-r)}, Y\in F^{(n-r)\times r}$. Then

$$P^{\sim}Q^{\sim}=B^{\sim}\begin{pmatrix} Q_1^{*} & -Q_1^{*}X \\ 0 & 0 \end{pmatrix}\left(B^{*}\right)^{\sim},\quad Q^{\sim}P^{\sim}=\left(A^{*}\right)^{\sim}\begin{pmatrix} Q_1^{*} & 0 \\ -YQ_1^{*} & 0 \end{pmatrix}A^{\sim}$$

Since $rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)$ we have that Q_1^{*} is invertible. By using Lemma 2 and 3 we get

$$\left(P^{\sim}Q^{\sim}\right)^{\#}=B^{\sim}\begin{pmatrix} \left(Q_1^{*}\right)^{-1} & -\left(Q_1^{*}\right)^{-1}X \\ 0 & 0 \end{pmatrix}\left(B^{*}\right)^{\sim},$$

$$\left(Q^{\sim}P^{\sim}\right)^{\#}=\left(A^{*}\right)^{\sim}\begin{pmatrix} \left(Q_1^{*}\right)^{-1} & 0 \\ -Y\left(Q_1^{*}\right)^{-1} & 0 \end{pmatrix}A^{\sim}$$

Then, 1)

$$\begin{aligned}
&P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}\\
&=B^{\sim}\begin{pmatrix} Q_1^{*} & -Q_1^{*}X \\ 0 & 0 \end{pmatrix}\left(B^{*}\right)^{\sim}B^{\sim}\begin{pmatrix} \left(Q_1^{*}\right)^{-1} & -\left(Q_1^{*}\right)^{-1}X \\ 0 & 0 \end{pmatrix}\left(B^{*}\right)^{\sim}B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}\\
&=B^{\sim}\begin{pmatrix} Q_1^{*}\left(Q_1^{*}\right)^{-1} & -Q_1^{*}\left(Q_1^{*}\right)^{-1}X \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}\\
&=B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}\\
&=P^{\sim}
\end{aligned}$$

Similarly we can prove 2) - 5).

3. Main Results

Theorem 1. *Let* $M=\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix}$ *where* $P^{\sim},Q^{\sim}\in F^{n\times n}\left(\mathcal{C}\right), rank\left(Q^{\sim}\right)\geq rank\left(P^{\sim}\right)=r$, *then*

1) The group inverse of M exists in $\mathcal{M}$ if and only if

$$rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)=rank\left(Q^{\sim}P^{\sim}\right).$$

2) If the group inverse of M exists in $\mathcal{M}$, then $M^{\#} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$, where

$$\begin{aligned} M_{11} &= \left(P^{\sim}Q^{\sim}\right)^{\#} P^{\sim} - \left(P^{\sim}Q^{\sim}\right)^{\#} \left(P^{\sim}\right)^{2} \left(Q^{\sim}P^{\sim}\right)^{\#} Q^{\sim}, \\ M_{12} &= \left(P^{\sim}Q^{\sim}\right)^{\#} P^{\sim}, \\ M_{21} &= \left(Q^{\sim}P^{\sim}\right)^{\#} Q^{\sim} - Q^{\sim} \left(P^{\sim}Q^{\sim}\right)^{\#} \left(P^{\sim}\right)^{2} \left(Q^{\sim}P^{\sim}\right)^{\#} \\ &\quad + Q^{\sim} \left(P^{\sim}Q^{\sim}\right)^{\#} P^{\sim} \left(P^{\sim}Q^{\sim}\right)^{\#} \left(P^{\sim}\right)^{2} \left(Q^{\sim}P^{\sim}\right)^{\#} Q^{\sim}, \\ M_{22} &= -Q^{\sim} \left(P^{\sim}Q^{\sim}\right)^{\#} \left(P^{\sim}\right)^{2} \left(Q^{\sim}P^{\sim}\right)^{\#}. \end{aligned}$$

Proof. 1) Given $rank\left(Q^{\sim}\right) \ge rank\left(P^{\sim}\right) = r$. Suppose $rank\left(P^{\sim}\right) = rank\left(Q^{\sim}\right)$ then, $rank\left(P^{\sim}\right)^{2} = rank\left(P^{\sim}Q^{\sim}\right)$. We know that

$$rank\left(P^{\sim}Q^{\sim}\right) = rank\left(P^{\sim}\right) \text{ so, } rank\left(P^{\sim}\right)^{2} = rank\left(P^{\sim}\right).$$

Therefore the group inverse of M exists. Now we show that the condition is necessary,

$$rank(M) = rank\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} = rank\begin{pmatrix} 0 & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} = rank\left(P^{\sim}\right) + rank\left(Q^{\sim}\right).$$

$$rank(M)^{2} = rank\begin{pmatrix} \left(P^{\sim}\right)^{2} + P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2} \\ Q^{\sim}P^{\sim} & Q^{\sim}P^{\sim} \end{pmatrix} = rank\begin{pmatrix} P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2} \\ 0 & Q^{\sim}P^{\sim} \end{pmatrix}.$$

Since the group inverse of M exists in $\mathcal{M}$ if and only if $rank(M) = rank(M)^{2}$, we have

$$\begin{aligned} rank\left(P^{\sim}\right) + rank\left(Q^{\sim}\right) &= rank(M)^{2} \\ &\le rank\left(P^{\sim}Q^{\sim}\right) + rank\begin{pmatrix} \left(P^{\sim}\right)^{2} \\ Q^{\sim}P^{\sim} \end{pmatrix} \\ &\le rank\left(P^{\sim}Q^{\sim}\right) + rank\left(P^{\sim}\right). \end{aligned}$$

Also

$$\begin{aligned} rank\left(P^{\sim}\right) + rank\left(Q^{\sim}\right) &= rank(M)^{2} \\ &\le rank\begin{pmatrix} P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2} \end{pmatrix} + rank\left(Q^{\sim}P^{\sim}\right) \\ &\le rank\left(Q^{\sim}P^{\sim}\right) + rank\left(P^{\sim}\right). \end{aligned}$$

Then $rank\left(P^{\sim}\right) \le rank\left(P^{\sim}Q^{\sim}\right)$ and $rank\left(Q^{\sim}\right) \le rank\left(Q^{\sim}P^{\sim}\right)$. Therefore,

$$rank\left(Q^{\sim}\right) = rank\left(P^{\sim}Q^{\sim}\right) = rank\left(Q^{\sim}P^{\sim}\right).$$

From

$$rank\left(Q^{\sim}\right) = rank\left(P^{\sim}Q^{\sim}\right) \le rank\left(P^{\sim}\right)$$

and

$$rank\left(P^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)\leq rank\left(Q^{\sim}\right),$$

we have

$$rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)$$

Since

$$rank\left(P^{\sim}\right)+rank\left(Q^{\sim}\right)\leq rank\left(P^{\sim}Q^{\sim}\quad \left(P^{\sim}\right)^{2}\right)+rank\left(Q^{\sim}P^{\sim}\right)$$

and

$$rank\left(P^{\sim}Q^{\sim}\quad \left(P^{\sim}\right)^{2}\right)\leq rank\left(P^{\sim}\right)\leq rank\left(P^{\sim}Q^{\sim}\quad \left(P^{\sim}\right)^{2}\right),$$

we get

$$rank\left(P^{\sim}Q^{\sim}\quad \left(P^{\sim}\right)^{2}\right)=rank\left(P^{\sim}\right).$$

Thus

$$rank\left(P^{\sim}Q^{\sim}\quad \left(P^{\sim}\right)^{2}\right)=rank\left(P^{\sim}Q^{\sim}\right).$$

Then there exists a matrix $U^{\sim}\in F^{n\times n}(\mathcal{C})$ such that $P^{\sim}Q^{\sim}U^{\sim}=\left(P^{\sim}\right)^{2}$. Then

$$rank\left(M\right)^{2}=rank\begin{pmatrix} P^{\sim}Q^{\sim} & 0 \\ 0 & Q^{\sim}P^{\sim}\end{pmatrix}=rank\left(P^{\sim}Q^{\sim}\right)+rank\left(Q^{\sim}P^{\sim}\right).$$

So, we get

$$rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)=rank\left(Q^{\sim}P^{\sim}\right).$$

2) Let $X=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{pmatrix}$, we will prove that the matrix X satisfies the conditions of the group inverse in $\mathcal{M}$. Firstly we compute

$$MX=\begin{pmatrix} P^{\sim}M_{11}+P^{\sim}M_{21} & P^{\sim}M_{12}+P^{\sim}M_{22} \\ Q^{\sim}M_{11} & Q^{\sim}M_{12}\end{pmatrix}$$

$$XM=\begin{pmatrix} M_{11}P^{\sim}+M_{12}Q^{\sim} & M_{11}P^{\sim} \\ M_{21}Q^{\sim}+M_{22}Q^{\sim} & M_{21}P^{\sim}\end{pmatrix}$$

Applying Lemma 4 1), 2) and 5) we have

$$\begin{aligned} P^{\sim}M_{11}+P^{\sim}M_{21} &= P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}+P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &\quad -P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &\quad +P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}+P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}-P^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &= P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}\end{aligned}$$

$$\begin{aligned} M_{11}P^{\sim}+M_{12}Q^{\sim} &= \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}-\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{\#}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim}+\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}Q^{\sim} \\ &= \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}-\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}P^{\sim}+P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \end{aligned}$$

$$\begin{aligned} P^{\sim}M_{12}+P^{\sim}M_{22} &= P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &= P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-P^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &= P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} \\ &= 0 \end{aligned}$$

$$\begin{aligned} M_{11}P^{\sim} &= \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}-\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P \\ &= \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}-\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2} \\ &= 0 \end{aligned}$$

$$Q^{\sim}M_{11}=Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}$$

$$\begin{aligned} M_{21}P^{\sim}+M_{22}Q^{\sim} &= \left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)P^{\sim} \\ &\quad +Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim} \\ &\quad -Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim} \\ &\quad +Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim} \\ &\quad -Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \end{aligned}$$

$$Q^{\sim}M_{12}=Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}$$

$$\begin{aligned} M_{21}P^{\sim} &= Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim} \\ &\quad +Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim} \\ &= Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim} \\ &\quad +Q^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim} \\ &= Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim} \end{aligned}$$

$$\Rightarrow MX=XM=\begin{pmatrix} P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & 0 \\ Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}-Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim} \end{pmatrix}$$

$$XMX = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & 0 \\ Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

$$MXM = \begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} \begin{pmatrix} P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & 0 \\ Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} & Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim} \end{pmatrix}$$

$$= \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & 0 \end{pmatrix}$$

$$\begin{aligned} Y_{11} &= \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} - \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &\quad + \left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} - \left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= \left(P^{\sim}Q^{\sim}\right)P^{\sim} - \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= M_{11} \end{aligned}$$

$$Y_{12} = \left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} = \left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} = M_{12}$$

$$\begin{aligned} Y_{21} &= \left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &\quad + Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}\left(Q^{\sim}P^{\sim}\right)\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &\quad - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} \\ &\quad + Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= \left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &\quad + Q^{\sim}\left(P^{\sim}Q^{\sim}\right)P^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \end{aligned}$$

$$\begin{aligned} Y_{22} &= -Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} \\ &= -Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)\left(Q^{\sim}P^{\sim}\right)^{\#} \\ &= M_{22} \end{aligned}$$

$$\Rightarrow XMX = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = X$$

Now

$$\begin{aligned} X_{11} &= \left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} + P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} - P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= \left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} + P^{\sim} - \left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} \\ &= P^{\sim} \end{aligned}$$

$$X_{12} = P^{\sim}Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}P^{\sim} = P^{\sim}$$

$$X_{21} = Q^{\sim}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim} = Q^{\sim}$$

$$\Rightarrow MXM = \begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} = M$$

$$\Rightarrow X = M^{\#}. \qquad \square$$

Theorem 2. *Let* $M = \begin{pmatrix} P^{\sim} & Q^{\sim} \\ P^{\sim} & 0 \end{pmatrix}$ *in* $\mathcal{M}$, *where* $P, Q \in F^{n\times n}(\mathcal{C})$,

$$rank\left(Q^{\sim}\right) \geq rank\left(P^{\sim}\right) = r.$$

Then,

1) the group inverse of *M* exists in $\mathcal{M}$ if and only if

$$rank\left(P^{\sim}\right) = rank\left(Q^{\sim}\right) = rank\left(P^{\sim}Q^{\sim}\right) = rank\left(Q^{\sim}P^{\sim}\right).$$

2) if the group inverse of *M* exists in $\mathcal{M}$, then $M^{\#} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$, where

$$Z_{11} = \left(P^{\sim}Q^{\sim}\right)^{\#} P^{\sim} - Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}$$

$$Z_{12} = Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#} - \left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)Q^{\sim} + Q^{\sim}\left(P^{\sim}Q^{\sim}\right)^{\#}(P^{\sim})^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}P^{\sim}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}$$

$$Z_{21} = \left(P^{\sim}Q^{\sim}\right)^{\#} P^{\sim}$$

$$Z_{22} = -\left(P^{\sim}Q^{\sim}\right)^{\#}\left(P^{\sim}\right)^{2}\left(Q^{\sim}P^{\sim}\right)^{\#}Q^{\sim}$$

Proof. 1) Given $rank\left(Q^{\sim}\right) \geq rank\left(P^{\sim}\right) = r$. Suppose $rank\left(P^{\sim}\right) = rank\left(Q^{\sim}\right)$ then,

$$rank\left(P^{\sim}\right)^{2} = rank\left(P^{\sim}Q^{\sim}\right).$$

We know that

$$rank\left(P^{\sim}Q^{\sim}\right) = rank\left(P^{\sim}\right)$$

so,

$$rank\left(P^{\sim}\right)^{2} = rank\left(P^{\sim}\right).$$

Therefore the group inverse of *M* exists in $\mathcal{M}$. Now we show that the condition is necessary,

$$rank(M) = rank\begin{pmatrix} P^{\sim} & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} = rank\begin{pmatrix} 0 & P^{\sim} \\ Q^{\sim} & 0 \end{pmatrix} = rank\left(P^{\sim}\right) + rank\left(Q^{\sim}\right)$$

$$M^{2} = \begin{pmatrix} P^{\sim} & Q^{\sim} \\ P^{\sim} & 0 \end{pmatrix} = \begin{pmatrix} \left(P^{\sim}\right)^{2} + Q^{\sim}P^{\sim} & P^{\sim}Q^{\sim} \\ \left(P^{\sim}\right)^{2} & P^{\sim}Q^{\sim} \end{pmatrix}$$

$$\Rightarrow rank\left(M^{2}\right) = rank\begin{pmatrix} \left(P^{\sim}\right)^{2} + Q^{\sim}P^{\sim} & P^{\sim}Q^{\sim} \\ \left(P^{\sim}\right)^{2} & P^{\sim}Q^{\sim} \end{pmatrix} = rank\begin{pmatrix} Q^{\sim}P^{\sim} & 0 \\ \left(P^{\sim}\right)^{2} & P^{\sim}Q^{\sim} \end{pmatrix}$$

Since the group inverse of *M* exists in $\mathcal{M}$ if and only if $rank(M) = rank\left(M^{2}\right)$. We know

$$rank\left(P^{\sim}\right)+rank\left(Q^{\sim}\right)=rank\left(M^{2}\right)$$
$$\le rank\begin{pmatrix} Q^{\sim}P^{\sim} \\ \left(P^{\sim}\right)^{2}\end{pmatrix}+rank\left(P^{\sim}Q^{\sim}\right)$$
$$\le rank\left(P^{\sim}\right)+rank\left(P^{\sim}Q^{\sim}\right)$$

Also

$$rank\left(P^{\sim}\right)+rank\left(Q^{\sim}\right)=rank\left(M^{2}\right)$$
$$\le rank\left(Q^{\sim}P^{\sim}\right)+rank\begin{pmatrix}\left(P^{\sim}\right)^{2} & P^{\sim}Q^{\sim}\end{pmatrix}$$
$$\le rank\left(Q^{\sim}P^{\sim}\right)+rank\left(P^{\sim}\right).$$

Then $rank\left(Q^{\sim}\right)\le rank\left(P^{\sim}Q^{\sim}\right)$ and $rank\left(Q^{\sim}\right)\le rank\left(Q^{\sim}P^{\sim}\right)$. Therefore

$$rank\left(Q^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)=rank\left(Q^{\sim}P^{\sim}\right)$$

From

$$rank\left(Q^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)\le rank\left(P^{\sim}\right)$$

and

$$rank\left(P^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)\le rank\left(Q^{\sim}\right)$$

we have

$$rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)$$

Since

$$rank\left(P^{\sim}\right)+rank\left(Q^{\sim}\right)\le rank\begin{pmatrix}P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2}\end{pmatrix}+rank\left(Q^{\sim}P^{\sim}\right)$$

and

$$rank\begin{pmatrix}P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2}\end{pmatrix}\le rank\left(P^{\sim}\right)\le rank\begin{pmatrix}P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2}\end{pmatrix},$$

we get

$$rank\begin{pmatrix}P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2}\end{pmatrix}=rank\left(P^{\sim}\right).$$

Thus

$$rank\begin{pmatrix}P^{\sim}Q^{\sim} & \left(P^{\sim}\right)^{2}\end{pmatrix}=rank\left(P^{\sim}Q^{\sim}\right).$$

Then there exist a matrix $U^{\sim}\in F^{n\times n}(\mathcal{C})$ such that $P^{\sim}Q^{\sim}U^{\sim}=\left(P^{\sim}\right)^{2}$. Thus

$$rank\left(M^{\sim}\right)^{2}=rank\begin{pmatrix}P^{\sim}Q^{\sim} & 0\\ 0 & Q^{\sim}P^{\sim}\end{pmatrix}=rank\left(P^{\sim}Q^{\sim}\right)+rank\left(Q^{\sim}P^{\sim}\right)$$

So, we get $rank\left(P^{\sim}\right)=rank\left(Q^{\sim}\right)=rank\left(P^{\sim}Q^{\sim}\right)=rank\left(Q^{\sim}P^{\sim}\right)$.

2) Proof is same as Theorem 1 2).

Theorem 3. *Let* $P, Q \in F^{n\times n}(\mathcal{C})$, *if*

$$rank\left(Q^{\sim}\right) = rank\left(P^{\sim}Q^{\sim}\right) = rank\left(Q^{\sim}P^{\sim}\right).$$

Then $P^{\sim}Q^{\sim}$ *and* $Q^{\sim}P^{\sim}$ *are similar.*

Proof. Suppose $rank(P) = r$, then by using Lemma 1, there are unitary matrices $A, B \in F^{n\times n}(\mathcal{C})$ such that

$$P^{\sim} = B^{\sim}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}A^{\sim}, \quad Q^{\sim} = \left(A^{*}\right)^{\sim}\begin{pmatrix} Q_1^{*} & -YQ_1^{*} \\ -Q_1^{*}X & YQ_1^{*}X \end{pmatrix}\left(B^{*}\right)^{\sim}$$

where $Q_1^{*} \in F^{r\times r}, X \in F^{r\times(n-r)}, Y \in F^{(n-r)\times r}$. Hence

$$\begin{aligned} P^{\sim}Q^{\sim} &= B^{\sim}\begin{pmatrix} Q_1^{*} & -YQ_1^{*} \\ 0 & 0 \end{pmatrix}\left(B^{*}\right)^{\sim} \\ &= B^{\sim}\begin{pmatrix} I_r & Y \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} Q_1^{*} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & -Y \\ 0 & I_{n-r} \end{pmatrix}\left(B^{*}\right)^{\sim} \end{aligned}$$

$$\begin{aligned} Q^{\sim}P^{\sim} &= \left(A^{*}\right)^{\sim}\begin{pmatrix} Q_1^{*} & 0 \\ -Q_1^{*}X & 0 \end{pmatrix}A^{\sim} \\ &= \left(A^{*}\right)^{\sim}\begin{pmatrix} I_r & 0 \\ -X & I_{n-r} \end{pmatrix}\begin{pmatrix} Q_1^{*} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ X & I_{n-r} \end{pmatrix}A^{\sim} \end{aligned}$$

So $P^{\sim}Q^{\sim}$ and $Q^{\sim}P^{\sim}$ are similar.

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