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# Matrix Inequalities for the Fan Product and the Hadamard Product of Matrices

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## Abstract

**A new inequality on the minimum eigenvalue for the Fan product of nonsingular *M*-matrices is given. In addition, a new inequality on the spectral radius of the Hadamard product of nonnegative matrices is also obtained. These inequalities can improve considerably some previous results.**

## Keywords

***M*-Matrix, Nonnegative Matrix, Fan Product, Hadamard Product, Spectral Radius, Minimum Eigenvalue**

## 1. Introduction

Let $A=(a_{ij})\in\mathbb{R}^{n\times n}$, and $N=\{1,2,\cdots,n\}$. We write $A\ge 0$ $(A>0)$ if $a_{ij}\ge 0$ $(a_{ij}>0)$ for any $i,j\in N$. If $A\ge 0$, $A$ is called a nonnegative matrix, and if $A>0$, $A$ is called a positive matrix. The spectral radius of a nonnegative matrix $A$ is denoted by $\rho(A)$.

We denote by $Z_n$ the class of all $n\times n$ real matrices, all of whose off-diagonal entries are nonpositive. A matrix $A=(a_{ij})\in Z_n$ is called an $M$-matrix if there exists a nonnegative matrix $B$ and a nonnegative real number $s$, such that $A=sI-B$ with $s\ge\rho(B)$, where $I$ is the identity matrix. If $s>\rho(B)$ (*resp.*, $s=\rho(B)$), then the $M$-matrix $A$ is nonsingular (*resp.*, *singular*) (see [1] [2]). Denote by $M_n$ the set of nonsingular $M$-matrices. We define $\tau(A)=\min\{Re(\lambda):\lambda\in\sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of $A$.

The Fan product of two matrices $A=(a_{ij})\in\mathbb{C}^{n\times n}$ and $B=(b_{ij})\in\mathbb{C}^{n\times n}$ is the matrix $A\star B=(c_{ij})\in\mathbb{C}^{n\times n}$, where

$$c_{ij}=\begin{cases}-a_{ij}b_{ij}, & \text{if } i\ne j,\\ a_{ii}b_{ii}, & \text{if } i=j.\end{cases}$$

If $A,B\in M_n$, then so is $A\star B$. In ([2], p. 359), a lower bound for $\tau(A\star B)$ was given: if $A,B\in M_n$,

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then $\tau(A\star B)\ge\tau(A)\tau(B)$.

If $A=(a_{ij})\in M_n$, and $a_{ii}>0$, we write $Q=D-A$, where $D=\operatorname{diag}(a_{ii})$. Thus we define $J_A=D^{-1}Q$. Obviously, $J_A$ is nonnegative. Recently, some authors gave some lower bounds of $\tau(A\star B)$ (see [3]-[8]). In [4], Huang obtained the following result for $\tau(A\star B)$,

$$\tau(A\star B)\ge\left(1-\rho(J_A)\rho(J_B)\right)\min_{1\le i\le n}\{a_{ii}b_{ii}\}. \tag{1}$$

The bound of (1) is better than the bound $\tau(A)\tau(B)$ in ([2], p. 359).

In [7], Liu gave a lower bound of $\tau(A\star B)$,

$$\tau(A\star B)\ge\min_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\Gamma_{ij}\right]^{\frac{1}{2}}\right\}, \tag{2}$$

where $\Gamma_{ij}=a_{ii}a_{jj}b_{ii}b_{jj}\rho^2(J_A)\rho^2(J_B)$. The bound of (2) is better than the one of (1).

For a nonnegative matrix $A=(a_{ij})$, let $N=A-D$, where $D=\operatorname{diag}(a_{ii})$. We denote $J'_A=D_1^{-1}N$, where $D_1=\operatorname{diag}(d_{ii})$,

$$d_{ii}=\begin{cases}a_{ii}, & \text{if } a_{ii}\ne 0,\\ 1, & \text{if } a_{ii}=0.\end{cases}$$

The Hadamard product of two matrices $A=(a_{ij})\in\mathbb{C}^{n\times n}$ and $B=(b_{ij})\in\mathbb{C}^{n\times n}$ is the matrix $A\circ B=(a_{ij}b_{ij})\in\mathbb{C}^{n\times n}$. For two nonnegative matrices *A* and *B*, recently, some authors gave several new upper bounds of $\rho(A\circ B)$ (see [3]-[7] [9]). In [4], Huang obtained the following result for $\rho(A\circ B)$,

1) If $a_{ii}b_{ii}\ne 0,\forall i\in N$, then

$$\rho(A\circ B)\le\left(1+\rho(J'_A)\rho(J'_B)\right)\max_{1\le i\le n}\{a_{ii}b_{ii}\}. \tag{3}$$

2) If $a_{i_0i_0}\ne 0$ or $b_{i_0i_0}\ne 0$ for some $i_0$, but $a_{ii}b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\rho(J'_A)\rho(J'_B)\max_{1\le i\le n}\{a_{ii},b_{ii}\}. \tag{4}$$

3) If $a_{ii}=0$ and $b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\rho(J'_A)\rho(J'_B). \tag{5}$$

4) If $a_{i_0i_0}b_{i_0i_0}\ne 0$ and $a_{j_0j_0}b_{j_0j_0}\ne 0$ for some $i_0$, $j_0$, then the upper bound of $\rho(A\circ B)$ is the maximum value of the upper bounds of the inequalities in (3)-(5).

The bound of $\rho(A\circ B)$ in [4] is better than that in ([2], p. 358).

In [7], Liu gave a new upper bound of $\rho(A\circ B)$,

1) If $a_{ii}b_{ii}\ne 0,\forall i\in N$, then

$$\rho(A\circ B)\le\max_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\Omega_{ij}\right]^{\frac{1}{2}}\right\}, \tag{6}$$

where $\Omega_{ij}=a_{ii}a_{jj}b_{ii}b_{jj}\rho^2(J'_A)\rho^2(J'_B)$.

2) If $a_{i_0i_0}\ne 0$ and $a_{j_0j_0}\ne 0$ or $b_{i_0i_0}\ne 0$ and $b_{j_0j_0}\ne 0$ for some $i_0,j_0$, but $a_{ii}b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\rho(J'_A)\rho(J'_B)\max_{i\ne j}\left\{\left(a_{ii}a_{jj}\right)^{\frac{1}{2}},\left(b_{ii}b_{jj}\right)^{\frac{1}{2}}\right\}. \tag{7}$$

3) If $a_{ii}=0$ and $b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\rho(J'_A)\rho(J'_B). \tag{8}$$

4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} \neq 0$ for some $i_0$, $j_0$, then the upper bound of $\rho(A \circ B)$ is the maximum value of the upper bounds of the inequalities in (6)-(8).

The bound of $\rho(A \circ B)$ in [7] is better than that in [4].

The paper is organized as follows. In Section 2, we give a new lower bound of $\tau(A \star B)$. In Section 3, we present a new upper bound of $\rho(A \circ B)$.

## 2. Inequalities for the Fan Product of Two *M*-Matrices

In this section, we will give a new lower bound of $\tau(A \star B)$.

If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $k \geq 0$, we write $A^{(k)} = (a_{ij}^k)$ for the $k$-th Hadamard power of $A$. If $x = (x_i) \in \mathbb{R}^n$ and $k \geq 0$, we write $x^{(k)} = (x_i^k)$.

**Lemma 1.** [7] *Let* $A, B \in M_n$, *and let* $D, E \in \mathbb{R}^{n\times n}$ *be two positive diagonal matrices. Then*

$$D(A \star B)E = (DAE) \star B = (DA) \star (BE) = (AE) \star (DB) = A \star (DBE).$$

**Lemma 2.** [2] *If* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *is a nonnegative matrix and* $k \geq 1$, *then*

$$\rho\left(A^{(k)}\right) \leq \rho^k(A).$$

**Theorem 1.** *Let* $A = (a_{ij})$ *and* $B = (b_{ij}) \in M_n$. *Then*

$$\tau(A \star B) \geq \min_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[\left(a_{ii}b_{ii} - a_{jj}b_{jj}\right)^2 + 4\gamma_{ij}^{(2)}\right]^{\frac{1}{2}}\right\},$$

where $\gamma_{ij}^{(2)} = a_{ii}a_{jj}b_{ii}b_{jj}\rho\left(J_A^{(2)}\right)\rho\left(J_B^{(2)}\right), i, j \in N$.

It is evident that the Theorem holds with equality for $n = 1$. Next, we assume that $n \geq 2$.

(1) First, we assume that $A \star B$ is irreducible matrix, then $A$ and $B$ are irreducible. Obviously $J_A$ and $J_B$ are also irreducible and nonnegative, so $J_A^{(2)}$ and $J_B^{(2)}$ are nonnegative irreducible matrices. Then there exist two positive vectors $\overline{u} = (\overline{u}_i)$ and $\overline{v} = (\overline{v}_i)$ such that $J_A^{(2)}\overline{u} = \rho\left(J_A^{(2)}\right)\overline{u}$ and $J_B^{(2)}\overline{v} = \rho\left(J_B^{(2)}\right)\overline{v}$. Let

$$u = (u_i) = \overline{u}^{\left(\frac{1}{2}\right)} = \left(\overline{u}_i^{\frac{1}{2}}\right), \quad v = (v_i) = \overline{v}^{\left(\frac{1}{2}\right)} = \left(\overline{v}_i^{\frac{1}{2}}\right).$$

Then we have $J_A^{(2)}u^{(2)} = \rho\left(J_A^{(2)}\right)u^{(2)}$ and $J_B^{(2)}v^{(2)} = \rho\left(J_B^{(2)}\right)v^{(2)}$, that is

$$\sum_{j\neq i}\frac{\left|a_{ij}\right|^2 u_j^2}{u_i^2} = a_{ii}^2\rho\left(J_A^{(2)}\right), \quad \sum_{j\neq i}\frac{\left|b_{ij}\right|^2 v_j^2}{v_i^2} = b_{ii}^2\rho\left(J_B^{(2)}\right), \quad i \in N.$$

Let $\tilde{A} = (\tilde{a}_{ij}) = U^{-1}AU$ and $\tilde{B} = (\tilde{b}_{ij}) = V^{-1}BV$ in which $U$ and $V$ are the nonsingular diagonal matrices $U = \operatorname{diag}(u_1, u_2, \cdots, u_n)$ and $V = \operatorname{diag}(v_1, v_2, \cdots, v_n)$. Then, we have

$$\tilde{A} = (\tilde{a}_{ij}) = U^{-1}AU = \begin{pmatrix} a_{11} & \frac{a_{12}u_2}{u_1} & \cdots & \frac{a_{1n}u_n}{u_1} \\ \frac{a_{21}u_1}{u_2} & a_{22} & \cdots & \frac{a_{2n}u_n}{u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}u_1}{u_n} & \frac{a_{n2}u_2}{u_n} & \cdots & a_{nn} \end{pmatrix},$$

$$\tilde{B}=\left(\tilde{b}_{ij}\right)=V^{-1}BV=\begin{pmatrix} b_{11} & \frac{b_{12}v_2}{v_1} & \cdots & \frac{b_{1n}v_n}{v_1} \\ \frac{b_{21}v_1}{v_2} & b_{22} & \cdots & \frac{b_{2n}v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}v_1}{v_n} & \frac{b_{n2}v_2}{v_n} & \cdots & b_{nn} \end{pmatrix}.$$

It is easy to see that $\tilde{A}$, $\tilde{B}$, and $VU$ are nonsingular since $V$ and $U$ are. From Lemma 1, we have

$$(VU)^{-1}(A\star B)(VU)=U^{-1}V^{-1}(A\star B)VU=\left(U^{-1}AU\right)\star\left(V^{-1}BV\right)=\tilde{A}\star\tilde{B}.$$

Thus, we obtain $\tau(A\star B)=\tau\left(\tilde{A}\star\tilde{B}\right)$, and

$$\tilde{A}\star\tilde{B}=\left(c_{ij}\right)=\begin{pmatrix} a_{11}b_{11} & -\frac{a_{12}b_{12}u_2v_2}{u_1v_1} & \cdots & -\frac{a_{1n}b_{1n}u_nv_n}{u_1v_1} \\ -\frac{a_{21}b_{21}u_1v_1}{u_2v_2} & a_{22}b_{22} & \cdots & -\frac{a_{2n}b_{2n}u_nv_n}{u_2v_2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}b_{n1}u_1v_1}{u_nv_n} & -\frac{a_{n2}b_{n2}u_2v_2}{u_nv_n} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

We next consider the minimum eigenvalue $\tau\left(\tilde{A}\star\tilde{B}\right)$ of $\tilde{A}\star\tilde{B}$. Let $\lambda=\tau\left(\tilde{A}\star\tilde{B}\right)$. Then we have that $0<\lambda<a_{ii}b_{ii},\forall i\in N$. By Theorem 1.23 of [10], there exist $i_0,j_0\in N$, $i_0\neq j_0$, such that

$$\left|\lambda-a_{i_0i_0}b_{i_0i_0}\right|\left|\lambda-a_{j_0j_0}b_{j_0j_0}\right|\le\sum_{t\neq i_0}\left|c_{i_0t}\right|\sum_{t\neq j_0}\left|c_{j_0t}\right|.$$

By Hölder's inequality, we have

$$\begin{aligned}\sum_{t\neq i_0}\left|c_{i_0t}\right|\sum_{t\neq j_0}\left|c_{j_0t}\right| &=\sum_{t\neq i_0}\left|\frac{a_{i_0t}b_{i_0t}u_tv_t}{u_{i_0}v_{i_0}}\right|\sum_{t\neq j_0}\left|\frac{a_{j_0t}b_{j_0t}u_tv_t}{u_{j_0}v_{j_0}}\right| \\ &\le\left(\sum_{t\neq i_0}\frac{\left|a_{i_0t}\right|^2u_t^2}{u_{i_0}^2}\sum_{t\neq i_0}\frac{\left|b_{i_0t}\right|^2v_t^2}{v_{i_0}^2}\sum_{t\neq j_0}\frac{\left|a_{j_0t}\right|^2u_t^2}{u_{j_0}^2}\sum_{t\neq j_0}\frac{\left|b_{j_0t}\right|^2v_t^2}{v_{j_0}^2}\right)^{\frac{1}{2}} \\ &=\left(a_{i_0i_0}^2a_{j_0j_0}^2b_{i_0i_0}^2b_{j_0j_0}^2\rho^2\left(J_A^{(2)}\right)\rho^2\left(J_B^{(2)}\right)\right)^{\frac{1}{2}}=\gamma_{i_0j_0}^{(2)}.\end{aligned}$$

Then, we have

$$\left|\lambda-a_{i_0i_0}b_{i_0i_0}\right|\left|\lambda-a_{j_0j_0}b_{j_0j_0}\right|\le\gamma_{i_0j_0}^{(2)}.$$

Since $0<\lambda<a_{ii}b_{ii},\forall i\in N$, then

$$\left(\lambda-a_{i_0i_0}b_{i_0i_0}\right)\left(\lambda-a_{j_0j_0}b_{j_0j_0}\right)\le\gamma_{i_0j_0}^{(2)}.$$

Hence,

$$\lambda\ge\frac{1}{2}\left\{a_{i_0i_0}b_{i_0i_0}+a_{j_0j_0}b_{j_0j_0}-\left[\left(a_{i_0i_0}b_{i_0i_0}-a_{j_0j_0}b_{j_0j_0}\right)^2+4\gamma_{i_0j_0}^{(2)}\right]^{\frac{1}{2}}\right\},$$

*i.e.*,

$$\tau(A\star B)\ge\frac{1}{2}\left\{a_{i_0i_0}b_{i_0i_0}+a_{j_0j_0}b_{j_0j_0}-\left[\left(a_{i_0i_0}b_{i_0i_0}-a_{j_0j_0}b_{j_0j_0}\right)^2+4\gamma_{i_0j_0}^{(2)}\right]^{\frac{1}{2}}\right\}$$
$$\ge\min_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\gamma_{ij}^{(2)}\right]^{\frac{1}{2}}\right\}.$$

(2) Now, assume that $A\star B$ is reducible. It is well known that a matrix in $Z_n$ is a nonsingular $M$-matrix if and only if all its leading principal minors are positive (see [11]). If we denote by $T=(t_{ij})$ the $n\times n$ permutation matrix with $t_{12}=t_{23}=\cdots=t_{n-1,n}=t_{n1}=1$, the remaining $t_{ij}$ zero, then both $A-\epsilon T$ and $B-\epsilon T$ are irreducible nonsingular $M$-matrix for any chosen positive real number $\epsilon$, sufficiently small such that all the leading principal minors of both $A-\epsilon T$ and $B-\epsilon T$ are positive. Now, we substitute $A-\epsilon T$ and $B-\epsilon T$ for $A$ and $B$, respectively, in the previous case, and then letting $\epsilon\to 0$, the result follows by continuity.

**Remark 1.** *By Lemma* 2, *the bound in Theorem* 1 *is better than that in Theorem* 4 *of* [8] *and Theorem* 2 *of* [7].

**Example 1.** Let

$$A=\begin{pmatrix}4&-1&-1&-1\\-2&5&-1&-1\\0&-2&4&-1\\-1&-1&-1&4\end{pmatrix},\quad B=\begin{pmatrix}1&-0.5&0&0\\-0.5&1&-0.5&0\\0&-0.5&1&-0.5\\0&0&-0.5&1\end{pmatrix}.$$

By calculating with Matlab 7.1, it is easy to show that $\tau(A\star B)=3.2296$.

Applying Theorem 4 of [4], Theorem 3.1 of [5], Theorem 2 of [7], and Theorem 3.1 of [8], we have $\tau(A\star B)\ge 1.5239$, $\tau(A\star B)\ge 2.4333$, $\tau(A\star B)\ge 1.5239$, and $\tau(A\star B)\ge 2.9779$, respectively. But, if we apply Theorem 1, we have

$$\tau(A\star B)\ge\min_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\gamma_{ij}^{(2)}\right]^{\frac{1}{2}}\right\}=2.9833.$$

The numerical example shows that the bound in Theorem 1 is better than that in Theorem 4 of [4], Theorem 3.1 of [5], Theorem 2 of [7], and Theorem 3.1 of [8].

## 3. Inequalities for the Hadamard Product of Two Nonnegative Matrices

In this section, we will give a new upper bound of $\rho(A\circ B)$ for nonnegative matrices $A$ and $B$. Similar to [7], for $A=(a_{ij})\ge 0$, write $Q=A-D$, where $D=\operatorname{diag}(a_{ii})$. We denote $J'_A=D_1^{-1}Q$ with $D_1=\operatorname{diag}(d_{ii})$, where

$$d_{ii}=\begin{cases}a_{ii}, & \text{if } a_{ii}\ne 0,\\ 1, & \text{if } a_{ii}=0.\end{cases}$$

Note that $J'_A$ is nonnegative, and $J'_A=A$ if $a_{ii}=0$, $\forall i\in N$. For $B=(b_{ij})\ge 0$, let $D_2=\operatorname{diag}(\delta_{ii})$, where

$$\delta_{ii}=\begin{cases}b_{ii}, & \text{if } b_{ii}\ne 0,\\ 1, & \text{if } b_{ii}=0.\end{cases}$$

Similarly, the nonnegative matrix $J'_B$ is defined.

**Lemma 3.** [2] *Let* $A,B\in\mathbb{R}^{n\times n}$, *and let* $D,E\in\mathbb{R}^{n\times n}$ *be diagonal matrices. Then*

$$D(A\circ B)E=(DAE)\circ B=(DA)\circ(BE)=(AE)\circ(DB)=A\circ(DBE).$$

**Lemma 4.** [12] *Let* $A=(a_{ij})\in\mathbb{R}^{n\times n}$ *be a nonnegative matrix. Then*

$$\rho(A)\le\max_{i\ne j}\frac{1}{2}\left\{a_{ii}+a_{jj}+\left[\left(a_{ii}-a_{jj}\right)^2+4\sum_{k\ne i}a_{ik}\sum_{k\ne j}a_{jk}\right]^{\frac{1}{2}}\right\}.$$

**Theorem 2.** *Let* $A=(a_{ij}), B=(b_{ij})\in\mathbb{R}^{n\times n}$, $A\geq 0$ *and* $B\geq 0$. *Then*

1) If $a_{ii}b_{ii}\neq 0, \forall i\in N$, then

$$\rho(A\circ B)\leq \max_{i\neq j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\eta_{ij}^{(2)}\right]^{\frac{1}{2}}\right\}, \tag{9}$$

where $\eta_{ij}^{(2)}=a_{ii}a_{jj}b_{ii}b_{jj}\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right), i,j\in N$.

2) If $a_{i_0i_0}\neq 0$ and $a_{j_0j_0}\neq 0$ or $b_{i_0i_0}\neq 0$ and $b_{j_0j_0}\neq 0$ for some $i_0, j_0$, but $a_{ii}b_{ii}=0, \forall i\in N$, then

$$\rho(A\circ B)\leq\left(\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right)\right)^{\frac{1}{2}}\max_{i\neq j}\left\{\left(a_{ii}a_{jj}\right)^{\frac{1}{2}},\left(b_{ii}b_{jj}\right)^{\frac{1}{2}}\right\}. \tag{10}$$

3) If $a_{ii}=0$ and $b_{ii}=0, \forall i\in N$, then

$$\rho(A\circ B)\leq\left(\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right)\right)^{\frac{1}{2}}. \tag{11}$$

4) If $a_{i_0i_0}b_{i_0i_0}\neq 0$ and $a_{j_0j_0}b_{j_0j_0}\neq 0$ for some $i_0$, $j_0$, then the upper bound of $\rho(A\circ B)$ is the maximum value of the upper bounds of the inequalities in (9)-(11).

*Proof.* It is evident that 4) holds with equality for $n = 1$. Next, we assume that $n\geq 2$.

(1) First, we assume that $A\circ B$ is irreducible matrix, then $A$ and $B$ are irreducible. Obviously $J_A'$ and $J_B'$ are also irreducible and nonnegative, so $J_A'^{(2)}$ and $J_B'^{(2)}$ are nonnegative irreducible matrices. Then there exist two positive vectors $\hat{u}=(\hat{u}_i)$ and $\hat{v}=(\hat{v}_i)$ such that $J_A'^{(2)}\hat{u}=\rho\left(J_A'^{(2)}\right)\hat{u}$ and $J_B'^{(k)}\hat{v}=\rho\left(J_B'^{(k)}\right)\hat{v}$. Let

$$u=(u_i)=\hat{u}^{\left(\frac{1}{2}\right)}=\left(\hat{u}_i^{\frac{1}{2}}\right),\quad v=(v_i)=\hat{v}^{\left(\frac{1}{2}\right)}=\left(\hat{v}_i^{\frac{1}{2}}\right).$$

Then we have $J_A'^{(2)}u^{(2)}=\rho\left(J_A'^{(2)}\right)u^{(2)}$ and $J_B'^{(2)}v^{(2)}=\rho\left(J_B'^{(2)}\right)v^{(2)}$, that is

$$\sum_{j\neq i}\frac{a_{ij}^2u_j^2}{u_i^2}=d_{ii}^2\rho\left(J_A'^{(2)}\right),\qquad \sum_{j\neq i}\frac{b_{ij}^2v_j^2}{v_i^2}=\delta_{ii}^2\rho\left(J_B'^{(2)}\right).$$

Let $\hat{A}=(\hat{a}_{ij})=U^{-1}AU$ and $\hat{B}=(\hat{b}_{ij})=V^{-1}BV$ in which $U$ and $V$ are the nonsingular diagonal matrices $U=\operatorname{diag}(u_1,u_2,\cdots,u_n)$ and $V=\operatorname{diag}(v_1,v_2,\cdots,v_n)$. Then we have

$$\hat{A}=(\hat{a}_{ij})=U^{-1}AU=\begin{pmatrix} a_{11} & \frac{a_{12}u_2}{u_1} & \cdots & \frac{a_{1n}u_n}{u_1}\\ \frac{a_{21}u_1}{u_2} & a_{22} & \cdots & \frac{a_{2n}u_n}{u_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{a_{n1}u_1}{u_n} & \frac{a_{n2}u_2}{u_n} & \cdots & a_{nn}\end{pmatrix},$$

$$\hat{B}=(\hat{b}_{ij})=V^{-1}BV=\begin{pmatrix} b_{11} & \frac{b_{12}v_2}{v_1} & \cdots & \frac{b_{1n}v_n}{v_1}\\ \frac{b_{21}v_1}{v_2} & b_{22} & \cdots & \frac{b_{2n}v_n}{v_2}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{b_{n1}v_1}{v_n} & \frac{b_{n2}v_2}{v_n} & \cdots & b_{nn}\end{pmatrix}.$$

It is easy to see that $\hat{A}$, $\hat{B}$, and $VU$ are nonsingular since $V$ and $U$ are. From Lemma 4, we have

$$(VU)^{-1}(A\circ B)(VU)=U^{-1}V^{-1}(A\circ B)VU=\left(U^{-1}AU\right)\circ\left(V^{-1}BV\right)=\hat{A}\circ\hat{B}.$$

Thus, we obtain $\rho(A\circ B)=\rho\left(\hat{A}\circ\hat{B}\right)$, and

$$\hat{A}\circ\hat{B}=\left(\hat{c}_{ij}\right)=\begin{pmatrix} a_{11}b_{11} & \frac{a_{12}b_{12}u_2v_2}{u_1v_1} & \cdots & \frac{a_{1n}b_{1n}u_nv_n}{u_1v_1} \\ \frac{a_{21}b_{21}u_1v_1}{u_2v_2} & a_{22}b_{22} & \cdots & \frac{a_{2n}b_{2n}u_nv_n}{u_2v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}b_{n1}u_1v_1}{u_nv_n} & \frac{a_{n2}b_{n2}u_2v_2}{u_nv_n} & \cdots & a_{nn}b_{nn} \end{pmatrix}.$$

We next consider the minimum eigenvalue $\rho\left(\hat{A}\circ\hat{B}\right)$ of $\hat{A}\circ\hat{B}$. For nonnegative irreducible matrices $\hat{A}$ and $\hat{B}$, by definition of the Hadamard product of $\hat{A}$ and $\hat{B}$, Hölder's inequality, and Lemma 5, we have

$$\begin{aligned}
\rho\left(\hat{A}\circ\hat{B}\right) &\le \max_{i\ne j}\frac{1}{2}\left\{\hat{c}_{ii}+\hat{c}_{jj}+\left[\left(\hat{c}_{ii}-\hat{c}_{jj}\right)^2+4\sum_{t\ne i}\hat{c}_{it}\sum_{t\ne j}\hat{c}_{jt}\right]^{\frac{1}{2}}\right\} \\
&= \max_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\sum_{t\ne i}\frac{a_{it}b_{it}u_tv_t}{u_iv_i}\sum_{t\ne j}\frac{a_{jt}b_{jt}u_tv_t}{u_jv_j}\right]^{\frac{1}{2}}\right\} \\
&\le \max_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\left(\sum_{t\ne i}\frac{a_{it}^2u_t^2}{u_i^2}\sum_{t\ne i}\frac{b_{it}^2v_t^2}{v_i^2}\sum_{t\ne j}\frac{a_{jt}^2u_t^2}{u_j^2}\sum_{t\ne j}\frac{b_{jt}^2v_t^2}{v_j^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\} \\
&= \max_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4d_{ii}d_{jj}\delta_{ii}\delta_{jj}\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right)\right]^{\frac{1}{2}}\right\}.
\end{aligned}$$

Thus, we obtain

1) If $a_{ii}b_{ii}\ne 0,\forall i\in N$, then

$$\rho(A\circ B)\le\max_{i\ne j}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left[\left(a_{ii}b_{ii}-a_{jj}b_{jj}\right)^2+4\eta_{ij}^{(2)}\right]^{\frac{1}{2}}\right\}.$$

2) If $a_{i_0i_0}\ne 0$ and $a_{j_0j_0}\ne 0$ or $b_{i_0i_0}\ne 0$ and $b_{j_0j_0}\ne 0$ for some $i_0, j_0$, but $a_{ii}b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\left(\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right)\right)^{\frac{1}{2}}\max_{i\ne j}\left\{\left(a_{ii}a_{jj}\right)^{\frac{1}{2}},\left(b_{ii}b_{jj}\right)^{\frac{1}{2}}\right\}.$$

3) If $a_{ii}=0$ and $b_{ii}=0,\forall i\in N$, then

$$\rho(A\circ B)\le\left(\rho\left(J_A'^{(2)}\right)\rho\left(J_B'^{(2)}\right)\right)^{\frac{1}{2}}.$$

4) If $a_{i_0i_0}b_{i_0i_0}\ne 0$ and $a_{j_0j_0}b_{j_0j_0}\ne 0$ for some $i_0$, $j_0$, then the upper bound of $\rho(A\circ B)$ is the maximum value of the upper bounds of the inequalities in (9)-(11).

(2) Now, we assume that $A \circ B$ is reducible. If we denote by $T = (t_{ij})$ the $n \times n$ permutation matrix with $t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1$, the remaining $t_{ij} = 0$, then both $A + \epsilon T$ and $B + \epsilon T$ are irreducible nonsingular matrices for any chosen positive real number $\epsilon$. Now, we substitute $A + \epsilon T$ and $B + \epsilon T$ for $A$ and $B$, respectively, in the previous case, and then letting $\epsilon \to 0$, the result follows by continuity.

**Remark 2.** *By Lemma* 2, *the bound in Theorem* 2 *is better than that in Theorem* 6 *of* [6] *and Theorem* 3 *of* [9].

**Example 2.** Let

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 4 & 0.5 & 0.5 \\ 1 & 0 & 3 & 0.5 \\ 0.5 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0 & 2 & 0.5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

By calculation with Matlab 7.1, we have $\rho(J'_A) = 0.8182$, $\rho(J'_B) = 1.1258$, $\rho\left(J'^{(2)}_A\right) = 0.3047$, $\rho\left(J'^{(2)}_B\right) = 0.6263$, and $\rho(A \circ B) = 6.3365$.

If we apply Theorem 6 of [4], Theorem 3 of [7], and Theorem 2.2 of [9], we have $\rho(A \circ B) \le 11.5266$, $\rho(A \circ B) \le 9.6221$, and $\rho(A \circ B) \le 9.4116$, respectively. But, if we apply Theorem 2, we have

$$\rho(A \circ B) \le \max_{i \ne j} \frac{1}{2} \left\{ a_{ii} b_{ii} + a_{jj} b_{jj} + \left[ \left( a_{ii} b_{ii} - a_{jj} b_{jj} \right)^2 + 4\eta_{ij}^{(2)} \right]^{\frac{1}{2}} \right\} = 7.3620.$$

The numerical example shows that the bound in Theorem 2 is better than that in Theorem 6 of [4], Theorem 3 of [7], and Theorem 2.2 of [9].

## References


[1] Berman, A. and Plemmons, R.J. (1979) Nonnegaive Matrices in the Mathematical Sciences. Academic Press, New York.

[2] Horn, R.A. and Johnson, C.R. (1985) Topics in Matrix Analysis. Cambridge University Press, Cambridge. http://dx.doi.org/10.1017/CBO9780511810817

[3] Fang, M.Z. (2007) Bounds on Eigenvalues for the Hadamard Product and the Fan Product of Matrices. *Linear Algebra and its Applications*, **425**, 7-15. http://dx.doi.org/10.1016/j.laa.2007.03.024

[4] Huang, R. (2008) Some Inequalities for the Hadamard Product and the Fan Product of Matrices. *Linear Algebra and its Applications*, **428**, 1551-1559. http://dx.doi.org/10.1016/j.laa.2007.10.001

[5] Li, Y.T., Li, Y.Y., Wang, R.W. and Wang, Y.Q. (2010) Some New Lower Bounds on Eigenvalues of the Hadamard Product and the Fan Product of Matrices. *Linear Algebra and its Applications*, **432**, 536-545. http://dx.doi.org/10.1016/j.laa.2009.08.036

[6] Liu, Q.B. and Chen, G.L. (2009) On Two Inequalities for the Hadamard Product and the Fan Product of Matrices. *Linear Algebra and its Applications*, **431**, 974-984. http://dx.doi.org/10.1016/j.laa.2009.03.049

[7] Liu, Q.B., Chen, G.L. and Zhao, L.L. (2010) Some New Bounds on the Spectral Radius of Matrices. *Linear Algebra and its Applications*, **432**, 936-948. http://dx.doi.org/10.1016/j.laa.2009.10.006

[8] Zhou, D.M., Chen, G.L., Wu, G.X. and Zhang, X.Y. (2013) On Some New Bounds for Eigenvalues of the Hadamard Product and the Fan Product of Matrices. *Linear Algebra and its Applications*, **438**, 1415-1426. http://dx.doi.org/10.1016/j.laa.2012.09.013

[9] Zhao, L.L. (2012) Two Inequalities for the Hadamard Product of Matrices. *Journal of Inequalities and Applications*, **2012**, 1-7. http://dx.doi.org/10.1186/1029-242X-2012-122

[10] Varga, R.S. (1962) Matrix Iterative Analysis. Prentice-Hall, Englewood Cliffs.

[11] Berman, A. and Plemmons, R.J. (1994) Nonnegaive Matrices in the Mathematical Sciences. SIAM, Philadelphia. http://dx.doi.org/10.1137/1.9781611971262

[12] Brauer, A. (1947) Limits for the Characteristic Roots of a Matrix Ⅱ. *Duke Mathematical Journal*, **14**, 21-26. http://dx.doi.org/10.1215/S0012-7094-47-01403-8