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Nonlinear Jordan Triple Derivations of Triangular Algebras

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Abstract

In this paper, it is proved that every nonlinear Jordan triple derivation on triangular algebra is an additive derivation.

Keywords

Nonlinear Jordan Triple Derivations, Triangular Algebras, Derivation

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ be an $\mathcal{R}$-algebra. A linear map $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. Additive (linear) derivations are very important maps both in theory and applications, and were studied intensively. More generally, we say that δ is a Jordan triple derivation if $\delta\left(\frac{1}{2}(ABC + CBA)\right) = \frac{1}{2}\left[\delta(A)BC + A\delta(B)C + AB\delta(C) + \delta(C)BA + C\delta(B)A + CB\delta(A)\right]$

for all $A, B, C \in \mathcal{A}$. If the linearity in the definition is not required, the corresponding map is said to be a nonlinear Jordan triple derivation. It should be remarked that there are several definitions of linear Jordan derivations and all of them are equivalent as long as the algebra $\mathcal{A}$ is 2-torsion free. We refer the reader to [1] for more details and related topics. But one can ask whether the equivalence is also true on the condition of nonlinear, and we are still unable to answer this question.

The structures of derivations, Jordan derivations and Jordan triple derivations were systematically studied. Herstein [2] proved that any Jordan derivation from a 2-torsion free prime ring into itself is a derivation, and the famous result of Brešar ([1], Theorem 4.3) states that every Jordan triple derivation from a 2-torsion free semi-prime ring into itself is a derivation. For other results, see [3]-[9] and the references therein.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital algebras over a commutative ring $\mathcal{R}$, and let $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bi-

module, which is faithful as a left $\mathcal{A}$-bimodule, that is, for $A \in \mathcal{A}, A\mathcal{M} = 0 \Rightarrow A = 0$ and a right $\mathcal{B}$-bimodule,

that is, for $B \in \mathcal{A}, \mathcal{M}B = 0 \Rightarrow B = 0$. Recall the algebra $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}$

under the usual matrix addition and formal matrix multiplication is called a triangular algebra [10]. Recently, Zhang [11] characterized that any Jordan derivation on a triangular algebra is a derivation. In this paper we present result corresponding to [11] (Theorem 2.1) for non-linear Jordan triple derivations (there is no linear or additive assumption) on an important algebra: triangular algebra.

As a notational convenience, we will adopt the traditional representations. Let us write $P = \begin{pmatrix} I_\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix}$,

$Q = \begin{pmatrix} 0 & 0 \\ 0 & I_\mathcal{B} \end{pmatrix}$ and $I = \begin{pmatrix} I_\mathcal{A} & 0 \\ 0 & I_\mathcal{B} \end{pmatrix}$ for the identity matrix of the triangular algebra $\mathcal{T}$.

2. The Main Results

In this note, our main result is the following theorem.

Theorem 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a 2-torsion free commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-bimodule and a right $\mathcal{B}$-bimodule. Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular algebra; if δ is a nonlinear Jordan triple derivation on $\mathcal{T}$, δ is an additive derivation.

Lemma 2.1. If δ is a nonlinear Jordan triple derivation on an upper triangular algebra $\mathcal{T}$ generated by $P = P^2 \in \mathcal{T}$, $\delta(P) = PT - TP$ with $T = \delta(P) \in \mathcal{T}$.

Proof. It follows from the fact $P = P^2$ that $\delta(P) = \delta(P)P + P\delta(P)P + P\delta(P)$, which implies that $P\delta(P)P = Q\delta(P)Q = 0$. Thus we have from the fact that $Q\delta(P)P = 0$ that $\delta(P) = P\delta(P)Q = PT - TP$, where $T = \delta(P) \in \mathcal{T}$.

Now define $d(A) = \delta(A) - (AT - TA)$ for each $A \in \mathcal{T}$. Clearly, d is also a nonlinear Jordan triple derivation from $\mathcal{T}$ into itself. It follows from Lemma 2.1 that $d(P) = d(Q) = 0$.

Lemma 2.2. $d(0) = 0$.

Proof. Clearly, $d(0) = d(0)00 + 0d(0)0 + 00d(0) = 0$.

Lemma 2.3. $d(\mathcal{T}_{ij}) \subseteq \mathcal{T}_{ij}, 1 \le i \le j \le 2$.

Proof. Firstly, we prove that $d(2P) \in \mathcal{T}_{11}$. It is clear that

$$d(2P) = d\left(\frac{1}{2}(P \cdot 2P \cdot P + P \cdot 2P \cdot P) \right) = Pd(2P)P, \quad \text{which implies that} \quad d(2P) \in \mathcal{T}_{11}.$$

Let $A_{12} \in \mathcal{T}_{12}$, $d(A_{12}) = d\left(\frac{1}{2}(2P \cdot A_{12} \cdot Q + Q \cdot A_{12} \cdot 2P) \right) = \frac{1}{2}\left[d(2P)A_{12} + 2Pd(A_{12})Q \right]$. Since

$d(2P)A_{12} \in \mathcal{T}_{12}$, we get $d(A_{12}) \in \mathcal{T}_{12}$.

Let $A_{11} \in \mathcal{T}_{11}$, $d(A_{11}) = d\left(\frac{1}{2}(P \cdot A_{11} \cdot P + P \cdot A_{11} \cdot P) \right) = Pd(A_{11})P$, and thus $d(A_{11}) \in \mathcal{T}_{11}$.

Similarly, one can check that $d(\mathcal{T}_{22}) \subseteq \mathcal{T}_{22}$.

Lemma 2.4. $d(2P) = d(2Q) = 0$.

Proof. For any $A_{12} \in \mathcal{T}_{12}$, it follows from Lemma 2.3 that

$$d(A_{12}) = d\left(\frac{1}{2}(2P \cdot P \cdot A_{12} + A_{12} \cdot P \cdot 2P) \right) = \frac{1}{2}\left[d(2P)A_{12} + 2Pd(A_{12}) \right]. \quad \text{This implies that} \quad Pd(2P)A_{12} = 0. \quad \text{Since}$$

$P\mathcal{T}Q$ is a faithful left $P\mathcal{T}P$-module, we have that

$$Pd(2P)P = 0.$$

It follows from $d(2P) \in \mathcal{T}_{11}$, we have $d(2P) = 0$. Similarly, we can get that $d(2Q) = 0$.

Lemma 2.5. For any $A_{11}, B_{11} \in \mathcal{T}_{11}, A_{12}, B_{12} \in \mathcal{T}_{12}, A_{22}, B_{22} \in \mathcal{T}_{22}$, we have

(1) $d(A_{11}B_{12}) = d(A_{11})B_{12} + A_{11}d(B_{12})$, (2) $d(A_{12}B_{22}) = d(A_{12})B_{22} + A_{12}d(B_{22})$,

(3) $d(A_{11}B_{11}) = d(A_{11})B_{11} + A_{11}d(B_{11})$, (4) $d(A_{22}B_{22}) = d(A_{22})B_{22} + A_{22}d(B_{22})$.

Proof. (1) For any $A_{11} \in \mathcal{T}_{11}, B_{12} \in \mathcal{T}_{12}$, it follows from Lemma 2.3 and 2.4; we have

$$d(A_{11}B_{12}) = d\left(\frac{1}{2}(2P \cdot A_{11} \cdot B_{12} + B_{12} \cdot A_{11} \cdot 2P)\right)$$

$$= \frac{1}{2}\Big[d(2P)A_{11}B_{12} + 2Pd(A_{11})B_{12} + 2A_{11}d(B_{12})$$

$$+ 2d(B_{12})A_{11} + 2B_{12}d(A_{11})P + B_{12}A_{11}d(2P)\Big]$$

$$= d(A_{11})B_{12} + A_{11}d(B_{12}).$$

(2) is proved similarly.

(3) For any $A_{11}, B_{11} \in \mathcal{T}_{11}, M_{12} \in \mathcal{T}_{12}$, by Lemma 2.5 (1), we get that

$$d(A_{11}B_{11}M_{12}) = d(A_{11}B_{11})M_{12} + A_{11}B_{11}d(M_{12}). \tag{1}$$

On the other hand,

$$d(A_{11}B_{11}M_{12}) = d(A_{11}(B_{11}M_{12})) = d(A_{11})B_{11}M_{12} + A_{11}d(B_{11}M_{12})$$

$$= d(A_{11})B_{11}M_{12} + A_{11}d(B_{11})M_{12} + A_{11}B_{11}d(M_{12}).$$

This and Equation (1) imply that

$$(d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}))M_{12} = 0.$$

Since PTQ is a faithful left PTP-module and $d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}) \in \mathcal{T}_{11}$, we get $d(A_{11}B_{11}) - d(A_{11})B_{11} - A_{11}d(B_{11}) = 0$, that is

$$d(A_{11}B_{11}) = d(A_{11})B_{11} + A_{11}d(B_{11}).$$

Similarly, (4) is true for all $A_{22}, B_{22} \in \mathcal{T}_{22}$.

Lemma 2.6. $d(A_{11} + A_{12}) = d(A_{11}) + d(A_{12})$ and $d(A_{12} + A_{22}) = d(A_{12}) + d(A_{22})$.

Proof. Let $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}$, it follows from Lemma 2.2 and 2.4, we have that

$$0 = d\left(\frac{1}{2}(2Q \cdot (A_{11} + A_{12}) \cdot Q + Q \cdot (A_{11} + A_{12}) \cdot 2Q)\right) = Qd(A_{11} + A_{12})Q, \text{ that is, } d(A_{11} + A_{12}) \in \mathcal{T}_{11} + \mathcal{T}_{12}.$$

For any $X_{12} \in \mathcal{T}_{12}$, it follows from Lemma 2.5 (1), we have

$$d(A_{11}X_{12}) = d(A_{11})X_{12} + A_{11}d(X_{12}). \tag{2}$$

On the other hand,

$$d(A_{11}X_{12}) = d\left(\frac{1}{2}(2P \cdot (A_{11} + A_{12}) \cdot X_{12} + X_{12} \cdot (A_{11} + A_{12}) \cdot 2P)\right)$$

$$= \frac{1}{2}\Big[d(2P)(A_{11} + A_{12})X_{12} + 2Pd(A_{11} + A_{12})X_{12} + 2(A_{11} + A_{12})d(X_{12})$$

$$+ 2d(X_{12})(A_{11} + A_{12})P + 2X_{12}d(A_{11} + A_{12})P + X_{12}(A_{11} + A_{12})d(2P)\Big]$$

$$= d(A_{11} + A_{12})X_{12} + A_{11}d(X_{12}).$$

This and Equation (2) imply that $\big[d(A_{11} + A_{12}) - d(A_{11})\big]X_{12} = 0$. Since PTQ is a faithful left PTP-module; hence

$$P\big[d(A_{11} + A_{12}) - d(A_{11})\big]P = 0.$$

Similarly, let $A_{12} \in \mathcal{T}_{12}$, for any $X_{22} \in \mathcal{T}_{22}$, then

$$d(A_{12}X_{22}) = d(A_{12})X_{22} + A_{12}d(X_{22}).$$

On the other hand,

$$d(A_{12}X_{22}) = d\left(\frac{1}{2}(2P \cdot (A_{11} + A_{12}) \cdot X_{22} + X_{22} \cdot (A_{11} + A_{12}) \cdot 2P)\right)$$

$$= d(A_{11} + A_{12})X_{22} + A_{12}d(X_{22}).$$

Therefore, we get $\left[d\left(A_{11}+A_{12}\right)-d\left(A_{12}\right)\right]X_{22}=0,$ that is

$P\left[d\left(A_{11}+A_{12}\right)-d\left(A_{12}\right)\right]Q+Q\left[d\left(A_{11}+A_{12}\right)-d\left(A_{12}\right)\right]Q=0.$ So

$P\left[d\left(A_{11}+A_{12}\right)-d\left(A_{12}\right)\right]Q=0$ and $Q\left[d\left(A_{11}+A_{12}\right)-d\left(A_{12}\right)\right]Q=0.$ Therefor combining Lemma 2.3, we have

$d\left(A_{11}+A_{12}\right)-d\left(A_{11}\right)-d\left(A_{12}\right)=P\left[d\left(A_{11}+A_{12}\right)-d\left(A_{11}\right)-d\left(A_{12}\right)\right]P$

$+P\left[d\left(A_{11}+A_{12}\right)-d\left(A_{11}\right)-d\left(A_{12}\right)\right]Q+Q\left[d\left(A_{11}+A_{12}\right)-d\left(A_{11}\right)-d\left(A_{12}\right)\right]Q=0.$ that is,

$d\left(A_{11}+A_{12}\right)=d\left(A_{11}\right)+d\left(A_{12}\right).$

Similarly, (2) is true for all $A_{12}\in\mathcal{T}_{12}$ and $A_{22}\in\mathcal{T}_{22}$.

Lemma 2.7. $d\left(A_{11}+A_{12}+A_{22}\right)=d\left(A_{11}\right)+d\left(A_{12}\right)+d\left(A_{22}\right).$

Proof. For any $A_{11}\in\mathcal{T}_{11},A_{12}\in\mathcal{T}_{12},A_{22}\in\mathcal{T}_{22}$,

$d\left(A_{11}\right)=d\left(\frac{1}{2}\left(P\left(A_{11}+A_{12}+A_{22}\right)P+P\left(A_{11}+A_{12}+A_{22}\right)P\right)\right)=Pd\left(A_{11}+A_{12}+A_{22}\right)P.$

$d\left(A_{12}\right)=d\left(\frac{1}{2}\left(2P\left(A_{11}+A_{12}+A_{22}\right)Q+Q\left(A_{11}+A_{12}+A_{22}\right)2P\right)\right)=Pd\left(A_{11}+A_{12}+A_{22}\right)Q.$

$d\left(A_{22}\right)=d\left(\frac{1}{2}\left(Q\left(A_{11}+A_{12}+A_{22}\right)Q+Q\left(A_{11}+A_{12}+A_{22}\right)Q\right)\right)=Qd\left(A_{11}+A_{12}+A_{22}\right)Q.$

Thus,

$$d\left(A_{11}+A_{12}+A_{22}\right)=Pd\left(A_{11}+A_{12}+A_{22}\right))P+Pd\left(A_{11}+A_{12}+A_{22}\right)Q$$
$$+Qd\left(A_{11}+A_{12}+A_{22}\right)Q$$
$$=d\left(A_{11}\right)+d\left(A_{12}\right)+d\left(A_{22}\right).$$

Lemma 2.8. For any $A_{12},B_{12}\in\mathcal{T}_{12}$, we have $d\left(A_{12}+B_{12}\right)=d\left(A_{12}\right)+d\left(B_{12}\right).$

Proof. For any $A_{12},B_{12}\in\mathcal{T}_{12}$, from Lemma 2.3 and 2.6, we have

$$d\left(A_{12}+B_{12}\right)=d\left(\frac{1}{2}\left(2P\cdot\left(P+A_{12}\right)\cdot\left(Q+B_{12}\right)+\left(Q+B_{12}\right)\cdot\left(P+A_{12}\right)\cdot 2P\right)\right)$$
$$=Pd\left(P+A_{12}\right)\left(Q+B_{12}\right)+P\left(P+A_{12}\right)d\left(Q+B_{12}\right)$$
$$+d\left(Q+B_{12}\right)\left(P+A_{12}\right)P+\left(Q+B_{12}\right)d\left(P+A_{12}\right)P$$
$$=Pd\left(A_{12}\right)Q+Pd\left(B_{12}\right)=d\left(A_{12}\right)+d\left(B_{12}\right).$$

Lemma 2.9. d is additive on $\mathcal{T}_{11}$ and $\mathcal{T}_{22}$ respectively.

Proof. For any $A_{11},B_{11}\in\mathcal{T}_{11}$, by Lemma 2.5 (1), we have

$$d\left(\left(A_{11}+B_{11}\right)B_{12}\right)=d\left(A_{11}+B_{11}\right)B_{12}+\left(A_{11}+B_{11}\right)d\left(B_{12}\right). \tag{3}$$

on the other hand, from Lemma 2.5 (1) and 2.8, we get that

$$d\left(\left(A_{11}+B_{11}\right)B_{12}\right)=d\left(A_{11}B_{12}+B_{11}B_{12}\right)=d\left(A_{11}\right)B_{12}+A_{11}d\left(B_{12}\right)+d\left(B_{11}\right)B_{12}+B_{11}d\left(B_{12}\right).$$

This and Equation (3) imply that

$$\left(d\left(A_{11}+B_{11}\right)-d\left(A_{11}\right)-d\left(B_{11}\right)\right)B_{12}=0.$$

Since $P\mathcal{T}Q$ is a faithful left $P\mathcal{T}P$-module and $d\left(A_{11}+B_{11}\right)-d\left(A_{11}\right)-d\left(B_{11}\right)\in\mathcal{T}_{11}$, we have that $d\left(A_{11}+B_{11}\right)-d\left(A_{11}\right)-d\left(B_{11}\right)=0,$ that is

$$d\left(A_{11}+B_{11}\right)=d\left(A_{11}\right)+d\left(B_{11}\right).$$

Similarly, we can also get the additivity of d on $\mathcal{T}_{22}$.

Lemma 2.10. d is additivity.

Proof. For any $A,B\in\mathcal{T}$, write $A=A_{11}+A_{12}+A_{22},B=B_{11}+B_{12}+B_{22}$, where $A_{ij},B_{ij}\in\mathcal{T}_{ij},1\le i\le j\le 2.$ Then Lemma 2.7-2.9 are all used in seeing the equation

$$d(A+B) = d(A_{11}+A_{12}+A_{22}+B_{11}+B_{12}+B_{22})$$
$$= d(A_{11}+B_{11})+d(A_{12}+B_{12})+d(A_{22}+B_{22})$$
$$= d(A_{11})+d(B_{11})+d(A_{12})+d(B_{12})+d(A_{22})+d(B_{22})$$
$$= d(A_{11}+A_{12}+A_{22})+d(B_{11}+B_{12}+B_{22})$$
$$= d(A)+d(B).$$

Lemma 2.11. $d(AB) = d(A)B + Ad(B)$ for all $A, B \in \mathcal{T}$.

Proof. For any $A, B \in \mathcal{T}$, let $A = A_{11}+A_{12}+A_{22}, B = B_{11}+B_{12}+B_{22}$, where $A_{ij}, B_{ij} \in \mathcal{T}_{ij}, 1 \le i \le j \le 2$. Now we have that by Lemma 2.5 (1)-(4), Lemma 2.7 and 2.8

$$d(AB) = d(A_{11}B_{11}+A_{11}B_{12}+A_{12}B_{22}+A_{22}B_{22})$$
$$= d(A_{11}B_{11})+d(A_{11}B_{12})+d(A_{12}B_{22})+d(A_{22}B_{22})$$
$$= d(A_{11})B_{11}+A_{11}d(B_{11})+d(A_{11})B_{12}+A_{11}d(B_{12})$$
$$+ d(A_{12})B_{22}+A_{12}d(B_{22})+d(A_{22})B_{22}+A_{22}d(B_{22}).$$

On the other hand, it follows from Lemma 2.3, 2.7; we get that

$$d(A)B + Ad(B)$$
$$= d(A_{11}+A_{12}+A_{22})(B_{11}+B_{12}+B_{22})+(A_{11}+A_{12}+A_{22})d(B_{11}+B_{12}+B_{22})$$
$$= d(A_{11})B_{11}+A_{11}d(B_{11})+d(A_{11})B_{12}+A_{11}d(B_{12})$$
$$+ d(A_{12})B_{22}+A_{12}d(B_{22})+d(A_{22})B_{22}+A_{22}d(B_{22}).$$

It is clear that $d(AB) = d(A)B + Ad(B)$ for all $A, B \in \mathcal{T}$.

Proof of Theorem 2.1. From the above lemmas, we have proved that d is an additive derivation on $\mathcal{T}$. Since $d(A) = \delta(A) - (AT - TA)$ for each $A \in \mathcal{T}$, by a simple calculation, we see that δ is also an additive derivation. The proof is completed.

Acknowledgements

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