Modulus-Based Matrix Splitting Iteration Methods for a Class of Stochastic Linear Complementarity Problem

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Abstract

For the expected value formulation of stochastic linear complementarity problem, we establish modulus-based matrix splitting iteration methods. The convergence of the new methods is discussed when the coefficient matrix is a positive definite matrix or a positive semi-definite matrix, respectively. The advantages of the new methods are that they can solve the large scale stochastic linear complementarity problem, and spend less computational time. Numerical results show that the new methods are efficient and suitable for solving the large scale problems.

Keywords

Stochastic Linear Complementarity Problem, Modulus-Based Matrix Splitting, Expected Value Formulation, Positive Semi-Definite Matrix

1. Introduction

The complementarity problems have been widely used in the engineering design, information technology, economic equilibrium, etc. Since some elements may involve uncertain data in practical applications, many problems can be attributed to stochastic variational inequality problems or stochastic linear complementarity problems, and arouse the attention of many researchers. Gurkan *et al.* [1] proposed the expected value (EV) formulation of stochastic variational inequality by using the sample-path method. Chen and Fukushima [2] proposed the expected residual minimization (ERM) formulation for stochastic linear complementarity problems by quasi-Monte Carlo methods. Lin and Fukushima [3] proposed the stochastic mathematical programs with equilibrium constraints

(SMPEC) for the stochastic nonlinear complementarity problems. Zhou and Cacceta [4] transformed the monotone stochastic linear complementarity problem (SLCP) in finite sample space into a constrained minimization problem, and solved it with the Feasible Semi-smooth Newton Method. Mangasarian and Ren [5] given a new error bound for the monotone LCP based on the error bounds. Chen *et al.* [6] studied the SLCP involving a random matrix whose expectation matrix is positive semi-definite. Zhang and Chen [7] proposed a smooth projection gradient algorithm to solve the SLCP. However, these methods are only suitable for solving the small-scale SLCP.

In recent years, some scholars have proposed a series of methods for the study of large-scale complementarity problems. Dong and Jiang [8] proposed a class of modified modulus-based method. Bai [9] presented a class of modulus-based matrix splitting iteration methods. Bai and Evans [10] [11] also proposed a class of modulus-based synchronous multi-splitting (MSM) iteration methods. Bai and Zhang [12] further proposed a synchronous two-stage multi-splitting iteration method, which can be applied to solving the large-scale linear complementarity problems. Zhang [13] summarized the latest development and achievements of the modulus-based matrix splitting iteration methods, including the corresponding multi-splitting iteration methods, etc. Zhang [14] improved the convergence theorem of matrix multi-splitting methods for linear complementarity problems. Such methods are easy to be implemented and very efficient in practical applications, and there is no need to project iteration results into space R_+^n. Li *et al.* [15] [16] [17] [18] applied a class of modulus-based matrix splitting iteration methods to solving the nonlinear complementarity problem. Numerical results show that the methods are efficient. In the past decade many scholars have made many new achievements in this field, see the literatures [19]-[30].

In this paper, we extend the modulus-based matrix splitting iteration methods to solve the large-scale stochastic linear complementarity problems. We also prove the convergence of these methods when the coefficient matrix is a positive definite matrix or a positive semi-definite matrix. The numerical results show that these methods are efficient.

The outline of the paper is as follows. In Section 2 we present some necessary results and lemmas. In Section 3 we establish the modulus-based matrix splitting iteration methods for solving the SLCP. The convergence of the methods is proved in Section 4. The numerical results are shown in Section 5. Finally, in Section 6, we give some concluding remarks.

2. Preliminaries

In this section, we briefly introduce some necessary results and lemmas.

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$, A is said to be positive semi-definite if $x^{\mathrm{T}} A x \geq 0$ for all $x \in \mathbb{R}^n$, and positive definite if $x^{\mathrm{T}} A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. $A \in R^{n \times n}$ is called a P_0-matrix if all of its principle minors are nonnegative.

Let (Ω, F, P) be a probability space, where Ω is a sample subset of $\mathbb{R}^m$.

Suppose that probability distribution is known, we consider the stochastic linear complementarity problem (SLCP): finding a vector $z \in \mathbb{R}^n$ such that

$$M(\omega)z + q(\omega) \geq 0, \ z \geq 0, \ z^\mathrm{T}\left(M(\omega)z + q(\omega)\right) = 0, \ \omega \in \Omega. \tag{1}$$

where $M(\omega) \in \mathbb{R}^{n \times n}$ and $q(\omega) \in \mathbb{R}^n$ are the rand matrices and vectors for $\omega \in \Omega$, respectively.

Usually there not exists z for all $\omega \in \Omega$ for Problem (1). In order to get a reasonable solution of (1), in this paper we use the EV formulation proposed by Gurkan *et al.* [1].

The Expected Value (EV) Formulation [1]:

Let $F(z, \omega) = M(\omega)z + q(\omega)$, $\bar{M} = \mathrm{E}\left[M(\omega)\right]$, $\mathrm{q} = \mathrm{E}\left[q(\omega)\right]$, and E be the expectation. We consider the following EV formulation: finding a vector $z \in \mathbb{R}^n$ such that

$$\bar{F}(z) = E\left[F(z, \omega)\right] = \bar{M}z + \mathrm{q} \geq 0, \ z \geq 0, z^\mathrm{T}\bar{F}(z) = 0. \tag{2}$$

We briefly denote it as $\mathrm{LCP}(\mathrm{q}, \bar{M})$.

Define

$$\mathrm{RES}(z) := \min\left(z, \bar{M}z + \mathrm{q}\right)$$

where the min operator denotes the componentwise minimum of two vectors. It is generally known that z^* solves the $\mathrm{LCP}(\mathrm{q}, \bar{M})$ if and only if z^* solves the equations

$$\mathrm{RES}(z) = 0$$

The function RES is called the natural residual of the $\mathrm{LCP}(\mathrm{q}, \bar{M})$ and is often used in error analysis.

Lemma 1 (see [8]) Let $\alpha \in (0, +\infty)$ be a scalar, then the $\mathrm{LCP}(\mathrm{q}, \bar{M})$ (2) is equivalent to the following fixed-point problem: finding $x \in \mathbb{R}^n$, satisfying that

$$\left(\alpha I + \bar{M}\right)x = \left(\alpha I - \bar{M}\right)|x| - \mathrm{q} \tag{3}$$

Moreover, if x is the solution of (3), then

$$r := \alpha\left(|x| - x\right), \ z := |x| + x \tag{4}$$

define a solution pair of Problem (2). On the other hand, if the vector pair z and r solves Problem (2), then $x := 1/2\left(z - r/\alpha\right)$ solves the fixed-point problem (3).

3. Modulus-Based Matrix Splitting Iteration Methods

In this section, we aim at the EV formulation of the stochastic linear complementarity problem (2). We give some corresponding modulus-based matrix splitting iteration methods.

For the strong monotone stochastic linear complementarity problem, the coefficient matrix is positive definite. For this case, we can apply the method proposed by Dong and Jiang [8].

Method 3.1

Step 1: Select an arbitrary initial vector $x^{(0)} \in \mathbb{R}^n$ and set $k := 0$;

Step 2: Calculate $x^{(k+1)}$ through the iteration scheme

$$\left(\alpha I+\bar{M}\right)x^{(k+1)}=\left(\alpha I-\bar{M}\right)\left|x^{(k)}\right|-q$$

Step 3: Let $z^{(k+1)}=\left|x^{(k+1)}\right|+x^{(k+1)}$, if $z^{(k+1)}$ satisfies the termination rule, then stop; otherwise, set $k:=k+1$ and return to Step 2.

Unfortunately, the coefficient matrices of some stochastic linear complementarity problems are positive semi-definite, Method 3.1 is not suitable for solving the problem (2). Cottle *et al.* [31] presented a regularization method. Based on this method, we establish a regularized modulus-based matrix splitting iteration method. To simplify the notation, we will denote $\{\varepsilon\}$ and $\left\{x_\varepsilon^k\right\}$ for $\{\varepsilon_k\}$ and $\left\{x_{\varepsilon_k}^k\right\}$, and denote the regularization problem for $\text{LCP}\left(q,\bar{M}+\varepsilon I\right)$.

Method 3.2

Step 1: Select a positive number $\varepsilon_0\in R$ and an arbitrary initial vector $x_\varepsilon^{(0)}\in\mathbb{R}^n$, and set $k:=0$;

Step 2: Generate the iteration sequence $x_\varepsilon^{(k+1)}$ through solving the following equations

$$\left(\alpha I+\bar{M}+\varepsilon I\right)x_\varepsilon^{(k+1)}=\left(\alpha I-\bar{M}-\varepsilon I\right)\left|x_\varepsilon^{(k)}\right|-q.$$

Let $z_\varepsilon^{(k+1)}=\left|x_\varepsilon^{(k+1)}\right|+x_\varepsilon^{(k+1)}$.

Step 3: Set $\varepsilon=\alpha\varepsilon$, where $\alpha\in(0,1)$ is a positive number, $k:=k+1$, and return to Step 2.

4. Convergence

In this section, we analyze the convergence of Method 3.1 and Method 3.2 when the coefficient matrix of the $\text{LCP}\left(q,\bar{M}\right)$ is a symmetric positive definite matrix and a symmetric positive semi-definite matrix.

4.1. The Case of Symmetric Positive Definite Matrix

We first discuss the convergence of Method 3.1 when the coefficient matrix is symmetric positive definite.

Theorem 1 Suppose that the system matrix $\bar{M}\in\mathbb{R}^{n\times n}$ is symmetric positive definite, then the sequence $\left\{x^{(k)}\right\}$ generated by Method 3.1 converges to x^*.

Proof. By Lemma 1 we get

$$x^{(k+1)}=\left(\alpha I+\bar{M}\right)^{-1}\left(\alpha I-\bar{M}\right)\left|x^{(k)}\right|-\left(\alpha I+\bar{M}\right)^{-1}q.$$

If x^* is a solution of (3), then

$$x^*=\left(\alpha I+\bar{M}\right)^{-1}\left(\alpha I-\bar{M}\right)\left|x^*\right|-\left(\alpha I+\bar{M}\right)^{-1}q.$$

We can get that

$$\begin{aligned}
\left\|x^{(k+1)}-x^*\right\|&=\left\|\left(\alpha I+\bar{M}\right)^{-1}\left(\alpha I-\bar{M}\right)\left(\left|x^{(k)}\right|-\left|x^*\right|\right)\right\|\\
&\leq\left\|\left(\alpha I+\bar{M}\right)^{-1}\left(\alpha I-\bar{M}\right)\right\|\left\|\left|x^{(k)}\right|-\left|x^*\right|\right\|\\
&\leq\left\|\left(\alpha I+\bar{M}\right)^{-1}\left(\alpha I-\bar{M}\right)\right\|\left\|x^{(k)}-x^*\right\|
\end{aligned}$$

Since matrix $\bar{M}$ is a symmetric positive definite, we have

$$\left\|\left(\alpha I + \bar{M}\right)^{-1}\left(\alpha I - \bar{M}\right)\right\| = \max_{\lambda_i \in \lambda(\bar{M})} \left|\frac{\alpha - \lambda_i}{\alpha + \lambda_i}\right| := \sigma(\alpha).$$

where $\lambda(\bar{M})$ denotes the set of all the eigenvalues of $\bar{M}$. As $\lambda_i > 0$, it follows that

$$\left|\frac{\alpha - \lambda_i}{\alpha + \lambda_i}\right| < 1$$

and thus

$$\left\|\left(\alpha I + \bar{M}\right)^{-1}\left(\alpha I - \bar{M}\right)\right\| = \sigma(x) < 1.$$

Hence, by the Banach contraction mapping theorem, we have the convergence of the infinite sequence $\left\{x^{(k)}\right\}$ to the unique solution x^* of the fixed-point equation.

4.2. The Case of Symmetric Positive Semi-Definite Matrix

We now discuss the convergence of Method 3.2 when the coefficient matrix is symmetric positive semi-definite.

Lemma 2 (See [31]) Let $\bar{M}$ be a P_0 matrix, $\{\varepsilon\}$ be a decreasing sequence, where ε is a positive scalar with $\varepsilon \to 0$. For each k, let z^k be the unique solution of the $\mathrm{LCP}\left(q, \bar{M} + \varepsilon I\right)$.

1) If $\bar{M} \in R_0$, then the sequence $\left\{z^k\right\}$ is bounded; moreover, every accumulation point of $\left\{z^k\right\}$ solves the $\mathrm{LCP}\left(q, \bar{M}\right)$;

2) If $\bar{M}$ is positive semi-definite and the $\mathrm{LCP}\left(q, \bar{M}\right)$ is solvable, then the sequence $\left\{z^k\right\}$ converges to the least l_2-norm solution of $\left(q, \bar{M}\right)$.

Theorem 2 Suppose that the system matrix $\bar{M}$ is symmetric positive semi-definite, $\{\varepsilon\}$ is a decreasing sequence, then the infinite sequence $\left\{x_\varepsilon^{(k)}\right\}$ produced by Method 3.2 is bounded. Moreover, every accumulation point of $\left\{z_\varepsilon^k\right\}$ solves the $\mathrm{LCP}\left(q, \bar{M}\right)$;

Proof Note that $\mathrm{E}\left[M(\omega) + \varepsilon I\right] = \bar{M} + \varepsilon I$ is symmetric positive definite. By the Step (3) of Method 3.2, we can get

$$x_\varepsilon^{(k+1)} = \left(\alpha I + \bar{M} + \varepsilon I\right)^{-1}\left(\alpha I - \bar{M} - \varepsilon I\right)\left|x_\varepsilon^{(k)}\right| - \left(\alpha I + \bar{M} + \varepsilon I\right)^{-1} q$$

Then

$$\left\|x_\varepsilon^{(k+1)}\right\| = \left\|\left(\alpha I + \bar{M} + \varepsilon I\right)^{-1}\left(\alpha I - \bar{M} - \varepsilon I\right)\left|x_\varepsilon^{(k)}\right| - \left(\alpha I + \bar{M} + \varepsilon I\right)^{-1} q\right\|$$

$$\leq \left\|\left(\alpha I + \bar{M} + \varepsilon I\right)^{-1}\left(\alpha I - \bar{M} - \varepsilon I\right)\right\|\left\|x_\varepsilon^{(k)}\right\| + \left\|\left(\alpha I + \bar{M} + \varepsilon I\right)^{-1} q\right\|.$$

Let $m_\varepsilon = \left\|\left(\alpha I + \bar{M} + \varepsilon I\right)^{-1}\left(\alpha I - \bar{M} - \varepsilon I\right)\right\|$ and $q_\varepsilon = \left\|\left(\alpha I + \bar{M} + \varepsilon I\right)^{-1} q\right\|$, there exist any positive numbers $m_1 < 1$ and m_2, we have

$$m_\varepsilon \le m_1, \quad q_\varepsilon \le m_2,$$

Moreover,

$$\left\| x_\varepsilon^{(k+1)} \right\| \le m_1 \left\| x_\varepsilon^{(k)} \right\| + m_2$$

$$\left\| x_\varepsilon^{(k)} \right\| \le m_1 \left\| x_\varepsilon^{(k-1)} \right\| + m_2$$

$$\vdots$$

$$\left\| x_\varepsilon^{(1)} \right\| \le m_1 \left\| x_\varepsilon^0 \right\| + m_2$$

Therefore

$$\left\| x_\varepsilon^{(k+1)} \right\| \le m_1 \left(m_1 \left\| x_\varepsilon^{(k-1)} \right\| + m_2 \right) + m_2$$

$$= m_1^2 \left\| x_\varepsilon^{(k-1)} \right\| + m_1 m_2 + m_2$$

$$\le m_1^2 \left(m_1 \left\| x_\varepsilon^{(k-2)} \right\| + m_2 \right) + m_1 m_2 + m_2$$

$$= m_1^3 \left\| x_\varepsilon^{(k-2)} \right\| + m_1^2 m_2 + m_1 m_2 + m_2$$

$$\vdots$$

$$\le m_1^k \left\| x_\varepsilon^{(1)} \right\| + m_1^{k-1} m_2 + m_1^{k-2} m_2 + \cdots + m_1 m_2 + m_2$$

$$= m_1^{k+1} \left\| x_\varepsilon^{(0)} \right\| + m_2 \frac{1 - m_1^k}{1 - m_1}$$

When $k \to +\infty$, the infinite sequence $\left\{ x_\varepsilon^{(k)} \right\}$ is bounded. Since $z_\varepsilon^k = \left| x_\varepsilon^{(k)} \right| + x_\varepsilon^{(k)}$, we get that the sequence $\left\{ z_\varepsilon^k \right\}$ is bounded.

By Lemma 2, we have that every accumulation point of $\left\{ z_\varepsilon^k \right\}$ solves the LCP $\left(q, \bar{M} \right)$. The proof is completed.

5. Numerical Results

In this section, we test some numerical results to show the efficiency of our methods. Let $\mathrm{RES} \le 10^{-5}$, n be the order of the matrix $\bar{M}$, IT denote the average iteration steps, and CPU denote the average iteration time.

Let $p_j = \mathrm{P}\left(\omega_j \in \Omega \right) = \frac{1}{m}, j = 1, 2, \cdots, m$. The steps to generate test problems can be found in the literature [4]. Numerical experimental results are shown in Tables 1-3.

Table 1 shows that Method 3.1 is effective when the coefficient matrix is symmetric positive definite.

Table 2 and Table 3 list the numerical experimental results of Method 3.2 when the coefficient matrix is symmetric positive semi-definite. We know that Method 3.2 is effective. (In the following, we briefly denote that Feasible Semi-smooth Newton Method is FSNM.)

Table 4 shows that, Method 3.2 is more effective than FSNM [19]. By Tables 1-4, we know that Method 3.2 improves the computational efficiency and is suitable for solving the large scale problems.

Table 1. The numerical results of Methods 3.1 ($\alpha = 0.9, m = 50$).

n	IT	CPU	RES
500	39	0.100	3.60e-06
1000	37	0.825	1.31e-06
2000	38	4.320	3.81e-06
3000	43	10.677	8.57e-06
3300	42	12.448	1.38e-06

Table 2. The numerical results of Methods 3.2 ($\alpha = 1.4, m = 50, \varepsilon = 10^{-7}$).

n	IT	CPU	RES
500	47	0.139	2.88e-06
1000	41	0.933	6.99e-07
2000	50	5.200	4.24e-06
3000	44	11.081	2.53e-06
3300	49	13.711	6.26e-07

Table 3. The numerical results of Methods 3.2 ($\alpha = 1.4, m = 50, \varepsilon = 10^{-9}$).

n	IT	CPU	RES
500	47	0.120	2.88e-06
1000	41	0.800	6.99e-07
2000	50	4.770	4.24e-06
3000	44	10.903	2.53e-06
3300	49	13.730	6.26e-07

Table 4. Comparison of numerical results of Method 3.2 and FSNM[19].

Method	$n = 30$	$n = 60$	$n = 150$
	CPU	CPU	CPU
Method 3.2	0.0078	0.0156	0.0188
FSNM[10]	0.0300	0.0499	1.5356

6. Conclusion

In this paper, we study the fast numerical methods for solving the stochastic linear complementarity problems. Firstly, we convert the expected value formulation of stochastic linear complementarity problems into the equivalent fixed point equations, then we establish a class of modulus-based matrix splitting iteration methods, and analyze the convergence of the method. These new methods can be applied to solve the large-scale stochastic linear complementarity problems. The numerical results also show the effectiveness of the new methods.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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