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Stochastic Reservoir Systems with Different Assumptions for Storage Losses

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Abstract

Moran considered a dam whose inflow in a given interval of time is a continuous random variable. He then developed integral equations for the probabilities of emptiness and overflow. These equations are difficult to solve numerically; thus, approximations have been proposed that discretize the input. In this paper, extensions are considered for storage systems with different assumptions for storage losses. We also develop discrete approximations for the probabilities of emptiness and overflow.

Keywords

Stochastic Storage Systems, Storage Losses, Probability of Emptiness and Overflow

1. Introduction

Moran [1] [2], Prabhu [3] [4] and Ghosal [5] all considered a finite dam whose input in a given interval of time is a continuous random variable. Integral equations are then developed that give the probability of emptiness and overflow. It is difficult to obtain exact numerical results from these equations. An analytic solution has only been obtained for an Erlang input. Klemes [6], Lochert and Phatarfod [7], Phatarfod and Srikanthan [8] and others have obtained approximations for these probabilities by discretizing the input. Following Bae and Devine [9], we consider reservoir systems with different assumptions for storage losses. We then obtain integral equations as above for the probability of emptiness and overflow, and develop discrete approximations to obtain numerical results for the probabilities of overflow and emptiness.

Moran considered a storage model of a dam in discrete time, $t = 0,1,2,\cdots$. Let Z_t be the level of the dam before input X_t, where the X's are i.i.d. random variables. Let

Y_t be the release at the end of the time period $(t, t+1)$, where the Y's are i.i.d. random variables independent of the *X*'s, and let $k < \infty$ be the capacity of the system. If $Z_t + X_t > k$, then there is an overflow of $X_t + Z_t - k$. If $Z_t + X_t \le k$ then no overflow occurs. At the end of the period, if there is an overflow, then $Z_{t+1} = k - Y_t$. If there is no overflow, then either $Z_{t+1} = Z_t + X_t - Y_t$ or $Z_{t+1} = 0$ if the storage system is empty.

Lindley [10] showed that if certain independence conditions are satisfied then

$$\begin{aligned} F_{t+1}(y) &= \Pr\left\{\text{storage level } Z_{t+1} \text{ of } (t+1)^{st} \text{ period} \le y\right\} \\ &= \Pr\{Z_{t+1} = 0\} + \Pr\{0 < Z_{t+1} \le y\} \\ &= \Pr\{Z_t + X_t - Y_t \le 0\} + \Pr\{0 < Z_t + X_t - Y_t \le y\} \\ &= \Pr\{Z_t + X_t - Y_t \le y\}. \end{aligned}$$

where $y \in [0, k]$. Further, define $H(\cdot)$ to be the *c.d.f.* of U_t where

$$U_t = X_t - Y_t$$

Then, by convolution

$$F_{t+1}(y) = \int_{-\infty}^{y} F_t(y-x)\,\mathrm{d}H(x), \qquad 0 \le y < \infty$$

Since the limiting distribution $F(y)$ of Z_t is independent of time *t* in the steady state, for the semi-infinite case (bounded below), we have:

$$F(y) = \begin{cases} \int_{0^-}^{\infty} F(y-x)\,\mathrm{d}H(x) & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases} \tag{1}$$

which is equal to

$$F(y) = -\int_{0^-}^{\infty} F(x)\,\mathrm{d}H(y-x), \qquad y \ge 0 \tag{2}$$

Equations (1) and (2) are known as Lindley's equations. Numerical solutions for specific input distributions to Lindley's equations are difficult to obtain. In Moran's original work, a solution for exponential inputs was found, but was strictly limited to that distribution.

It is not an easy task to obtain probabilities for emptiness and overflow in continuous time. In this regard. Moran [6] proposed a discrete approximation in order to obtain numerical results for the probabilies of emptiness and overflow. Modifications to this approach have been developed by Klemes [3], Lochert and Phatarfod [5], Phatarfod and Srikanthan [8]. In this paper, we model energy storage systems with different assumptions about storage losses, and develop similar discrete approximations to calculate the probabilities of emtiness and overflow.

2. Finite Model

Moran's model yields the following Markov chain:

$$Z_{t+1} = \begin{cases} 0 & \text{if } Z_t + X_t \le Y_t \\ Z_t + X_t - Y_t & \text{if } Y_t < Z_t + X_t < k \\ k - Y_t & \text{if } Z_t + X_t \ge k \end{cases}$$

For the case of a finite system of capacity $k < \infty$, $F(y) = 1$ for $y \ge k$ and Equation (2) becomes:

$$F(y) = -\int_0^k F(x)\,\mathrm{d}H(y-x) - \int_k^\infty \mathrm{d}H(y-x) \quad (3)$$

$$= H(y-k) - \int_0^k F(x)\,\mathrm{d}H(y-x) \quad 0 \le y \le k \quad (4)$$

If the system has a constant release, then $Y_t = m < k$ and

$$Z_{t+1} = \begin{cases} 0 & \text{if } Z_t + X_t \le m \\ Z_t + X_t - m & \text{if } m < Z_t + X_t < k \\ k - m & \text{if } Z_t + X_t \ge k \end{cases}$$

Thus, $Z_{t+1} = \min\{k - m, \max[0, Z_t + X_t - m]\}$. Figure 1 illustrates a single time period of the previous Markov chain:

Now, let $G(.)$ be the c.d.f. of X_t, then

$$H(u) = \Pr\{X_t - Y_t \le u\} = \Pr\{X_t - m \le u\} = G(u+m)$$

Since $F(x) = 1$ for $x \ge k - m$, Equation (3) becomes

$$F(y) = -\int_0^\infty F(x)\,\mathrm{d}G(y - x + m)$$

$$= -\int_0^{k-m} F(x)\,\mathrm{d}G(y - x + m) - \int_{k-m}^\infty \mathrm{d}G(y - x + m)$$

$$= G(y + 2m - k) - \int_0^{k-m} F(x)\,\mathrm{d}G(y - x + m), \quad 0 \le y \le k - m$$

3. Losses from Storage

Model 1

The most basic case of leakage occurs when a fixed amount q leaks from storage after the release Y_t at the end of each time interval. This type of system is shown below in Figure 2: (see Bae and Devine [9]):

Then, we have

$$\Pr\{Z_{t+1} \le y\} = \Pr\{Z_t + X_t - Y_t - q \le y\}$$

And

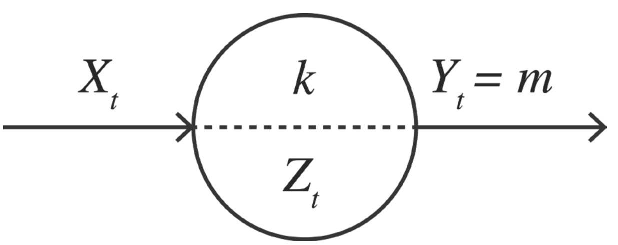

Figure 1. Simple storage system.

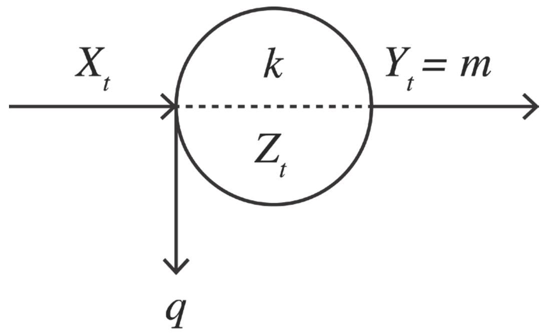

Figure 2. Fixed leakage from storage.

$$\begin{aligned} F(y) &= -\int_0^{\infty} F(x)\,\mathrm{d}H(y+q-x), \qquad y \geq 0 \\ &= -\int_0^{k-q} F(x)\,\mathrm{d}H(y+q-x) - \int_{k-q}^{\infty} F(x)\,\mathrm{d}H(y+q-x) \\ &= H(y+2q-k) - \int_0^{k-q} F(x)\,\mathrm{d}H(y+q-x) \end{aligned} \tag{5}$$

When the output is fixed, *i.e.*, $Y_t = m$, define $U_t = X_t - (m+q)$. Applying this to the previous equation gives

$$F(y) = G\left[y - k + 2(m+q)\right] - \int_0^{k-m-q} F(x)\,\mathrm{d}G(y - x + m + q) \tag{6}$$

Equations (5) and (6) indicate that leakage may be treated as a part of the output; thus, no separate analysis is needed in this case.

The second case represents a variable leakage whereby a quantity proportional to the amount stored is lost at the beginning of each time interval. Let $1 - e_0$ denote the fraction of Z_t lost in each period (*i.e.*, e_0 is a measure of the storage efficiency). The Markov chain corresponding to this case is

$$Z_{t+1} = \begin{cases} 0 & \text{if } e_0 Z_t + X_t \leq Y_t \\ e_0 Z_t + X_t - Y_t & \text{if } Y_t \leq e_0 Z_t + X_t < k \\ k - Y_t & \text{if } e_0 Z_t + X_t \geq k \end{cases}$$

Figure 3 illustrates this type of system:

$$\begin{aligned} F_{t+1}(y) &= \Pr\{e_0 Z_t + X_t - Y_t \leq y\} \\ &= \int_0^{\infty} \Pr\{X_t - Y_t \leq y - e_0 x \,|\, Z_t = x\} \Pr\{Z_t = x\} \\ &= \int_0^{\infty} H_t(y - e_0 x)\,\mathrm{d}F_t(x) \end{aligned}$$

We thus have

$$F(y) = H(y - e_0 k) - \int_0^{k} F(x)\,\mathrm{d}H(y - e_0 x) \tag{7}$$

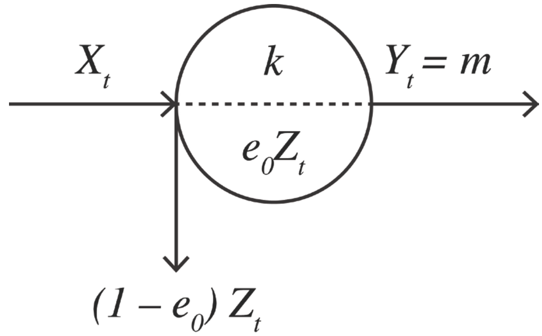

Figure 3. Fixed leakage from storage.

When the output is fixed, *i.e.*, $Y_t = m$, we again have $F(x) = 1$ for $x \ge k - m$. Then, the limiting c.d.f. of Z_t is given by

$$F(y) = G\left[y - e_0 k + (1 + e_0)m\right] - \int_0^{k-m} F(x)\,\mathrm{d}G(y - e_0 x + m) \tag{8}$$

Model 2

This model describes a system in which the input passes through a process having an efficiency e_1 before entering storage, and the quantity released from storage passes through an output process having an efficiency e_2 before leaving the system. The inputs X_t are independent random variables following a given c.d.f. and each $Y_t = m$. A diagram of this system is given in **Figure 4**.

Now we define

$$S_t = \begin{cases} X_t - m & \text{if } X_t \ge m \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$T_i = \begin{cases} m - X_t & \text{if } X_t < m \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

When both the input and output devices have efficiency factors, the schematic changes to (**Figure 5**).

It is apparent from (9) and (10) that either $S_t' = e_1 S_t$ or $T_t' = \dfrac{T_t}{e_2}$ must be 0.

Similar to previous models, we also define

$$G(\cdot) = \text{the c.d.f. of } X_t$$
$$\hat{U}_t = S_t' - T_t'$$
$$H(\cdot) = \text{the c.d.f. of } \hat{U}_t$$

$\hat{H}(u)$ and $G(\cdot)$ are then given by

$$H(u) = \begin{cases} G\left(\dfrac{u}{e_1} + m\right) & \text{if } u \ge 0 \\ G(e_2 u + m) & \text{if } u < 0 \end{cases} = \begin{cases} \hat{H}(e_1 u) & \text{if } u \ge 0 \\ \hat{H}\left(\dfrac{u}{e_2}\right) & \text{if } u < 0 \end{cases}$$

As before,

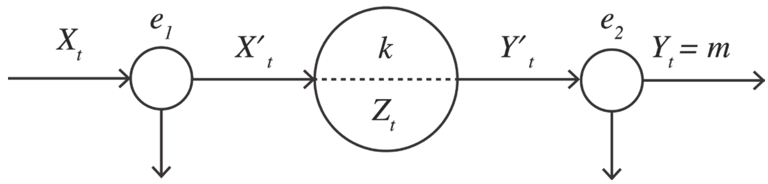

Figure 4. Input and output efficiencies.

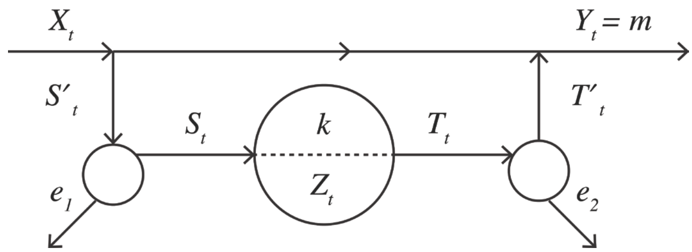

Figure 5. In-out devices efficiencies.

$$F(y) = \lim_{t\to\infty} F_t(y) = -\int_0^\infty F(x)\,\mathrm{d}H(y-x) \tag{11}$$

$$= H(y-k) - \int_0^k F(x)\,\mathrm{d}H(y-x), \qquad 0 \le y \le k \tag{12}$$

If

$$\hat{H}(u) = \Pr\{S'_t - T'_t \le u\} \tag{13}$$

then

$$\begin{aligned}
F(y) &= -\int_0^\infty F(x)\,\mathrm{d}\hat{H}(y-x) \\
&= -\int_0^y F(x)\,\mathrm{d}\hat{H}(y-x) - \int_y^k F(x)\,\mathrm{d}\hat{H}(y-x) - \int_k^\infty F(x)\,\mathrm{d}\hat{H}(y-x) \\
&= -\int_0^y F(x)\,\mathrm{d}\hat{H}(y-x) - \int_y^k F(x)\,\mathrm{d}\hat{H}(y-x) - \lim_{N\to\infty} \int_k^N F(x)\,\mathrm{d}\hat{H}(y-x) \\
&= -\int_0^y F(x)\,\mathrm{d}\hat{H}(y-x) - \int_y^k F(x)\,\mathrm{d}\hat{H}(y-x) - \lim_{N\to\infty} \hat{H}(y-N) + \hat{H}(y-k) \\
&= \hat{H}(y-k) - \int_0^y F(x)\,\mathrm{d}\hat{H}(y-x) - \int_y^k F(x)\,\mathrm{d}\hat{H}(y-x) \\
&= H\left(y - \frac{k}{e_2}\right) - \int_0^y F(x)\,\mathrm{d}H\left(e_1(y-x)\right) - \int_y^k F(x)\,\mathrm{d}H\left(\frac{y-x}{e_2}\right) \\
&= G\left(y - \frac{k}{e_2}\right) - \int_0^y F(x)\,\mathrm{d}G\left(e_1(y-x)\right) - \int_y^k F(x)\,\mathrm{d}G\left(\frac{y-x}{e_2}\right)
\end{aligned}$$

4. Methodology

We develop a discrete analogue by defining

$$F_i = \Pr(Z_t \le i) \tag{14}$$

$$g_i = \Pr(X_t = i) \tag{15}$$

$$G_j = \begin{cases} \sum_{i=0}^{j} g_i & \text{if } j \ge 0 \\ 0 & \text{if } j < 0 \end{cases} \tag{16}$$

Thus

$$F_i = \begin{cases} 0 & \text{if } i < 0 \\ G_{i+2m-k} + \sum_{j=0}^{k-m-1} g_{i+m-j} F_j & \text{if } i = 0,1,\cdots,k-m-1 \\ 1 & \text{if } i \ge k-m \end{cases}$$

where the probabilities of emptiness and overflow are

$$\Pr\{\text{emptiness}\} = \Pr\{Z_t \le 0\} = F(0) = F_0$$

$$\Pr\{\text{overflow}\} = \Pr\{Z_t \ge k-m\} = 1 - F(k-m-1) = 1 - F_{k-m-1}$$

4.1. Fixed Leakage

When the storage has a fixed quantity q that leaks as given in Equations (5) and (6), the discrete analogue is given by

$$F_i = \begin{cases} 0 & \text{if } i < 0 \\ G_{i+2(m-q)-k} + \sum_{j=0}^{k-m-q-1} g_{i+m+q-j} F_j & \text{if } i = 0,1,\cdots,k-m-1 \\ 1 & \text{if } i \ge k-m \end{cases}$$

where the probabilities of emptiness and overflow are

$$\Pr\{\text{emptiness}\} = \Pr\{Z_t \le 0\} = F(0) = F_0$$

$$\Pr\{\text{overflow}\} = \Pr\{Z_t \ge k-m\} = 1 - F(k-m-1) = 1 - F_{k-m-1}$$

4.2. Storage Leakage

A discrete analogue of (8) is

$$F_i = \begin{cases} 0 & \text{if } i < 0 \\ G_{i+(1-e_0)m-e_0 k} + \sum_{j=0}^{k-m-1} g_{i+m-e_0 j} F_j & \text{if } i = 0,1,\cdots,k-m-1 \\ 1 & \text{if } i \ge k-m \end{cases}$$

where the probabilities of emptiness and overflow are

$$\Pr\{\text{emptiness}\} = \Pr\{Z_t \le 0\} = F(0) = F_0$$

$$\Pr\{\text{overflow}\} = \Pr\{Z_t \ge k-m\} = 1 - F(k-m-1) = 1 - F_{k-m-1}$$

5. Constant Output Model

Using the definitions (14)-(16), we have

$$F_i = \begin{cases} 0 & \text{if } i < 0 \\ G_{i+m-k} + \sum_{j=0}^{k-1} g_{i+m-j} F_j & \text{if } i = 0,1,\cdots,k-1 \\ 1 & \text{if } i \geq k \end{cases}$$

where the probabilities of emptiness and overflow are

$$\Pr\{\text{emptiness}\} = \Pr\{Z_t \leq 0\} = F(0) = F_0$$

$$\Pr\{\text{overflow}\} = \Pr\{Z_t \geq k\} = 1 - F(k-1) = 1 - F_{k-1}$$

6. Numerical Results

Figure 6 and Figure 7 represent the impact of storage leakage on the basic system. The capacity of the following systems is determined in relation to the release amount. Figure 6 below gives the emptiness probabilities for an average input of 3.75 with a standard deviation of 1, and a storage efficiency e_0, of 75%.

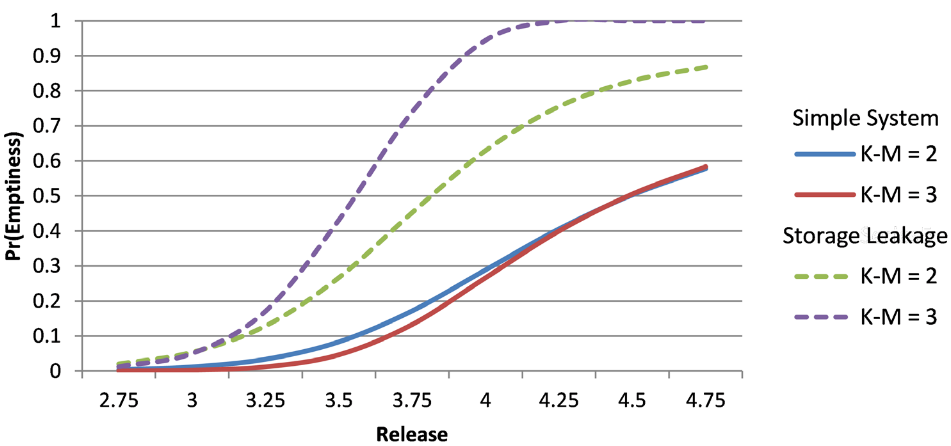

Figure 6. Comparative emptiness probabilities.

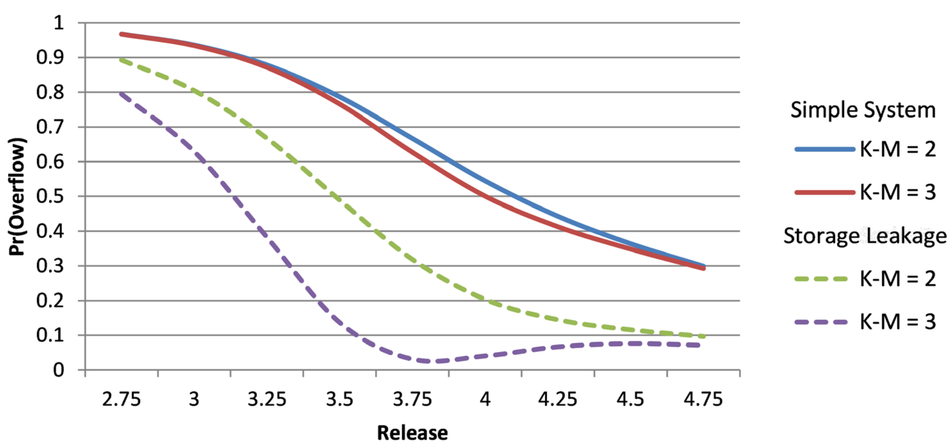

Figure 7. Comparative overflow probabilities.

Figure 8 represents the difference between Figure 6 and Figure 7. For both systems, there is an input of 4, with a release between 2 and 10, and a capacity determined by the relation of $k-m=3$.

Figure 9 represents the impact of a varying standard deviation on an input.

In this system, the input is 15, release is 15, and capacity ranges from 17 to 26. Additionally, each probability is determine with a standard deviation of 0.75, 1, 2 and 4.

Figure 10 represents the impacts of efficiency on the input and output process on the model obtained for the probability of overflow.

For this system, the input is 15 with a standard deviation of 4. The capacity for this system is 30. Additionally, the release changes from 7 to 21. "Series 1" has input/output efficiency of 1, "Series 2" has efficiency of 0.5/1, "Series 3" has efficiency of 1/0.5, and "Series 4" has efficiency of 0.7/0.9.

7. Sensitivity Analysis

Figure 6 and Figure 7 represent the impact of storage leakage on the basic system

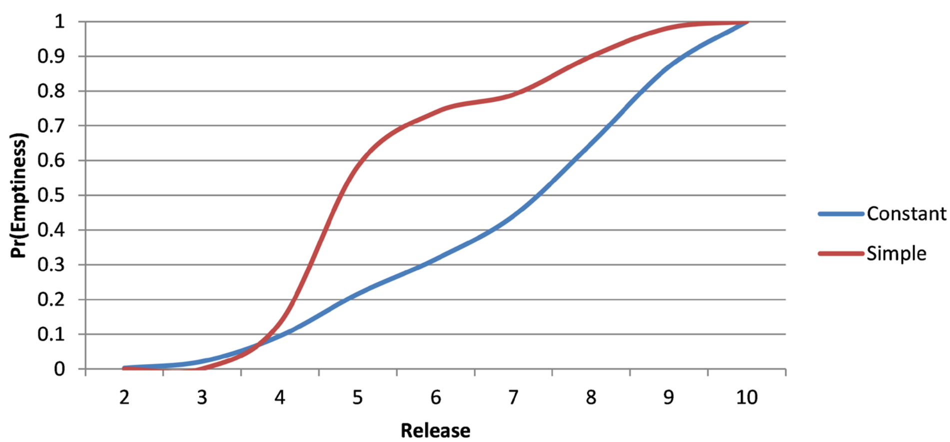

Figure 8. Comparative simple and constant output probabilities.

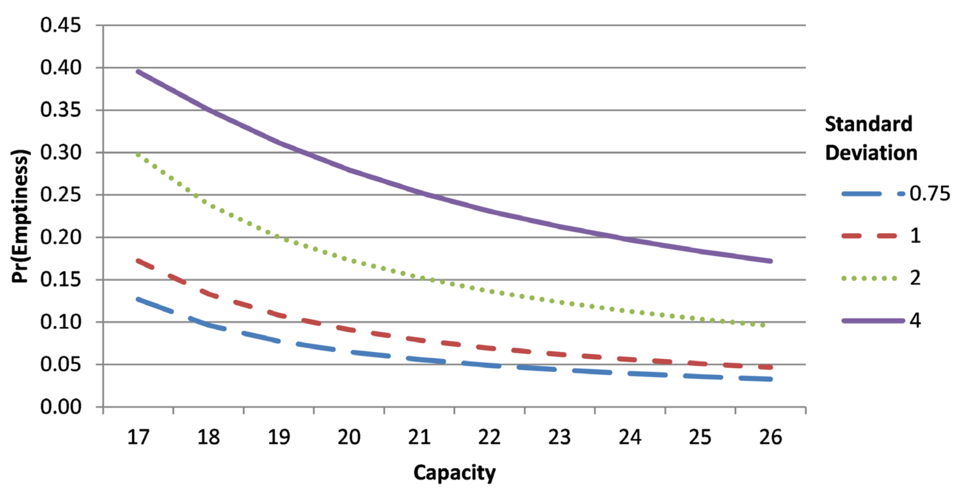

Figure 9. Represent the impact of a varying standard deviation on an input.

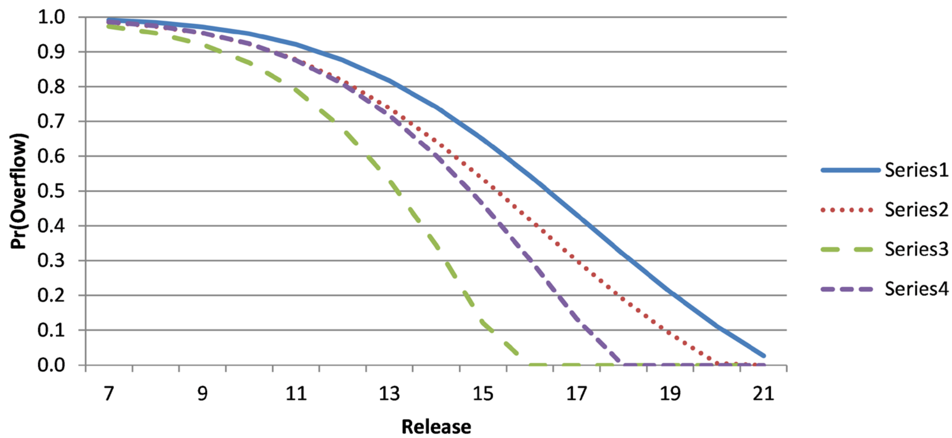

Figure 10. Impact of input/output efficiency.

when the basic discrete time model is considered. The capacity of the following system is determined in relation to the release amount. The input is 3.75 with a standard deviation of 1, and a storage efficiency, e_0, of 75%.

Figure 8 represents the difference between **Figure 6** and **Figure 7**. For both systems, there is an input of 4, with a release between 2 and 10, and a capacity determined by the relation of $k-m=3$.

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