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A Museum Cost Sharing Problem

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Abstract

Ginsburgh and Zang [1] consider a revenue sharing problem for the museum pass program, in which several museums jointly offer museum passes that allow visitors an unlimited access to participating museums in a certain period of time. We consider a cost sharing problem that can be regarded as the dual problem of the above revenue sharing problem. We assume that all museums are public goods and have various (e.g., service) costs. These costs must be shared by museum visitors. We propose a cost sharing method and provide an axiomatic characterization of the method. We then define a game for the problem and show that the cost sharing method is the Shapley value of the game. We also provide a comparative statics analysis for both the Shapley value of the museum pass game and the Shapley value for the cost sharing game when the number of museums and/or the number of visitors change.

Keywords: Cost Sharing, Shapley Value

1. Introduction

Ginsburgh and Zang [1] consider a revenue sharing problem for museums participating in a museum pass program. In the museum pass program, museum passes provide visitors an unlimited access to participating museums in a certain period of time. The total revenue from sales of museum passes must be shared among the participating museums. Since some museums may have more visitors than others, museums with more visitors should share more revenue. In general, each visitor may visit more than one museum. Thus, one cannot simply distribute the total revenue to museums in proportion to the number of visitors each museum has had. Ginsburgh and Zang define a cooperative game in which museums team up in offering the access passes. They apply the Shapley [2] value to allocate the joint income from the sale of passes among the museums.

We consider a cost sharing problem that can be regarded as the dual problem of the above revenue sharing problem. We assume that all museums are public goods and have various costs. These costs must be shared by museum users. Thus, museum users (customers) pay different fees for the services provided by different museums. This is in contrast to the museum pass program in which users pay the same fee for a pass. In the museum pass program, some users may subsidize for some other users. Here, we require that different users pay different fees depending on their different uses of the services provided by different museums.

We propose a cost sharing method for the cost sharing problem. The method simply allocates each museum's cost equally among the visitors who have visited the museum. We first provide an axiomatic characterization of the method by the axioms of Additivity, Anonymity and No Blind Cost. Additivity and Anonymity are the standard axioms in the cost sharing literature ([2,3]). No Blind Cost says that if the museums visited by a visitor all have zero costs, then this visitor shouldn't pay any cost. We then define a game for the cost sharing problem. The game can be considered as the dual game of the museum pass game. We show that the cost sharing method is the Shapley value of the game.

We also provide a comparative statics analysis for both the Shapley value of the museum pass game and the Shapley value of the cost sharing game when the number of museums and/or the number of visitors change. For the museum pass game, we demonstrate by an example that adding one more museum will decrease all museums' revenue if there is no increase in the number of visitors. But if the additional museum brings additional visitors to the museum itself as well as some other museums, then it is possible that some museums are better off and some are worse off. For the cost sharing game, we demonstrate by using a similar example that adding one more museum will increase all visitors' cost if there is no increase in the number of visitors. But if the additional museum brings additional visitors to the museum itself as well as some other museums, then it is possible that some visitors are better off and some are worse off.

In general, our model can be used to deal with cost sharing problems where a finite number of users use a finite number of production functions (or service facilities). There are many cost sharing problems involving two groups of agents in which each agent from the first group obtains services from some agents in the second group. And the total service costs incurred to the second group must be shared among the agents in the first group (see Moulin and Laigret [4]).

2. The Model and Proposed Method

Let $M=\{1,\dots,m\}$ denote the set of customers (e.g., museum users), and let $N=\{1,\dots,n\}$ denote the set of service providers (e.g., museums), where m,n are two positive integers. Let $C=\{c_1,\dots,c_n\}$ be the cost vector, where $c_i \ge 0$, $i\in N$ is the cost incurred by service provider i.

For each customer $j\in M$, let $K(j)$ denote the set of service providers whose services were utilized by customer j. Denote K such a mapping from M to 2^N. For each service provider $i\in N$, let $K^{-1}(i)$ be the set of customers who use the service of provider i. Denote K^{-1} such an inverse of K. Assume that for each service provider $i\in N$, $K^{-1}(i)\neq\varnothing$. Denote $\mathcal{K}$ the set of all possible mappings from M to 2^N. A *problem* is a list (M,N,K,C) where $K\in\mathcal{K}$ and $C\in R_+^n$. A solution is a vector $x=(x_1,\dots,x_m)\in R_+^m$ such that.

$$\sum_{j\in M} x_j=\sum_{i\in N} c_i$$

A method is a mapping that assigns to each problem (M,N,K,C) a solution $x(M,N,K,C)$. Except Section 5, through out the paper, we fix the sets M and N. Thus, we simply call (K,C) a problem and $x(K,C)$ a solution.

In this paper, we propose and study the following method which allocates each service provider's cost equally to the customers who have utilized that service:

$$x_j(K,C)=\sum_{i\in K(j)}\frac{c_i}{\left|K^{-1}(i)\right|}, j=1,\dots,m \qquad (1)$$

where $\left|K^{-1}(i)\right|$ is the number of elements in the set $\left|K^{-1}(i)\right|$.

Example 1. In **Figure 1**, there are three customers 1,2,3, and three service providers 1,2,3. Customer 1 uses all three service providers, customer 2 uses only service providers 1 and 2 and customer 3 uses service providers 2 and 3. Suppose that service providers' costs are given by the vector $C=(c_1,c_2,c_3)$.

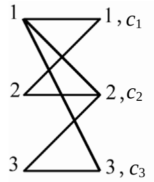


Figure 1. The left side denotes the three customers. The right side denotes the three service providers along with their costs, respectively.

Then the cost sharing method (1) gives the following solution:

$$x_1=\frac{1}{2}c_1+\frac{1}{3}c_2+\frac{1}{2}c_3,$$
$$x_2=\frac{1}{2}c_1+\frac{1}{3}c_2,$$
$$x_3=\frac{1}{3}c_2+\frac{1}{2}c_3,$$

3. Axiomatic Characterization of the Method

In this section, we provide an axiomatic characterization of the method (1). For this purpose, we require the following axioms.

Additivity: Fix K. For any $C^1=(c_1^1,\dots,c_n^1)\in R_+^n$ and $C^2=(c_1^2,\dots,c_n^2)\in R_+^n$, we have $x_j(K,C^1+C^2)=x_j(K,C^1)$ $+x_j(K,C^2)$ for all $j\in M$.

Additivity is a classical axiom in the cooperative game theory [2] and in the cost-sharing literature [3]. In the present context of museum cost sharing problem, we can provide the following interpretation. If museums' costs are split into two parts, for example, the capital cost and the operating cost, and the cost allocation of each part of these costs is computed, then the sum of these two cost allocations would be equal to the cost allocation obtained by applying the method to the unsplit total costs.

A permutation π of $M=\{1,\dots,M\}$ is a one-to-one mapping from M to M, *i.e.*, $\pi: M\to M$ and for all $i,j\in M$, $\pi(i)\neq\pi(j)$ if and only $i\neq j$.

Anonymity: For any permutation π of M and any $C\in R_+^n$,

$$\pi(x(K,C))=x(\pi(K,C))$$

where $\pi(K,C)=(\pi(K),C)$, $\pi(K)(i)=K(\pi(i))$, and $\pi(x)_j=x_{\pi(j)}$ for $x\in R_+^m$.

In words, Anonymity requires that the costs allocated to the customers do not depend on their names.

No Blind Cost: For any $j\in M$, if for all $i\in K(j)$, $c_i\in 0$, then $x_j(K,C)=0$.

In words, for any customer, if the costs of all museums he or she has visited are all zeros, then the customer shouldn't pay any cost. In other words, there are no cross-subsidizations.

Theorem 1 *The method defined in* (1) *is the only method that satisfies the axioms of Additivity, Anonymity, and No Blind Cost.*

Proof: It is easy to check that the method (1) satisfies the axioms of Additivity, Anonymity, and No Blind Cost.

Now we show that the method (1) is the *only* method that satisfies the three axioms. Suppose that a cost sharing method φ satisfies the three axioms. Fix an arbitrary K, and a cost vector $C=(c_1,\dots c_n)$. For any $i\in\{1,\cdots,n\}$, let $C^i=(0,\dots,0,1,0,\dots,0)$ where 1 is the i th component of the n-dimensional vector C^i. Note that $K^{-1}(i)$ is the set of customers that are served by museum i which has one unit cost in C^i. By No Blind Cost and Anonymity, we have $\varphi_j(K,C^i)=0$ for all $j\in M\setminus K^{-1}(i)$, and $\varphi_j(K,C^i)=1/\left|K^{-1}(i)\right|$ for all $j\in K^{-1}(i)$.

Now for any given $C\in R_+^n$, it can be uniquely written as $C=(c_1,\dots,c_n)=\sum_{i\in N}c_i\cdot C^i$. By Additivity,[1] we have

$$\varphi_j(K,C)=\sum_{i\in N\setminus K(j)}0+\sum_{i\in K(j)}\frac{c_i}{\left|K^{-1}(i)\right|}=\sum_{i\in K(j)}\frac{c_i}{\left|K^{-1}(i)\right|}$$

$$=\varphi_j\left(K,\sum_{i\in N}c_i\cdot C^i\right)=\sum_{i\in N}c_i\cdot\varphi_j\left(K,C^i\right)$$

for all $j\in M$.

This completes the proof of the theorem.

Q.E.D.

4. The Game and the Shapley Value

We now show that the method (1) defined above is, in fact, the Shapley value of the following game $c(\cdot)$ that is associated to the problem (K,C):

$$c(S)=\sum_{i\in K(S)}c_i,S\subseteq M, \tag{2}$$

Where $K(S)=\cup_{j\in S}K(j)$, *i.e.*, the set of service providers whose services have been utilized by the customers in S.

Recall that the Shapley value of a game $c(\cdot)$ is defined by

$$\phi_j(c)=\sum_{S\subseteq M:j\in S}\frac{(|S|-1)!(m-|S|)!}{m!}\left[c(S)-c(S\setminus\{j\})\right],j=1,\cdots,m. \tag{3}$$

Proposition 1. The Shapley value of the cost game $c(\cdot)$ defined in (2) coincides with the method (1).

Proof. For any $i\in N$, consider the unit vector in R^n

$$C^i=(0,\cdots,0,1,0,\cdots,0),$$

where 1 is the i-th component of the vector C^i. The corresponding cost game defined by (2) is given by

$$c^i(S)=\begin{cases}1 & if \quad K(S)\ni i\\ 0 & otherwise,\end{cases}\quad S\subseteq M. \tag{4}$$

Clearly, all agents in $M\setminus K^{-1}(i)\subseteq M$ are dummy agents in the game c^i and all agents in $K^{-1}(i)$ are symmetric. Thus the Shapley value of the game c^i is given by

$$\phi(c^i)=\begin{cases}\dfrac{1}{\left|K^{-1}(i)\right|}, & j\in K^{-1}(i)\\ 0, & otherwise\end{cases} \tag{5}$$

for all $j\in M$.

Since the cost vectors, $(C^i)(i\in\{1,\dots,n\})$, form a basis of R^n, for any $C\in R_+^n$, it can be uniquely written as $C=\sum_{i\in N}c_i\cdot C^i$. By the definition of the cost game, for $\varnothing\neq S\subseteq M$, we have

$$c(S)=\sum_{k\in K(S)}c_k=\sum_{k\in K(S)}\left(\sum_{i\in N}\left[c_i\cdot C^i\right]_k\right)$$

$$=\sum_{i\in N}c_i\cdot\left(\sum_{k\in K(S)}\left[C^i\right]_k\right)=\sum_{i\in N}c_i\cdot c^i(S)$$

where $[C]_k$ is the k th component of the vector C. Because the Shapley value satisfies Additivity (and thus, Linearity, see footnote 1), we have

$$\phi_j(c)=\sum_{i\in N}c_i\cdot\phi_j(c^i)=\sum_{i\in N\setminus K(j)}0+\sum_{i\in K(j)}\frac{c_i}{\left|K^{-1}(i)\right|}$$

$$=\sum_{i\in K(j)}\frac{c_i}{\left|K^{-1}(i)\right|}$$

For all $j\in M$. This completes the proof of the proposition.

Q.E.D.

5. Comparative Statics Analysis

Now we consider the effect of adding one more museum on the solution of the problem. This is important because when several museums consider the museum pass program, they need to decide which museums shall be included themselves. If each museum makes its decision about joining the program independently, it becomes a *strategic game* for all potential museums in a given geographic region. Of course, the revenue sharing method they choose would eventually affect the equilibrium

[1]It can be easily shown that Additivity plus Positivity (non-negative cost shares) implies Linearity, *i.e.*, φ is linear with respect to the cost vector C. See Aczel [5].

number of museums joining the program. A complete analysis of this strategic game is beyond the scope of this paper. Nevertheless, it is helpful to consider the effect of adding one more museum (and likely generating more visitors) on the revenue (cost) allocation.

For completeness, we first provide a comparative statics analysis of the Shapley value of the museum pass game. We demonstrate by an example that adding one more museum will decrease all museums' revenue if there is no increase in the number of visitors. But if the additional museum brings additional visitors to the museum itself as well as some other museums, then it is possible that some museums are better off and some are worse off.

Next, we provide a parallel analysis for the Shapley value of the cost sharing game. We also demonstrate by using the same example as for the museum pass game that adding one more museum will increase all visitors' cost if there is no increase in the number of visitors. But if the additional museum brings additional visitors to the museum itself as well as some other museums, then it is possible that some visitors are better off and some are worse off.

Now consider the following example:

Assume that initially there are three museums and three visitors (**Figure 2(a)**). Suppose that visitor 1 visits museums 1, 2, and 3, visitor 2 visits museums 1 and 2, and visitor 3 visitor museum 3 only. The Shapley value solution of the revenue sharing problem is the following:

$$\phi(1)=\frac{1}{3}+\frac{1}{2},\ \phi(2)=\frac{1}{3}+\frac{1}{2},\ \phi(3)=\frac{1}{3}+1$$

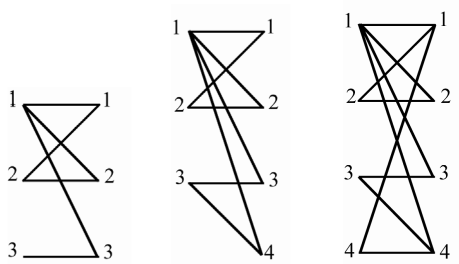
Suppose that one more museum, museum 4, joins the museum pass program as shown in **Figure 2(b)**. If museum 4 brings no additional visitors, then the solution is the following:

$$\phi(1)=\frac{1}{4}+\frac{1}{2},\phi(2)=\frac{1}{4}+\frac{1}{2},$$
$$\phi(3)=\frac{1}{4}+\frac{1}{2},\phi(4)=\frac{1}{4}+\frac{1}{2}.$$

Obviously, all museums except the newly added museum are worse off than before.

But as shown in **Figure 2(c)**, if the new museum 4 brings one additional visitor 4, who also visits other museums, e.g., museum 1, then museum 1 is better off while museums 2 and 3 are worse off as shown below (compared to the case without the museum 4 and the visitor 4):

$$\phi(1)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2},\phi(2)=\frac{1}{4}+\frac{1}{2},$$
$$\phi(3)=\frac{1}{4}+\frac{1}{2},\phi(4)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}.$$



(a) (b) (c)

Figure 2. In this revenue sharing problem, the left side denotes visitors and the right side museums. The Shapley value solution assigns to each museum a proportional share of the total revenue. The total revenue is omitted.

In general, adding one more museum without bringing in more visitors will make all existing museums worse off. But if more visitors are generated by the additional museum and these additional visitors also visit other museums in addition to the newly added museum, then some museums may be better off and some may be worse off, depending on how these visitors are distributed to the museums.

In parallel, we have similar results for the cost sharing problem. We use the same example as above but now assume that museums $1,\cdots,4$ have costs c_1,c_2,c_3 and c_4, respectively.

Again, initially there are three museums and three visitors as shown in **Figure 3(a)**. Also suppose that visitor 1 visits museums 1, 2, and 3, visitor 2 visits museums 1 and 2, and visitor 3 visitor museum 3 only. The Shapley value of the cost sharing problem is the following:

$$\psi(1)=\frac{1}{2}c_1+\frac{1}{2}c_2+\frac{1}{2}c_3,$$
$$\psi(2)=\frac{1}{2}c_1+\frac{1}{2}c_2,$$
$$\psi(3)=\frac{1}{2}c_3.$$

Suppose that one more museum, museum 4, joins with cost c_4 (**Figure 3(b)**). If museum 4 brings no additional visitors, then the new cost sharing problem has the following Shapley value:

$$\psi(1)=\frac{1}{2}c_1+\frac{1}{2}c_2+\frac{1}{2}c_3+\frac{1}{2}c_4,$$
$$\psi(2)=\frac{1}{2}c_1+\frac{1}{2}c_2,$$
$$\psi(3)=\frac{1}{2}c_3+\frac{1}{2}c_4.$$

Apparently all visitors are either worse off or no better off than before.

But as shown in **Figure 3(c)**, if the new museum 4 brings one additional visitor 4, who also visits museum 1, then visitor 2 is better off while visitor 3 is worse off, but it is unclear whether visitor 1 is better off or worse off as shown below (compared to the case without the museum 4 and the visitor 4)[2]:

$$\psi(1)=\frac{1}{3}c_1+\frac{1}{2}c_2+\frac{1}{2}c_3+\frac{1}{3}c_4,$$

$$\psi(2)=\frac{1}{3}c_1+\frac{1}{2}c_2,$$

$$\psi(3)=\frac{1}{2}c_3+\frac{1}{3}c_4,$$

$$\psi(4)=\frac{1}{3}c_1+\frac{1}{3}c_4.$$

In general, adding one more museum without bringing in more visitors will make all visitors worse off. But if more visitors are generated by the additional museum and these additional visitors also visit other museums in addition to the newly added museum, then some visitors may be better off and some may be worse off, depending on the new distribution of visitors to museums.

6. Discussion

It is straightforward to show that the game (2) is convex, and consequently its Shapley value is contained in its core [6].

Our model contributes to the recent literature on cost sharing problems involving cost functions that are generated from multiple technologies. In the traditional cost sharing models, it is usually assumed that all agents share a commonly owned technology given by a single cost function (or production function). And the cost function is independent of how the agents form coalitions. But in the cost sharing problem discussed in this paper, the coalitions of visitors determine the costs of the museums they have visited. In a recent paper, Trudeau [7] considers a cost sharing problem with multiple technologies, where gains come from the presence of agents rather than from the returns to scale. As pointed out in Trudeau [7], this literature on cost sharing problem with multiple technologies is closely related to network formation problems in which gains come from the presence of agents.

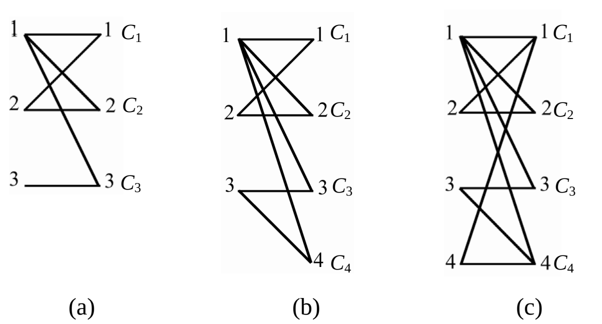


Figure 3. This is a cost sharing version of the comparative statics problem.

As we have pointed out in Section 5, the Shapley value (in fact, any revenue sharing method) of the museum pass game induces a strategic game for the museums with regard to whether or not they should join the museum pass program in the first place. For example, if the number of visitors has increased and more revenue has been received, the Shapley value allocation just distributes the additional revenue equally to museums that are visited by these visitors. However, if one more museum joins the museum pass program, a new game must be defined and the Shapley value must be recalculated. Depending on how the additional visitors are distributed among the museums, it is possible that some museums may be better off and some worse off. But this, in effect, will determine which museums actually join the program (or are allowed to join by existing museum members if they vote on accepting new members). Indeed, each revenue sharing method would generate a strategic game for all potential museums when they contemplate participating the museum pass program. An important question is: which revenue sharing method would generate an efficient coalition of museums?

For the cost sharing problem, if one more museum is added, its cost is equally allocated to those visitors who have visited the museum. If one more visitor has visited a number of museums, all existing visitors will benefit. In general, when one more museum is added it usually generates additional visitors to museums at the same time. In this case, as demonstrated by the example in Section 5, it is possible that some visitors are better off and some are worse off. When the number of visitors is large, choosing which museums to visit is an independent decision. Thus, unlike the museums in the museum pass program, visitors usually do not act strategically.

Finally, we point out that the cost sharing model proposed in this paper is more suitable in dealing with cost sharing problems involving two groups of agents in which each agent from the first group obtains services from some agents in the second group. And the total service costs incurred to the second group must be shared among the agents in the first group. When the number of agents (users and service providers) is not so numerous, our cost sharing method provides a straightforward calculation for the allocation of the costs. As agents are not so many, choosing which service providers once again becomes a strategic decision for the users, which, in turn, depends on the cost sharing method.

[2] For this example, it depends on whether or not $\frac{1}{2}c_1 > \frac{1}{3}c_1 + \frac{1}{3}c_4$.

7. References

[1] V. Ginsburgh and I. Zang, "The Museum Pass Game and Its Value," *Games and Economic Behavior*, Vol. 43, No. 2, 2003. pp. 322-325. doi:10.1016/S0899-8256(03)00013-7

[2] L. S. Shapley, "A Value for n-Person Games," In: H. W. Kuhn and A. W. Tucker, Eds., *Contributions to the Theory of Games II*, *Annals of Mathematics Studies*, Princeton University Press, Princeton, Vol. 28, 1953, pp. 307-317.

[3] H. Moulin, "Axiomatic Cost and Surplus Sharing," In: K. J. Arrow, A. K. Sen and K. Suzumura, Eds., *Handbook of Social Choice and Welfare*, North-Holland, Amsterdam, Ch.6, 2002, pp. 289-357

[4] H. Moulin and F. Laigret, "Equal-Need Sharing of a Network under Connectivity Constraints," *Games and Economic Behavior*, Vol. 72, No. 1, 2011, pp. 314-320. doi:10.1016/j.geb.2010.08.002

[5] J. Aczel, "Function Equations and Their Applications," Academic Press, New York, 1966.

[6] L. S. Shapley, "Cores of Convex Games," *International Journal of Game Theory*, Vol. 1, No. 1, 1971, pp. 11-26. doi:10.1007/BF01753431

[7] C. Trudeau, "Cost Sharing with Multiple Technologies," Working Paper, University of Montreal, Montreal, 2007.

