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Approximate Solutions for a Class of Fractional-Order Model of HIV Infection via Linear Programming Problem

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Abstract

In this paper, we provide a new approach to solve approximately a system of fractional differential equations (FDEs). We extend this approach for approximately solving a fractional-order differential equation model of HIV infection of CD4+T cells with therapy effect. The fractional derivative in our approach is in the sense of Riemann-Liouville. To solve the problem, we reduce the system of FDE to a discrete optimization problem. By obtaining the optimal solutions of new problem by minimization the total errors, we obtain the approximate solution of the original problem. The numerical solutions obtained from the proposed approach indicate that our approximation is easy to implement and accurate when it is applied to a systems of FDEs.

1. Introduction

In recent years, scientists have been interested in studying the fractional calculus and the FDEs in different fields of engineering, physics, mathematics, biology, finance, biomechanics and electrochemical processes (see [1]-[8], for more details). Also, it has been shown that modelling the behavior of many biological systems that governed by FDEs has more advantages than classical integer-order modelling [9]. Readers interested in FDEs are referred to [10]-[17]. Although great efforts have been made to find numerical and analytical techniques for

solving FDE, for example, predictor-corrector method [18], the Adomian decomposition [19], the variational iteration method [20], collocation using spline functions [21] and matrix expression given by [22] [23], but most of these FDEs do not have analytic solutions.

In this paper, at first, we approximate the fractional derivative by a finite difference method and then use the AVK approach [24] to obtain a new approximate solution for the FDEs. This approach substitutes the FDEs with an equivalent minimization problem in which the optimal solution of this problem is the approximate solution of the original FDE. Moreover, since the error of this approach is minimized, the approximate solutions are the best solutions for the original problem. We employ this approximation to get numerical solution of a system of FDEs which has been used for modelling HIV infection of CD4$^+$T cells.

The discussion of paper will be as follows: in the next section, we express the fractional HIV model and introduce the notations that used in the rest of this paper. In Section 3, we design an efficient approach to approximate the fractional derivative and use it in our numerical method for solving FDEs. Some numerical examples are displayed in Section 4. Finally, conclusions are included in the last section.

2. The Problem

Consider the following fractional-order differential equation model of HIV infection of CD4$^+$T cells [25]:

$$\begin{cases} D^\alpha T = s - \mu_T T + rT\left(1 - \dfrac{T+I}{T_{\max}}\right) - k_V T, \\ D^\alpha I = k_I' VT - \mu_I I, \\ D^\alpha V = N\mu_b I - k_I VT - \mu_V V, \end{cases} \tag{1}$$

with the initial conditions $T(0) = T_0$, $I(0) = 0$ and $V(0) = 0$, in which the parameter values reported by **Table 1**.

Following Theorem 1 of [25], we note that (1) along with its initial conditions possesses a unique solution which is non-negative. Throughout this paper, we set D^α ($0 < \alpha < 1$) as the Riemann-Liouville derivative of order α defined by [26]:

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{-\alpha} f(\tau) \mathrm{d}\tau. \tag{2}$$

The aim of this paper is to extend the application of the AVK approach to solve a fractional order model for this HIV infection model of CD4$^+$T cells. So, in the next section, at first we convert the original FDE to an

Table 1. Variables and parameters for HIV infection model.

Parameter	Value/unit
μ_T (Natural death rate of CD4$^+$T)	0.02 day^{-1}
μ_I (Blanket death rate of infected CD4$^+$T)	0.26 day^{-1}
μ_v (Death rate of free virus)	2.4 day^{-1}
μ_b (Lytic death rate for infected cells)	0.24 day^{-1}
k_I (Rate CD4$^+$T become infected with virus)	2.4×10^{-5} mm$^{-3} \cdot$ day^{-1}
k_I' (Rate infected cells become active)	2×10^{-5} mm$^{-3} \cdot$ day^{-1}
r (Rowth rate of CD4$^+$T population)	0.03 day^{-1}
N (Number of virions produced by infected CD4$^+$T)	Varies
$T_{\max}$ (Maximal population level of CD4$^+$T)	1500 mm^{-3}
s (Source term for uninfected CD4$^+$T)	10 day$^{-1} \cdot$ mm^{-3}
T_0 (CD4$^+$T population for HIV-negative persons)	1000 mm^{-3}

optimization problem based on minimization of error. By discretizing the new problem and approximating the Riemann-Liouville fractional derivative by a finite difference method, we obtaine the best approximate solution of the original FDE.

3. AVK Approach for Solving Approximately FDEs

Consider a general system of FDEs as follows:

$$D^\alpha x(t) = g(x, t),$$
$$x(0) = x_0, \tag{3}$$

where D^α $(0 < \alpha < 1)$ is the Riemann-Liouville derivative of order α, g is an riemann integrable time varying function, $g : A \times [0,1] \to \mathbb{R}^n$, $t \in [0,1] \subseteq \mathbb{R}$ and A is a compact subset in $\mathbb{R}^n$. Also $x(t) = (x_1(t), \cdots, x_n(t)) \in A$ called the state variable. We want to obtain an approximate solution of problem (3). Therefore, we need the following definition.

Definition 1. *For problem* (3) *we define the following functional* $E(.)$ *that is called the total error functional*:

$$E(D^\alpha x, x, t) = \int_0^1 \|D^\alpha x(t) - g(x, t)\| dt, \tag{4}$$

where $E : A \times [0,1] \to \mathbb{R}$ is a non-negative functional, $\|.\|$ is any norm in $\mathbb{R}^n$ space, such as $\|.\|_1$ where is defined as follows:

$$\|x(t)\|_1 = \|(x_1(t), \cdots, x_n(t))\|_1 = \sum_{i=1}^n |x_i(t)|. \tag{5}$$

Here, we convert the problem (4) to a nonlinear programming (NLP) as follow:

$$\begin{cases} \min & E(D^\alpha x, x, t) \\ s.t & x(0) = x_0, \\ & (x, t) \in A \times [0,1]. \end{cases} \tag{6}$$

Now, to reach the approximating solution for the original problem (3) it is sufficient to solve the minimization problem (6). Hence, we need the following mean theorem [27] and corollary.

Theorem 1. *Let h be a nonnegative continuous function on* $[a, b]$, *the necessary and sufficient condition for* $\int_a^b h dt = 0$ *is that* $h \equiv 0$, *on* $[a, b]$.

Corollary 1. *Necessary and sufficient condition for the trajectory* $x(t)$ *to be a solution of system* (3) *is that the optimal solution of* (6) *has zero objective function.*

To develop the numerical solution of problem (6) approximately, we defined the grid size in time by $\delta t = \dfrac{1}{m}$ for some positive integer *m*, so the grid points in the time interval $[0,1]$ is given by $t_k = k \delta t$, $k = 1, \cdots, m$. In order to illustrate the numerical approach better, we introduce the following notations:

$$x_i^k = x_i(t_k), g_i^k = g(x_i^k, t_k), i = 1, \cdots, n, k = 1, \cdots, m.$$

By the above notations, problem (6) is now approximated by the following optimization problem:

$$\begin{cases} \min J(x, t) = \sum_{i=1}^n \sum_{k=1}^m \int_{t_{k-1}}^{t_k} \left| D^\alpha x_i(t) - g(t, x_i(t)) \right| dt \\ s.t \quad x(0) = x_0, x_i^k \in A \subseteq \mathbb{R}^n, t \in (t_{k-1}, t_k) \subseteq \mathbb{R}. \end{cases} \tag{7}$$

By using the ending point in any subinterval for approximating integrals, problem (7) is now approximated by the following optimization problem:

$$\min_x J(x_i, t) = \sum_{i=1}^n \sum_{k=1}^m \frac{1}{m} \left| D^\alpha x_i^k - g_i^k \right|. \tag{8}$$

Now, we approximate fractional derivative $D^{\alpha} x_i$ as follows:

$$D^{\alpha} x_i = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{x_i(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau. \tag{9}$$

Define $y(t) = \int_0^t x_i(\tau)(t-\tau)^{-\alpha} \mathrm{d}\tau$. Then, Equation (9) yields to

$$D^{\alpha} x_i = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} y(t). \tag{10}$$

In order to better illustrate the numerical approach, we also introduce the following difference operator:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) \cong \frac{y(t+\delta t) - y(t)}{\delta t} = m(y(t+\delta t) - y(t)). \tag{11}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = m\left(\int_0^{t+\delta t} x_i(\tau)(t+\delta t-\tau)^{-\alpha} \mathrm{d}\tau - \int_0^t x_i(\tau)(t-\tau)^{-\alpha} \mathrm{d}\tau\right). \tag{12}$$

Hence δt or sampling time is very important, and must be chosen small, so the number of partitions is great. This is a trade off between sampling time and speed of problem solving. Using again trapezoidal rule in any subinterval for approximating integrals, except for the last interval that we use the midpoint approximation, and suppose $t_h = \frac{h}{m}$, $x_i^h = x_i\left(\frac{h}{m}\right)$ for $h = 1, 2, \cdots, k$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} y(t_k) &= m\left(\int_0^{t_{k+1}} x_i(\tau)(t_{k+1}-\tau)^{-\alpha} \mathrm{d}\tau - \int_0^{t_k} x_i(\tau)(t_k-\tau)^{-\alpha} \mathrm{d}\tau\right) \\
&\cong m\left(\sum_{h=1}^{k+1} \int_{\frac{h-1}{m}}^{\frac{h}{m}} x_i(\tau)(t_{k+1}-\tau)^{-\alpha} \mathrm{d}\tau - \sum_{h=1}^{k} \int_{\frac{h-1}{m}}^{\frac{h}{m}} x_i(\tau)(t_k-\tau)^{-\alpha} \mathrm{d}\tau\right) \\
&\cong m\left\{\left(\sum_{h=1}^{k} \frac{1}{m} x_i^h (t_{k+1}-t_h)^{-\alpha} + \frac{1}{m} x_i\left(\frac{2k+1}{2m}\right)\left(t_{k+1} - \frac{2k+1}{2m}\right)^{-\alpha}\right) \right. \\
&\quad \left. - \left(\sum_{h=1}^{k-1} \frac{1}{m} x_i^h (t_k-t_h)^{-\alpha} + \frac{1}{m} x_i\left(\frac{2k-1}{2m}\right)\left(t_k - \frac{2k-1}{2m}\right)^{-\alpha}\right)\right\}.
\end{aligned} \tag{13}$$

Thus, we simply get problem (8) in the following form:

$$\min_{x_i} \sum_{i=1}^{n} \sum_{k=1}^{m} \frac{1}{m} \left| \frac{1}{\Gamma(1-\alpha)} \left\{ \sum_{h=1}^{k} x_i^h (t_{k+1}-t_h)^{-\alpha} - \sum_{h=1}^{k-1} x_i^h (t_k-t_h)^{-\alpha} + \left(\frac{1}{2m}\right)^{-\alpha} \left(x_i^{k+\frac{1}{2}} - x_i^{k-\frac{1}{2}}\right)\right\} - g_i^k \right|, \tag{14}$$

in which, $x_i^{k\pm\frac{1}{2}} = x_i\left(t_k \pm \frac{\delta t}{2}\right)$ for $k = 1, 2, \cdots, m$.

We solved this optimization problem by linear programming (LP) formulation which is done in what follows.

Lemma 1. *Let pairs* (v_i^*, u_i^*), $i = 1, 2, \cdots, m$, *be the optimal solutions of the following LP problem*:

$$\begin{cases} \min \sum_{i=1}^{m} v_i \\ s.t \quad v_i \geq u_i, v_i \geq -u_i, v_i \geq 0, u_i \in I. \end{cases}$$

where *I* is a compact set. Then u_i^*, $i = 1, 2, \cdots, m$, is the optimal solution of the following NLP problem:

$$\min_{u \in I} \sum_{i=1}^{m} |u_i|.$$

Proof. Since, (v_i^*, u_i^*), $i = 1, 2, \cdots, m$, is the optimal solution of the LP problem, so they satisfy the con-

straints. Thus there is $v_i^* \geq u_i^*$ and $v_i^* \geq -u_i^*$ for $i = 1, 2, \cdots, m$. Hence, $\left| u_i^* \right| \leq v_i^*$, $i = 1, 2, \cdots, m$, and so $\sum_{i=1}^{m} \left| u_i^* \right| \leq \sum_{i=1}^{m} v_i^*$. Now, let there exist $\bar{u}_i^* \in I$, $i = 1, 2, \cdots, m$, such that $\sum_{i=1}^{m} \left| \bar{u}_i^* \right| < \sum_{i=1}^{m} \left| u_i^* \right|$. Define, $\bar{v}_i^* = \left| \bar{u}_i^* \right|$ for $i = 1, 2, \cdots, m$. Then $\bar{v}_i^* \geq \bar{u}_i^*$ and $\bar{v}_i^* \geq -\bar{u}_i^*$. Moreover, $\sum_{i=1}^{m} \bar{v}_i^* = \sum_{i=1}^{m} \left| \bar{u}_i^* \right|$ and hence

$$\sum_{i=1}^{m} \bar{v}_i^* = \sum_{i=1}^{m} \left| \bar{u}_i^* \right| < \sum_{i=1}^{m} \left| u_i^* \right| < \sum_{i=1}^{m} v_i^*,$$

So $\sum_{i=1}^{m} \bar{v}_i^* < \sum_{i=1}^{m} v_i^*$, which is a contradiction. See [28] more details.

Now, by lemma 1, problem (14) can be converted to the following equivalent LP problem:

$$\min \ \sum_{i=1}^{n} \sum_{k=1}^{m} \frac{1}{m\Gamma(1-\alpha)} \mu_i^k$$

$$s.t - \mu_i^k + \sum_{h=1}^{k} x_i^h \left(t_{k+1} - t_h \right)^{-\alpha} - \sum_{h=1}^{k-1} x_i^h \left(t_k - t_h \right)^{-\alpha} + \left(\frac{1}{2m} \right)^{-\alpha} \left(x_i^{k+\frac{1}{2}} - x_i^{k-\frac{1}{2}} \right) \leq \Gamma(1-\alpha) g_i^k$$

$$- \mu_i^k - \sum_{h=1}^{k} x_i^h \left(t_{k+1} - t_h \right)^{-\alpha} + \sum_{h=1}^{k-1} x_i^h \left(t_k - t_h \right)^{-\alpha} - \left(\frac{1}{2m} \right)^{-\alpha} \left(x_i^{k+\frac{1}{2}} - x_i^{k-\frac{1}{2}} \right) \leq -\Gamma(1-\alpha) g_i^k \tag{15}$$

$$i = 1, 2, \cdots, n, \ k = 1, 2, \cdots, m.$$

By obtaining the solution of this problem, we recognize the value of unknown admissible x_i^k, $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, m$.

4. Numerical Examples

In this section, we give some numerical examples and apply the method presented in the last sections for solving them. Moreover, we extend this approach for approximately solving a model of HIV infection of CD4$^+$T cells with therapy effect including a system of FDEs. These test problems demonstrate the validity and efficiency of this approximation.

Example 1. *As first example, we compute* $D^\alpha x(t)$, *with* $\alpha = \frac{1}{2}$, *for* $x(t) = t^4$. *The exact formulas of the derivatives are derived from*

$$D^{0.5}(t^s) = \frac{\Gamma(s+1)}{\Gamma(s+1-0.5)} t^{s-0.5}$$

Figure 1 shows the results by using approximation (10)-(13) for $\alpha = 0.5$ and various choices of *m*.

Now, assume that $\bar{x}_i(t_k)$, $i = 1, \cdots, n$, $k = 1, \cdots, m$ and $x_i(t_k)$ are the approximated and exact solutions of system (3), respectively. We defined the absolute error of approximation as follow:

$$E = \max_k \left(\left\| x_i(t_k) - \bar{x}_i(t_k) \right\|_1 \right), i = 1, \cdots, n, \ k = 1, 2, \cdots, m. \tag{16}$$

In this example, the maximum absolute errors computed by Equation (16) for $\alpha = 0.5$ and various choices of *m*, has been shown in **Table 2**.

Example 2. *Consider the following initial value problem*:

$$D^{0.5}x(t) = t^2 - x(t) + \frac{2}{\Gamma(2.5)} t^{\frac{3}{2}}, \tag{17}$$

with initial condition $x(0) = 0$.

We know that $D^{0.5}(t^2) = \frac{2}{\Gamma(2.5)} t^{\frac{3}{2}}$. Therefore, the analytic solution for system (17) is $x(t) = t^2$. Now we expand the fractional derivative up to the problem (15). The solution is drawn in **Figures 2-4** for *m* = 20, 50, 100 and $\alpha = 0.1, 0.5, 0.99$.
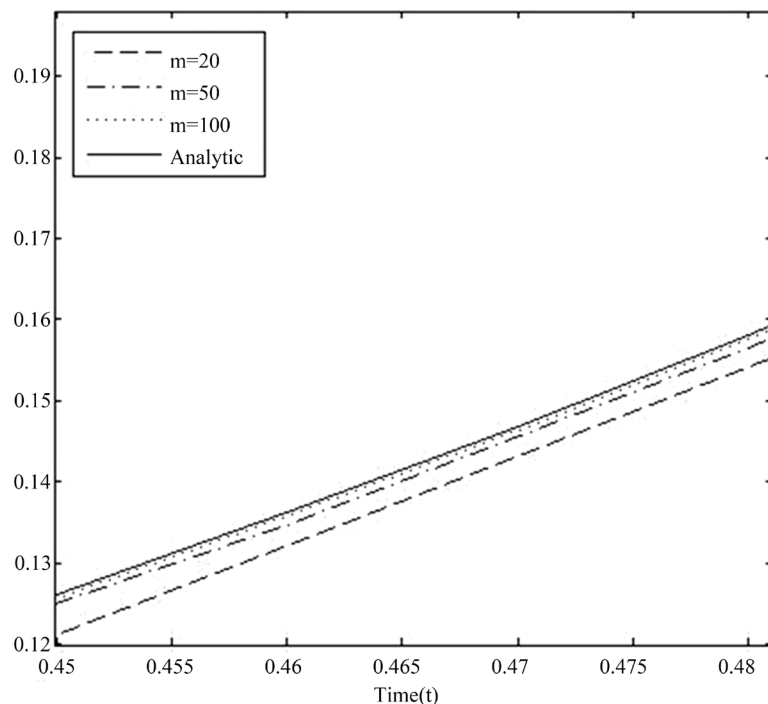
Figure 1. Analytic solution and numerical approximation (10), with various choices of *m* and $\alpha = 0.5$, for Example 1.

Table 2. Maximum absolute error for Example 1.

K	E_{appro}
5	2.65757×10^{-3}
10	6.47115×10^{-4}
20	4.09595×10^{-4}
40	9.77932×10^{-5}
50	2.27352×10^{-5}
100	5.04375×10^{-6}

In the case of $\alpha = 0.1, 0.5, 0.99$, the maximum absolute errors (16) with various choices of *m* is shown in **Table 3**.

From numerical results we can indicate that the solution of FDE approaches to the solution of integer order differential equation, whenever α approaches to its integer value.

Example 3. *Consider the following FDE*:

$$_0D_t^\alpha x(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - x(t) + t^2 - t, \tag{18}$$

where $0 < \alpha \le 1$ and $x(0) = 0$.

The exact solution of this equation is $x(t) = t^2 - t$. In **Figure 5** & **Figure 6**, we compare the exact solution with the numerical approximation (15) for two values of *m* and $\alpha = 0.5$.

Table 4 shows the exact solution and the approximate solution for equation (18) by solving problem (15) for $m = 100$ and $\alpha = 0.5, 0.99$. The results compare well with those obtained in [29].

Example 4. *Now we want to solve the fractional-order differential equation model of HIV infection of* CD4[+]T *cells* (1) *For the parameter values given in* **Table 1**. *The system* (1) *can be expressed in a vector form as follows*:
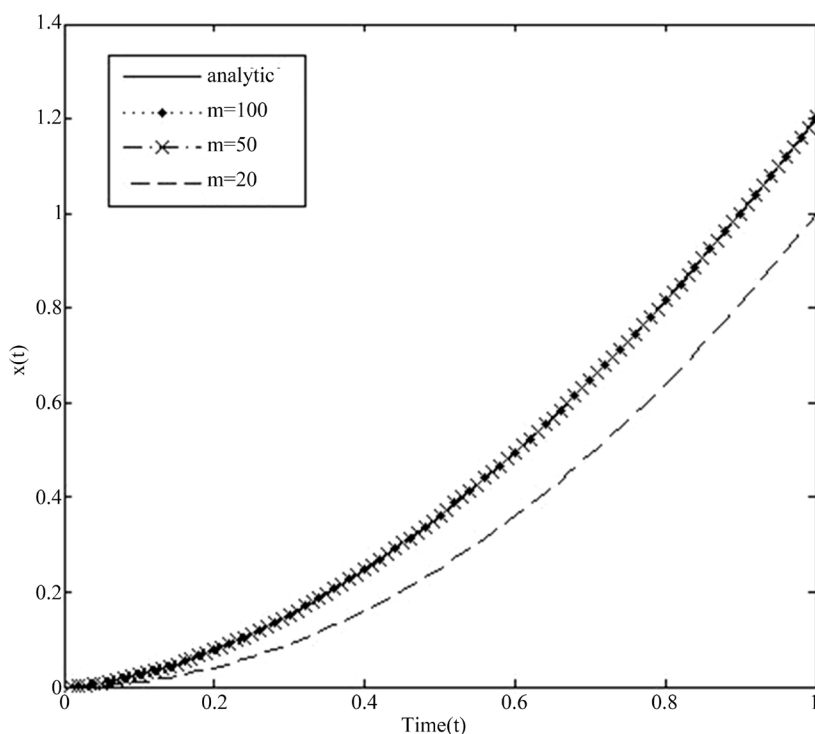
Figure 2. Exact and approximation solutions for problem in Example 2 with $\alpha = 0.1$ and different values of *m*.
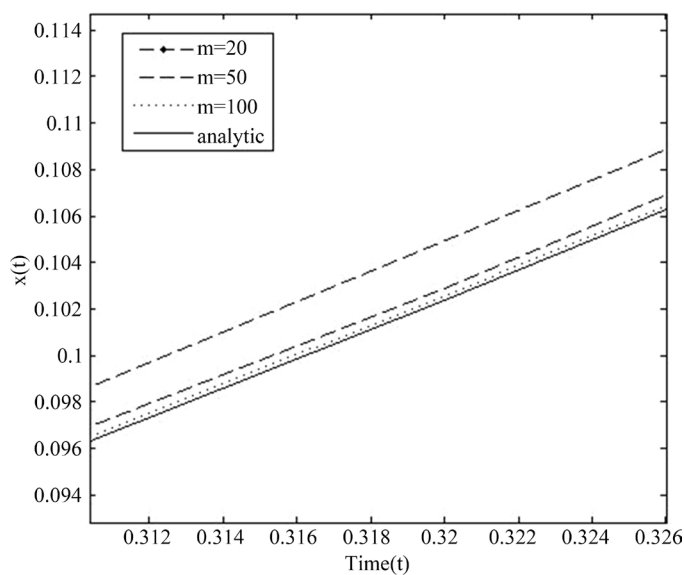

Figure 3. Exact and approximation solutions for problem in Example 2 with $\alpha = 0.5$ and different values of *m*.

$$D^{\alpha}x(t) = g(t, x(t)), \tag{19}$$

where $x(t) = (T(t), I(t), V(t))$ is the state vector and

$$x(0) = (T_0, 0, 0). \tag{20}$$

For numerical simulations we assumed 350 days for treatment period. With the change of variables $t = 350\tau$,
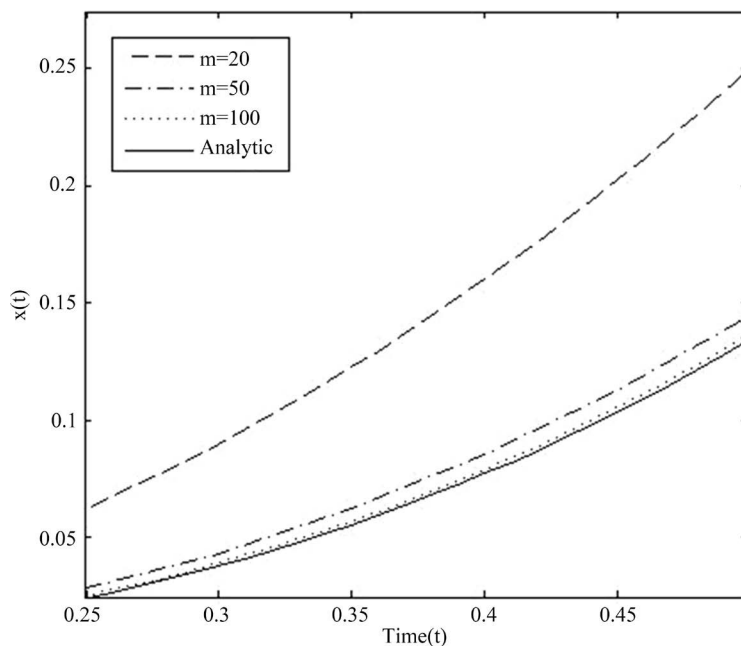
Figure 4. Exact and approximation solutions for problem in Example 2 with $\alpha = 0.99$ and different values of *m*.

Table 3. Maximum absolute error for different values of α, m for Example 2.

m	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.99$
10	0.26227	2.61278×10^{-4}	2.65757×10^{-3}
20	0.11574	2.14695×10^{-4}	6.47115×10^{-4}
40	0.08285	1.99832×10^{-4}	4.09595×10^{-4}
50	0.02621	1.55710×10^{-4}	9.77932×10^{-5}
100	0.00748	1.16467×10^{-4}	2.27352×10^{-5}

Table 4. Numerical values with $\alpha = 0.5, 0.99$ and $m = 100$ for Example 3.

| t	$x_{appro} \, (\alpha = 0.5)$	x_{exact}	$x_{appro} \, (\alpha = 0.99)$	x_{exact}
0.0	0.000000	0.000000	0.000000	0.000000
0.1	−0.089978	−0.090000	−0.089586	−0.090000
0.2	−0.159889	−0.160000	−0.159688	−0.160000
0.3	−0.209891	−0.210000	−0.209707	−0.210000
0.4	−0.239974	−0.240000	−0.239787	−0.240000
0.5	−0.249896	−0.250000	−0.249738	−0.250000
0.6	−0.239998	−0.240000	−0.239795	−0.240000
0.7	−0.199879	−0.210000	−0.209830	−0.210000
0.8	−0.160109	−0.160000	−0.159897	−0.160000
0.9	−0.096390	−0.090000	−0.100098	−0.090000
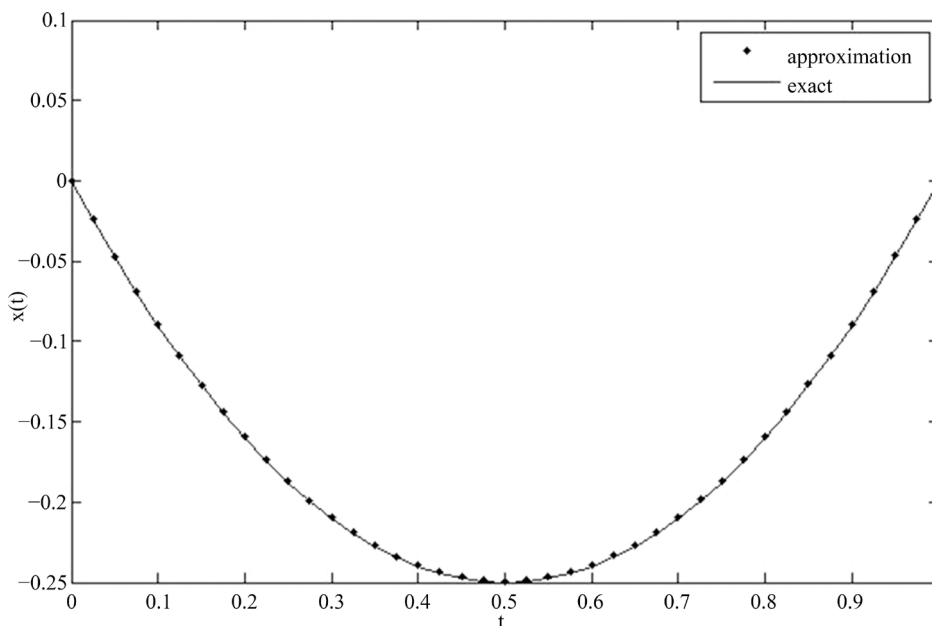
Figure 5. Analytic solution and numerical approximation (15) for Example 3 for $m = 40$.
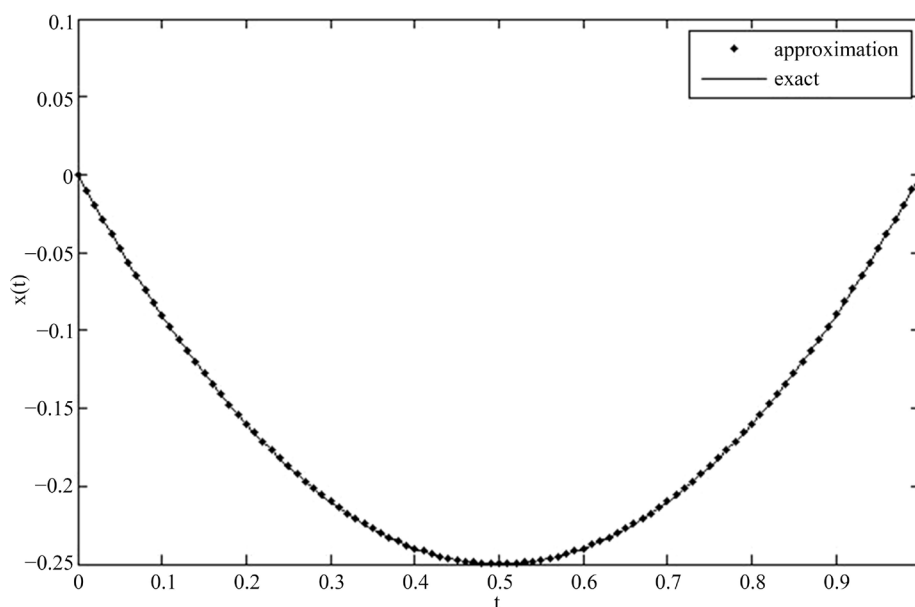

Figure 6. Analytic solution and numerical approximation (15) for Example 3 for $m = 100$.

we converted period $t \in [0,350]$ to $\tau \in [0,1]$. Based on concepts was said in the previous section, the key to the derivation of the approach is to replace the system (19) by the following equivalent optimization problem:

$$
\begin{aligned}
\min \sum_{k=1}^{m} \int_{\tau_{i-1}}^{\tau_i} & \left\{ \left| D^\alpha T(350\tau) - s - \mu_T T(350\tau) + rT(350\tau) \left(1 - \frac{T(350\tau) + I(350\tau)}{T_{\max}} \right) - k_V T(350\tau) \right| \right. \\
& + \left| D^\alpha I(350\tau) - k_1' V(350\tau) T(350\tau) - \mu_I I(350\tau) \right| \\
& + \left. \left| D^\alpha V(350\tau) - N\mu_b I(350\tau) - k_1 V(350\tau) T(350\tau) - \mu_V V(350\tau) \right| \right\} d\tau,
\end{aligned}
\tag{21}
$$

Table 5. Maximum absolute error for $m = 100$ and different values of α for Example 4.

P	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
T	$8.0224E-5$	$1.0253E-6$	$2.2102E-7$	$2.8532E-8$	$1.0929E-9$
I	$1.1031E-5$	$1.9756E-6$	$1.1102E-8$	$2.4527E-8$	$1.7608E-9$
V	$1.2226E-5$	$2.7756E-7$	$1.19553E-7$	$2.6392E-9$	$1.2079E-9$

with the initial condition (20). To solve this optimization problem, by approximating integrals as before, we transformed (21) to a discretized problem in the following form:

$$
\min \frac{1}{m} \sum_{k=1}^{m} \Bigg\{ \left| D^\alpha T\left(350\tau_k\right) - s - \mu_T T\left(350\tau_k\right) + rT\left(350\tau_k\right)\left(1 - \frac{T\left(350\tau_k\right) + I\left(350\tau_k\right)}{T_{\max}}\right) - k_V T\left(350\tau_k\right) \right|
$$

$$
+ \left| D^\alpha I\left(350\tau_k\right) - k_1' V\left(350\tau_k\right) T\left(350\tau_k\right) - \mu_I I\left(350\tau_k\right) \right| \tag{22}
$$

$$
+ \left| D^\alpha V\left(350\tau_k\right) - N\mu_b I\left(350\tau_k\right) - k_I V\left(350\tau_k\right) T\left(350\tau_k\right) - \mu_V V\left(350\tau_k\right) \right| \Bigg\}.
$$

In problem (21) and (22), the factor 350 is omitted because of having no effect on the solution of it. Then, the minimum problem (22) converted to a linear programming problem with the following change of variables:

$$
\min \frac{1}{m} \sum_{k=1}^{m} u_k + v_k + r_k + e_k + w_k + q_k
$$

$$
s.t.\ D^\alpha T\left(350\tau_k\right) - s - \mu_T T\left(350\tau_k\right) + rT\left(350\tau_k\right)\left(1 - \frac{T\left(350\tau_k\right) + I\left(350\tau_k\right)}{T_{\max}}\right) - k_V T\left(350\tau_k\right) = u_k - v_k,
$$

$$
D^\alpha I\left(350\tau_k\right) - k_1' V\left(350\tau_k\right) T\left(350\tau_k\right) - \mu_I I\left(350\tau_k\right) = r_k - e_k, \tag{23}
$$

$$
D^\alpha V\left(350\tau_k\right) - N\mu_b I\left(350\tau_k\right) - k_I V\left(350\tau_k\right) T\left(350\tau_k\right) - \mu_V V\left(350\tau_k\right) = w_k - q_k,
$$

$$
T\left(0\right) = T_0, I\left(0\right) = V\left(0\right) = 0, u_k, v_k, r_k, e_k, w_k, q_k \geq 0.
$$

Now, we approximate fractional derivatives from (10)-(13). Our approach introduces an approximate solution for the fractional HIV model based on minimization the total error. The maximum absolute errors (16) with $m = 100$ and different values of α that shown in **Table 5**, confirmed the efficacy of our approach in comparison with the result obtained by [25].

5. Conclusions

In this paper, the finite difference method discrete time AVK approach has been successfully used for finding the solutions of a system of FDEs such as a model for HIV infection of CD4+T cells. Our approach introduces an approximate solution for the FDEs based on the minimization of the total error. In the suggested method, the original problem reduces to an optimization problem. By discretizing the new problem and solving it, we obtain the best approximate solution of the original problem. Results represent a unifying approach for numerical approximation of differential equations of fractional order. Since this method is not based on point to point error, but according to its results, it is clear that there is no difference between the exact and approximate solutions in point to point case.

Three numerical examples are given and the results are compared with the exact solutions and with the other methods. It is shown that, as the order of fractional derivatives approaches to 1, the numerical solutions for the FDEs approach the clasicall solutions of the problem. Then we use this technique for finding approximate solutions of FDEs system of a model for HIV infection of CD4+T cells. The result demonstrates the validity of the approach.

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