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Random Crank-Nicolson Scheme for Random Heat Equation in Mean Square Sense

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Abstract

This paper deals with the construction of random Crank-Nicolson solution for heat equation containing uncertainty through the coefficient. Under suitable hypotheses on data, we prove that the constructed random Crank-Nicolson solution is satisfying mean square convergent through the whole space. Furthermore, the basic main statistical measures, like as the expected value and the variance, of the computationally solution obtained by the random Crank-Nicolson scheme are given. At the last, we apply the proposed method to illustrative cases.

Keywords

Random Partial Differential Equations (RPDEs), Mean Square Sense (m.s), Second Order Random Variable $(2r.v.'s)$, Random Crank-Nicolson Scheme, Convergence, Consistency, Stability

1. Introduction

The goal of computational science is to develop models that predict phenomena observed in nature. However, these models are often based on parameters that are uncertain. In recent decades, main numerical methods for solving SPDEs have been used such as, finite difference and finite element schemes [1]-[5]. Also, some practical techniques like the method of lines for boundary value problems have been applied to the linear stochastic partial differential equations, and the outcomes of these approaches have been experimented numerically [7]. In [8]-[10], the author discussed mean square convergent finite difference method for solving some random partial differential equations. Random numerical techniques for both ordinary and partial random differential equations are treated in [4] [10]. As regards applications using explicit analytic solutions or numerical methods, a few results may be found in [5] [6] [11]. This article focuses on solving random heat equation by using Crank-Nicol-

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son technique under mean square sense and it is organized as follows. In Section 2, the mean square calculus preliminaries that will be required throughout the paper are presented. In Section 3, the Crank-Nicolson scheme for solving the random heat equation is presented. In Section 4, some case studies are showed. Short conclusions are cleared in the end section.

2. Preliminaries

Definition 2.1. Let us take in to consideration that, the properties of a class of real random variables $r.v.'s\ X_1, X_2, \cdots, X_n$ whose second moments, $E\left\{X_1^2\right\}, E\left\{X_2^2\right\}, \cdots, E\left\{X_n^2\right\}$ are finite. In this case they are called second order random variables $(2r.v'.s)$.

Definition 2.2. A sequence of $2.r.v.'s\ \left\{X_n, n>0\right\}$ is mean square convergent to a random variable X if: $\lim_{n\to\infty} E\left|X_n - X\right|^2 = 0.$

3. Random Crank-Nicolson Scheme (RCNS)

If we have the linear random heat problem of the form:

$$u_t = \alpha u_{xx},\ t \in [0,\infty),\ x \in [a,b] \tag{1}$$

Where α is a second order random variable.

$$u(x,0) = u_0(x) \tag{2}$$

$$u(a,t) = g(t),\ \ u(b,t) = h(t) \tag{3}$$

Then, we can find the random Crank-Nicolson scheme for this problem as follows:

Take a uniform mesh with step size Δx and Δt on x-axis and t-axis respectively. Additionally, Let u_k^n approximates $u(x, t)$ at point $(k\Delta x, n\Delta t)$. Hence $u_k^0 = u_0(k\Delta x)$. On this mesh we have:

$$u_t(x,t) \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

Then,

$$u_t(k\Delta x, n\Delta t) \approx \frac{u_k^{n+1} - u_k^n}{\Delta t}$$

Similarly,

$$u_{xx}(x,t) \approx \frac{1}{2}\left(\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2} + \frac{u(x+\Delta x,t+1) - 2u(x,t+1) + u(x-\Delta x,t+1)}{\Delta x^2}\right)$$

Then,

$$u_{xx}(k\Delta x, n\Delta t) \approx \frac{1}{2}\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} + \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2}\right)$$

Hence for (1):

$$\frac{u_k^{n+1} - u_k^n}{\Delta t} = \frac{\alpha}{2}\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} + \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2}\right)$$

$$u_k^{n+1} - u_k^n = \frac{\alpha\Delta t}{2}\left(\frac{u_{k+1}^n - 2u_k^n + u_{k-1}^n}{\Delta x^2} + \frac{u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}}{\Delta x^2}\right)$$

$$u_k^{n+1} = u_k^n + \frac{\alpha\Delta t}{2\Delta x^2}\left(u_{k+1}^n - 2u_k^n + u_{k-1}^n + u_{k+1}^{n+1} - 2u_k^{n+1} + u_{k-1}^{n+1}\right)$$

Put $r = \frac{\Delta t}{2\Delta x^2}$

Hence, the RCNS for our problem is:

$$-r\alpha u_{k+1}^{n+1} + (1+2r\alpha) u_k^{n+1} - r\alpha u_{k-1}^{n+1} = r\alpha u_{k+1}^n + (1-2r\alpha) u_k^n + r\alpha u_{k-1}^n \tag{4}$$

$$u_k^0 = u_0(k\Delta x) = u_0(x_k) \tag{5}$$

$$u_a^n = g(n\Delta t) = g(t_n), u_b^n = h(n\Delta t) = h(t_n) \tag{6}$$

3.1. Consistency of RCNS

We can rewrite the above scheme as:

$$u_{k+1}^{n+1} = \frac{1+2r\alpha}{r\alpha} u_k^{n+1} - u_{k-1}^{n+1} - u_{k+1}^n + \frac{2r\alpha - 1}{r\alpha} u_k^n - u_{k-1}^n$$

The above scheme is a random Crank-Nicolson version of (1 - 3). For a RPDE, say $Lv = G$ where L is a differentiable operator and $G \in L^2(R)$. On the other hand, we represent finite difference scheme at the point $(k\Delta x, n\Delta t)$ by $L_k^n u_k^n = G_k^n$.

Definition 3.1.1. A random difference scheme $L_k^n u_k^n = G_k^n$ that approximating RPDE $Lv = G$ is **consistent** in mean square sense at time $t = (n+1)\Delta t$, if for any continuously differentiable function $\Phi = \Phi(x,t)$, we have in mean square:

$$E\left|(L\Phi - G)_k^n - \left(L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n\right)\right|^2 \to 0$$

$$\text{As} \quad \Delta t \to 0, \Delta x \to 0, k \to \infty, n \to \infty \quad \text{and} \quad (k\Delta x, n\Delta t) \to (x,t)$$

Theorem 3.1.1. The random Crank-Nicolson difference scheme (4)-(6) with second order random variable is to be consistent in mean square sense as: $\Delta t \to 0, \Delta x \to 0, n \to \infty$ and $(k\Delta x, n\Delta t) \to (x,t)$.

Proof. Assume that $\Phi(x,t)$ is a deterministic smooth function then:

$$L(\Phi)_k^n = \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \alpha \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,ds$$

$$\begin{aligned}
&L_k^n \Phi(k\Delta x, n\Delta t) \\
&= \Phi((k+1)\Delta x, (n+1)\Delta t) - \frac{1+2r\alpha}{r\alpha}\Phi(k\Delta x, (n+1)\Delta t) - \Phi((k-1)\Delta x, (n+1)\Delta t) \\
&\quad + \Phi((k+1)\Delta x, n\Delta t) - \frac{2r\alpha - 1}{r\alpha}\Phi(k\Delta x, n\Delta t) + \Phi((k-1)\Delta x, n\Delta t)
\end{aligned}$$

Then,

$$\begin{aligned}
&E\left|L(\Phi)_k^n - L_k^n \Phi\right|^2 \\
&= E\Bigg[\frac{1+3r\alpha}{r\alpha}\Phi(k\Delta x, (n+1)\Delta t) + \frac{r\alpha - 1}{r\alpha}\Phi(k\Delta x, n\Delta t) - \Phi((k+1)\Delta x, (n+1)\Delta t) \\
&\quad - \Phi((k-1)\Delta x, (n+1)\Delta t) - \Phi((k+1)\Delta x, n\Delta t) - \Phi((k-1)\Delta x, n\Delta t) - \alpha \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,ds\Bigg]^2 \\
&= E\Bigg[\left[-\alpha \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,ds\right]^2 + 2\left[-\alpha \int_{n\Delta t}^{(n+1)\Delta t} \Phi_{xx}(k\Delta x, s)\,ds\right]\Bigg[\frac{1+3r\alpha}{r\alpha}\Phi(k\Delta x, (n+1)\Delta t) \\
&\quad + \frac{r\alpha - 1}{r\alpha}\Phi(k\Delta x, n\Delta t) - \Phi((k+1)\Delta x, (n+1)\Delta t) - \Phi((k-1)\Delta x, (n+1)\Delta t) \\
&\quad - \Phi((k+1)\Delta x, n\Delta t) - \Phi((k-1)\Delta x, n\Delta t)\Bigg] + \Bigg[\frac{1+3r\alpha}{r\alpha}\Phi(k\Delta x, (n+1)\Delta t) + \frac{r\alpha - 1}{r\alpha}\Phi(k\Delta x, n\Delta t) \\
&\quad - \Phi((k+1)\Delta x, (n+1)\Delta t) - \Phi((k-1)\Delta x, (n+1)\Delta t) - \Phi((k+1)\Delta x, n\Delta t) - \Phi((k-1)\Delta x, n\Delta t)\Bigg]^2\Bigg]
\end{aligned}$$

As: $\Delta t \to 0, \Delta x \to 0, n \to \infty$, $(k\Delta x, n\Delta t) \to (x,t)$ and at time $t=(n+1)\Delta t$, Then we have

$$E\left|L(\Phi)_k^n - L_k^n\Phi\right|^2 \to 0$$

Hence, the random Crank-Nicolson scheme (4)-(6) is **consistent** in mean square sense.■

3.2. Exponential Stability Analysis of RCNS

Definition 3.2.1. A random Crank-Nicolson difference scheme $L_k^n u_k^n = G_k^n$ is **exponential stable** in mean square if there exist some positive constants *a*, *c* and constants *k*, *b*. Such that:

$$E\left|u_k^{n+1}\right|^2 \le k e^{-bt} E\left|u^0\right|^2$$

For: $t=(n+1)\Delta t \ge 0, 0 \le \Delta x \le a$ and $0 \le \Delta t \le c$

Theorem 3.2.1. The random Crank-Nicolson scheme (4)-(6) with second order random variable is ***unconditionally stable*** *in* mean square sense as with *k* = 1 and *b* = 0.

Proof: Since,

$$u_{k+1}^{n+1} = \frac{1+2r\alpha}{r\alpha}u_k^{n+1} - u_{k-1}^{n+1} - u_{k+1}^n + \frac{2r\alpha-1}{r\alpha}u_k^n - u_{k-1}^n$$

Then,

$$\begin{aligned}
E\left|u_{k+1}^{n+1}\right|^2 &= E\left|\frac{1+2r\alpha}{r\alpha}u_k^{n+1} - u_{k-1}^{n+1} - u_{k+1}^n + \frac{2r\alpha-1}{r\alpha}u_k^n - u_{k-1}^n\right|^2 \\
&= E\Bigg[\left(\frac{1+2r\alpha}{r\alpha}\right)^2\left|u_k^{n+1}\right|^2 + \left|u_{k-1}^{n+1}\right|^2 + \left|u_{k+1}^n\right|^2 + \left(\frac{2r\alpha-1}{r\alpha}\right)^2\left|u_k^n\right|^2 + \left|u_{k-1}^n\right|^2 - 2\left(\frac{1+2r\alpha}{r\alpha}\right)\left|u_k^{n+1}u_{k-1}^{n+1}\right| \\
&\quad - 2\left(\frac{1+2r\alpha}{r\alpha}\right)\left|u_k^{n+1}u_{k+1}^n\right| + 2\left(\frac{4r^2\alpha^2-1}{r^2\alpha^2}\right)\left|u_k^{n+1}u_k^n\right| - 2\left(\frac{1+2r\alpha}{r\alpha}\right)\left|u_k^{n+1}u_{k-1}^n\right| + 2\left|u_{k-1}^{n+1}u_{k+1}^n\right| \\
&\quad - 2\left(\frac{2r\alpha-1}{r\alpha}\right)\left|u_{k-1}^{n+1}u_k^n\right| + 2\left|u_{k-1}^{n+1}u_{k-1}^n\right| + 2\left|u_{k+1}^n u_{k-1}^n\right| - 2\left(\frac{2r\alpha-1}{r\alpha}\right)\left|u_{k+1}^n u_k^n\right| - 2\left(\frac{2r\alpha-1}{r\alpha}\right)\left|u_{k-1}^n u_k^n\right|\Bigg] \\
&= E\Bigg[\frac{1}{r^2\alpha^2}\left|u_k^{n+1}\right|^2 + \frac{4}{r\alpha}\left|u_k^{n+1}\right|^2 + 4\left|u_k^{n+1}\right|^2 + \left|u_{k-1}^{n+1}\right|^2 + \left|u_{k+1}^n\right|^2 + 4\left|u_k^n\right|^2 - \frac{4}{r\alpha}\left|u_k^n\right|^2 \\
&\quad + \frac{1}{r^2\alpha^2}\left|u_k^n\right|^2 + \left|u_{k-1}^n\right|^2 - \frac{2}{r\alpha}\left|u_k^{n+1}u_{k-1}^{n+1}\right| - 4\left|u_k^{n+1}u_{k-1}^{n+1}\right| - \frac{2}{r\alpha}\left|u_k^{n+1}u_{k+1}^n\right| - 4\left|u_k^{n+1}u_{k+1}^n\right| \\
&\quad + 8\left|u_k^{n+1}u_k^n\right| - \frac{2}{r^2\alpha^2}\left|u_k^{n+1}u_k^n\right| - \frac{2}{r\alpha}\left|u_k^{n+1}u_{k-1}^n\right| - 4\left|u_k^{n+1}u_{k-1}^n\right| + 2\left|u_{k-1}^{n+1}u_{k+1}^n\right| - 4\left|u_{k-1}^{n+1}u_k^n\right| \\
&\quad + \frac{2}{r\alpha}\left|u_{k-1}^{n+1}u_k^n\right| + 2\left|u_{k-1}^{n+1}u_{k-1}^n\right| + 2\left|u_{k+1}^n u_{k-1}^n\right| - 4\left|u_{k+1}^n u_k^n\right| + \frac{2}{r\alpha}\left|u_{k+1}^n u_k^n\right| - 4\left|u_{k-1}^n u_k^n\right| + \frac{2}{r\alpha}\left|u_{k-1}^n u_k^n\right|\Bigg] \\
&\le \frac{1}{r^2}\sup_{k,n} E\left[\left(\frac{1}{\alpha^2}\right)\left|u_k^n\right|^2\right] + \frac{4}{r}\sup_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] + 4\sup_{k,n} E\left[\left|u_k^n\right|^2\right] + \sup_{k,n} E\left[\left|u_k^n\right|^2\right] + \sup_{k,n} E\left[\left|u_k^n\right|^2\right] \\
&\quad + 4\sup_{k,n} E\left[\left|u_k^n\right|^2\right] - \frac{4}{r}\inf_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] + \frac{1}{r^2}\sup_{k,n} E\left[\left(\frac{1}{\alpha^2}\right)\left|u_k^n\right|^2\right] + \sup_{k,n} E\left[\left|u_k^n\right|^2\right] - \frac{2}{r}\inf_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] \\
&\quad - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] - \frac{2}{r}\inf_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] + 8\sup_{k,n} E\left[\left|u_k^n\right|^2\right] - \frac{2}{r^2}\inf_{k,n} E\left[\left(\frac{1}{\alpha^2}\right)\left|u_k^n\right|^2\right] \\
&\quad - \frac{2}{r}\inf_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] + 2\sup_{k,n} E\left[\left|u_k^n\right|^2\right] - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] + \frac{2}{r}\sup_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] \\
&\quad + 2\sup_{k,n} E\left[\left|u_k^n\right|^2\right] + 2\sup_{k,n} E\left[\left|u_k^n\right|^2\right] - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] + \frac{2}{r}\sup_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] \\
&\quad - 4\inf_{k,n} E\left[\left|u_k^n\right|^2\right] + \frac{2}{r}\sup_{k,n} E\left[\left(\frac{1}{\alpha}\right)\left|u_k^n\right|^2\right] = \sup_{k,n} E\left[\left|u_k^n\right|^2\right]
\end{aligned}$$

Finally, we have: $\sup_{k,n} E\left[\left|u_{k+1}^{n+1}\right|^2\right] \le \sup_{k,n} E\left[\left|u_k^n\right|^2\right] \le \cdots \le \sup_{k,n} E\left[\left|u_k^0\right|^2\right]$

At: $t=(n+1)\Delta t$ then, we have:

$$E\left[\left|u^{n+1}\right|^2\right] \le E\left[\left|u^0\right|^2\right]$$

Hence, the random Crank-Nicolson difference scheme with second order random variable is **unconditionally stable** with $k=1$ and $b=0$. ■

3.3. Convergence of RCNS

Definition 3.3.1. A random difference scheme $L_k^n u_k^n = G_k^n$ that approximating RPDE $Lv = G$ is **convergent** in mean square sense at time $t=(n+1)\Delta t$, if:

$$E\left|u_k^n - u\right|^2 \to 0 \text{, as } \Delta t \to 0, \Delta x \to 0, k \to \infty, n \to \infty \text{ and } (k\Delta x, n\Delta t) \to (x,t)$$

Theorem 3.3.1. The random Crank-Nicolson difference scheme (4)-(6) with second order random variables is **convergent** in mean square sense.

Proof.

Since, the RCNS is **consistent** and **unconditionally exponential stable,** thus, the scheme (4)-(6) is convergent in mean square sense.■

4. Case Studies

Consider the linear random parabolic partial differential equation:

$$u_t = \alpha u_{xx},\ t \in [0,\infty),\ x \in [0, X]$$

With initial condition

$$u(x,0) = \sin \pi x \tag{7}$$

and the boundary conditions

$$u(0,t) = u(X,t) = 0$$

And α is a second order random variable.

4.1. The Exact Relation

$$u(x,t) = e^{-i\alpha\pi^2 t} \sin(\pi x) \tag{8}$$

4.2. The Numerical Solution

The Random Crank-Nicolson Difference Scheme for this problem is

$$-r\alpha u_{k+1}^{n+1} + (1+2r\alpha)u_k^{n+1} - r\alpha u_{k-1}^{n+1} = r\alpha u_{k+1}^n + (1-2r\alpha)u_k^n + r\alpha u_{k-1}^n \tag{9}$$

where $r = \dfrac{\Delta t}{2\Delta x^2}$, $t_n = n\Delta t$ and $x_k = k\Delta x.$

Substituting by $k = 1,2,3$ in (9) we have:

$$\begin{bmatrix} 1+2r\alpha & -r\alpha & 0 \\ -r\alpha & 1+2r\alpha & -r\alpha \\ 0 & -r\alpha & 1+2r\alpha \end{bmatrix} \begin{bmatrix} u_{1,n+1} \\ u_{2,n+1} \\ u_{3,n+1} \end{bmatrix} = \begin{bmatrix} 1-2r\alpha & r\alpha & 0 \\ r\alpha & 1-2r\alpha & r\alpha \\ 0 & r\alpha & 1-2r\alpha \end{bmatrix} \begin{bmatrix} u_{1,n} \\ u_{2,n} \\ u_{3,n} \end{bmatrix}$$

Putting $n = 0$ in the above system then we have:

$$\begin{bmatrix} 1+2r\alpha & -r\alpha & 0 \\ -r\alpha & 1+2r\alpha & -r\alpha \\ 0 & -r\alpha & 1+2r\alpha \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 1-2r\alpha & r\alpha & 0 \\ r\alpha & 1-2r\alpha & r\alpha \\ 0 & r\alpha & 1-2r\alpha \end{bmatrix} \begin{bmatrix} \sin(\pi\Delta x) \\ \sin(2\pi\Delta x) \\ \sin(3\pi\Delta x) \end{bmatrix}$$

Then, we have the system:

$$(1+2r\alpha)u_{1,1} - r\alpha u_{2,1} = (1-2r\alpha)\sin(\pi\Delta x) + r\alpha\sin(2\pi\Delta x)$$

$$-r\alpha u_{1,1} + (1+2r\alpha)u_{2,1} - r\alpha u_{3,1} = r\alpha\sin(\pi\Delta x) + (1-2r\alpha)\sin(2\pi\Delta x) + r\alpha\sin(3\pi\Delta x)$$

$$-r\alpha u_{2,1} + (1+2r\alpha)u_{3,1} = r\alpha\sin(2\pi\Delta x) + (1-2r\alpha)\sin(3\pi\Delta x)$$

From this system we have:

$$u_{11} = \frac{\sin(\pi\Delta x)\left[8r\alpha^2\cos^2(\pi\Delta x) + (\cos(\pi\Delta x))(-2r\alpha^3 + 8r\alpha^2 + 4r\alpha) - 3r\alpha^3 - 6r\alpha^2 + r\alpha + 1\right]}{(2r\alpha^2 + 4r\alpha + 1)(1+2r\alpha)}$$

$$u_{21} = \frac{\sin(\pi\Delta x)\left[8r\alpha\cos^2(\pi\Delta x) - (\cos(\pi\Delta x))(6r\alpha^2 - 2) + r\alpha^2\right]}{(2r\alpha^2 + 4r\alpha + 1)}$$

$$u_{31} = \frac{\sin(\pi\Delta x)}{(2r\alpha^2 + 4r\alpha + 1)(1+2r\alpha)}\left[\cos^2(\pi\Delta x)(16r\alpha^3 + 16r\alpha^2 - 8r\alpha - 4) + (\cos(\pi\Delta x))(6r\alpha^3 - 2r\alpha) - 7r\alpha^3 - 10r\alpha^2 + r\alpha + 1\right]$$

Verification for the Convergence of Mean

(1) Changing step size Δt

- Choosing: $\Delta x = 0.01, \alpha \sim \text{Poisson}(0.05)$

k	n	t_n	$E(u_k^n)$	$E\left(u(x,t)_{x_k,t_n}\right)$	$\left\lvert E\left(u(x,t)_{x_k,t_n}\right) - E(u_k^n)\right\rvert$
1	1	0.05	0.002112928253	0.03065693462	0.02854400637
1	1	0.01	0.008963097412	0.03126858723	0.02230548982
1	1	0.005	0.01499503761	0.03134589691	0.01635085930

- Choosing: $\Delta x = 0.01, \alpha \sim \text{Exponential}(0.05)$

k	n	t_n	$E(u_k^n)$	$E\left(u(x,t)_{x_k,t_n}\right)$	$\left\lvert E\left(u(x,t)_{x_k,t_n}\right) - E(u_k^n)\right\rvert$
1	1	0.05	0.001977193183	0.03065693462	0.02867974144
1	1	0.01	0.008336142923	0.03126858723	0.02293244431
1	1	0.005	0.01386553736	0.03134589691	0.01748035955

(2) Changing step size Δx

- Choosing: $\Delta t = 0.01, \alpha \sim \text{Poisson}(0.05)$

k	n	t_n	$E(u_k^n)$	$E\left(u(x,t)_{x_k,t_n}\right)$	$\left\lvert E\left(u(x,t)_{x_k,t_n}\right) - E(u_k^n)\right\rvert$
1	1	0.005	0.001307560560	0.01563622416	0.01432866360
1	1	0.0005	0.000001382256	0.001563686128	0.001562303872
1	1	0.00005	0.00000000138304	0.0001563686765	0.0001563672935

- Choosing: $\Delta t = 0.01$, $\alpha \sim \text{Exponential}(0.05)$

k	n	t_n	$E(u_k^n)$	$E(u(x,t)_{x_k,t_n})$	$\lvert E(u(x,t)_{x_k,t_n}) - E(u_k^n)\rvert$
1	1	0.005	0.001222325088	0.01563622416	0.01441389907
1	1	0.0005	0.000001294722	0.001563686128	0.001562391406
1	1	0.00005	0.00000000129548	0.0001563686765	0.0001563673810

(3) Changing the expectations

- Choosing: $\Delta x = 0.01$, $\Delta t = \frac{1}{3}$, $\alpha \sim$ Poisson distribution

k	n	t_n	$E(u_k^n)$	$E(u(x,t)_{x_k,t_n})$	$\lvert E(u(x,t)_{x_k,t_n}) - E(u_k^n)\rvert$
1	1	0.001	0.01272168525	0.03132010580	0.01859842055
1	1	0.0008	0.01489325756	0.03134073699	0.01644747943
1	1	0.0004	0.02241624210	0.03138204014	0.00896579804

- Choosing: $\Delta x = 0.01$, $\Delta t = \frac{1}{3}$, $\alpha \sim$ Exponential distribution

k	n	t_n	$E(u_k^n)$	$E(u(x,t)_{x_k,t_n})$	$\lvert E(u(x,t)_{x_k,t_n}) - E(u_k^n)\rvert$
1	1	0.001	0.03132010580	0.01042366840	0.02089643740
1	1	0.0008	0.01219546475	0.03134073699	0.01914527224
1	1	0.0004	0.01837369513	0.03138204014	0.00896579804

From these tables we note that the error is acceptable if:

1) The changing happens in Δx when $\Delta t, E[\alpha]$ are constant values.
2) The changing happens in Δt when $\Delta x, E[\alpha]$ are constant values.
3) The changing happens in $E[\alpha]$ when $\Delta t, \Delta x$ are constant values.

5. Conclusion

The random heat equation can be solved numerically by using mean square convergent Crank-Nicolson scheme. The random variable in the Crank-Nicolson scheme is must second order random variable and the random Crank-Nicolson scheme is unconditionally stable in the area of mean square sense. Many complicated equations in linear and nonlinear parabolic partial differential problems can be discussed using finite difference schemes in mean square sense.

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