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A Simple Proof of Gustafsson's Conjecture in Case of Poisson Equation on Rectangular Domains

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Abstract

We consider the standard five-point finite difference method for solving the Poisson equation with the Dirichlet boundary condition. Its associated matrix is a typical ill-conditioned matrix whose size of the condition number is as big as $O(h^{-2})$. Among ILU, SGS, modified ILU (MILU) and other ILU-type preconditioners, Gustafson shows that only MILU achieves an enhancement of the condition number in different order as $O(h^{-1})$. His seminal work, however, is not for the MILU but for a perturbed version of MILU and he observes that without the perurbation, it seems to reach the same result in practice. In this work, we give a simple proof of Gustafsson's conjecture on the unnecessity of perturbation in case of Poisson equation on rectangular domains. Using the Cuthill-Mckee ordering, we simplify the recursive equation in two dimensional grid nodes into a recursive one in the level that is one-dimensional. Due to the simplification, our proof is easy to follow and very short.

Keywords

Five-Point Finite Difference Method, Modified ILU Preconditioning, Condition Number

1. Introduction

Consider the standard five-point finite difference method for solving the Poisson equation with the Dirichlet boundary condition. Its associated matrix is a typical ill-conditioned matrix whose condition number is of size $O(h^{-2})$, where h is the grid size. In mitigating the large size, Dupont, Kendall and Rachfold [1] propose a

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preconditioning technique which works quite well for elliptic problems with $O\left(h^{-1}\right)$ convergence rate, which is a simple modification of incomplete LU (ILU) and called the modified ILU (MILU) preconditioning technique. The MILU requires all the same row sums for the preconditioner and the original matrices. Also, Gustafsson [2] [3] shows that the MILU preconditiong reduces the size to $O\left(h^{-1}\right)$, while other popular preconditionings such as ILU and symmetric Gauss-Seidel (SGS) do not improve the order. Numerical study by Greenbaum and Rodrigue [4] indicates that further reduction is not possible with the same sparsity pattern.

The MILU preconditioing introduced by Axelsson [5] and developed by Gustafsson [2] adds some artificial diagonal perturbation on the orginal matrix. In [1] and [2], it is found that a small positive perturbation improves the convergence rate quite well for many elliptic problems. We refer to [6]-[9] and references therein for more results and details.

The numerical experiments [10] with Dirichlet boundary condition, however, suggest that the perturbation is unnecessary. It is Gustafsson's conjecture [2] [11] to prove the estimate $O\left(h^{-1}\right)$ for the unperturbed MILU preconditioing. Beauwens [12] considers a general setting that includes the five-point method, and proves the conjecture using the matrix-graph connectivity properties (see also [13]). Beauwens' proof deals with a Stieltjes matrix under several assumptions. Notay [14] also obtains an upper bound $O\left(h^{-1}\right)$ for the block MILU with the line partitioning. We also refer the reader to [15]-[18] for related works on Gustafsson's conjecture.

We introduce a novel and heuristic proof for the conjecture in case of Poisson equation with Dirichlet boundary condition on rectangular domains. The MILU preconditioner is obtained from recursively calculating the row-sum equation at each grid node in the lexicographical ordering. In the case of the five-point method, it is well known [19] that the same matrix can be obtained in the Cuthill-Mckee ordering. The matrix entry on the (ih, jh) node depends only on $((i-1)h, jh)$ and $(ih,(j-1)h)$ nodes, both of which lie on the same level $n=i+j-1$ of the Cuthill-Mckee ordering. So we can simplify the recursive equation in two dimensional grid nodes into a recursive one in the level that is one dimensional. Due to the simplification, our proof is easy to follow and very short.

2. MILU Preconditioning

Consider the Poisson equation $-\Delta u = f$ in a rectangular domain $\Omega=(0,a)\times(0,b)$ with the Dirichlet boundary condition $u=g$ on $\partial\Omega$. The standard five-point finite difference method approximates the equation as

$$4u_{ij}-u_{i+1,j}-u_{i-1,j}-u_{i,j+1}-u_{i,j-1}=f_{ij}\cdot h^2,$$

at each grid node $(ih, jh)\in\Omega_h=\Omega\cap\left(h\mathbb{Z}^2\right)$. The approximations constitute a linear system $A\left[u_{ij}\right]=b$. With the lexicographical ordering, we decompose the matrix as

$$A=L+D+U,$$

where L, U, and D are its strictly lower and upper, and diagonal parts, respectively. MILU preconditioner is the matrix of the form $M=(E+L)E^{-1}(E+U)$, where the diagonal matrix E is obtained recursively as follows.

for $i=1,2,\cdots,M$

for $j=1,2,\cdots,N$

$$e_{i,j}=4-\frac{l_{i-\frac{1}{2},j}}{e_{i-1,j}}\left(l_{i-\frac{1}{2},j}+l_{i-1,j+\frac{1}{2}}\right)-\frac{l_{i,j-\frac{1}{2}}}{e_{i,j-1}}\left(l_{i,j-\frac{1}{2}}+l_{i+\frac{1}{2},j-1}\right)$$

Here $e_{i,j}$ denotes the diagonal element of E corresponding to the node point (ih, jh), *i.e.* $E_{(i,j),(i,j)}$. $l_{i+\frac{1}{2},j}$ and $l_{i,j+\frac{1}{2}}$ denote the entry $A_{(i,j),(i+1,j)}$ and $A_{(i,j),(i,j+1)}$, respectively. Note that the above formula results from the row sum property, $Me=Ae$ with $e=(1,\cdots,1)^{\mathrm{T}}$. Due to the Dirichlet boundary condition, $l_{i+\frac{1}{2},j}$ and $l_{i,j+\frac{1}{2}}$ are either -1 or 0, and $e_{1,1}=4$.

Lemma 1. *Let* $\{c_n\}_{n=1}^{\infty}$ *be a sequence defined recursively as*

$$c_1 = 4 \quad \text{and} \quad c_{n+1} = 4 - \frac{4}{c_n}, \quad n \geq 1. \tag{1}$$

Then we have

$$c_n \geq 2 + \frac{2}{n}, \quad n \geq 1.$$

Proof. Let $\{c_n\}_{n=1}^{\infty}$ be the sequence defined as (1). The lemma is shown by the mathematical induction. Assume that $c_n \geq 2 + 2/n$, for $n = 1, 2, \cdots, k$. Then

$$c_{k+1} = 4 - \frac{4}{c_k} \geq 4 - \frac{2k}{k+1} = 2 + \frac{2}{k+1},$$

and this proves the lemma.

Theorem 1. *Let* $M = (L+E)E^{-1}(U+E)$ *be the MILU preconditioner for A. Then, for every diagonal element* $e_{(i,j)}$ *of E corresponding to the node* $(ih, jh) \in \Omega_h$*, we have*

$$e_{i,j} \geq 2 + \frac{2}{i+j} \quad \text{for } i, j = 1, 2, \cdots.$$

and, **therefore**,

$$\left\| e_{i,j} \right\|_{\infty} \geq 2 + \frac{2}{N+M}.$$

Proof. We shall show that $e_{i,j} \geq c_{i+j}$ for $i, j = 1, 2, \cdots$ by mathematical induction on $n = i + j$. Then follows the result from the previous lemma. When $n = 2$, $e_{1,1} = 4 \geq 3 = c_2$. Assume that $e_{i,j} \geq c_n$ for all $(ih, jh) \in \Omega_h$ with $i + j = n$. Then for any $(ih, jh) \in \Omega_h$ with $i + j = n + 1$,

$$\begin{aligned} e_{i,j} &= 4 - \frac{l_{i-\frac{1}{2},j}}{e_{i-1,j}} \left(l_{i-\frac{1}{2},j} + l_{i-1,j+\frac{1}{2}} \right) - \frac{l_{i,j-\frac{1}{2}}}{e_{i,j-1}} \left(l_{i,j-\frac{1}{2}} + l_{i+\frac{1}{2},j-1} \right) \\ &\geq 4 - \frac{2}{c_{i+j-1}} - \frac{2}{c_{i+j-1}} = 4 - \frac{4}{c_n} = c_{n+1}. \end{aligned}$$

Now, we are ready to estimate the condition number of the MILU preconditioned matrix $M^{-1}A$. The following analysis is a standard approach, for the details see [2]. Since $M^{-1}A$ is similar to $E^{\frac{1}{2}}(L+E)^{-1} A (L+U)^{-1} E^{\frac{1}{2}}$ that is symmetric and positive definite, all the eigenvalues of $M^{-1}A$ are real and positive. Moreover, the minimum and maximum eigenvalues of $M^{-1}A$ are given as

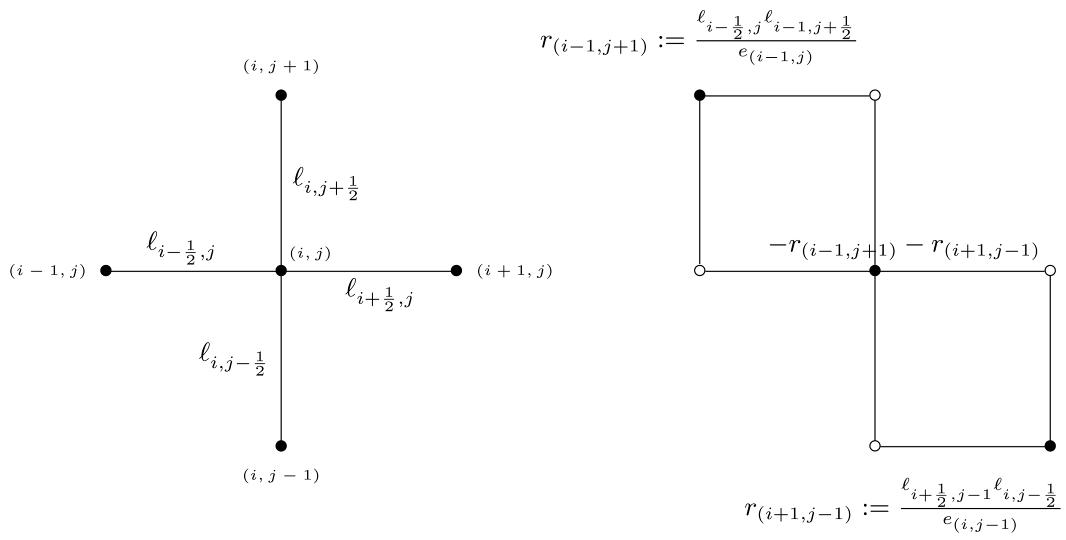
$$\lambda_{\min} = \min_{v \in \mathbb{R}^{|\Omega_h|}} \frac{\langle Av, v \rangle}{\langle Mv, v \rangle} \quad \text{and} \quad \lambda_{\max} = \max_{v \in \mathbb{R}^{|\Omega_h|}} \frac{\langle Av, v \rangle}{\langle Mv, v \rangle}, \tag{2}$$

and $\langle Av, v \rangle / \langle Mv, v \rangle$ is written in the form

$$\frac{\langle Av, v \rangle}{\langle Mv, v \rangle} = \frac{1}{1 + \langle Rv, v \rangle / \langle Av, v \rangle} \tag{3}$$

for the matrix $R = M - A$ (see (b) of **Figure 1** for its entries). For arbitrary $v \in \mathbb{R}^{|\Omega_h|}$, we have

$$\langle Av, v \rangle \geq \sum_{i=1}^{M} \sum_{j=1}^{N} \left(\left| l_{i+\frac{1}{2},j} \right| \left(v_{i+1,j} - v_{i,j} \right)^2 + \left| l_{i,j+\frac{1}{2}} \right| \left(v_{i,j+1} - v_{i,j} \right)^2 \right). \tag{4}$$



(a) (b)

Figure 1. Matrices *A* and *R*. (a) Matrix A; (b) Matrix B = M – A.

Using the inequality $(x+y)^2 \le 2(x-z)^2 + 2(y-z)^2$ and Theorem 1, we also have

$$
\begin{aligned}
-\langle Rv, v\rangle &= \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{l_{i+\frac{1}{2},j}\, l_{i,j+\frac{1}{2}}}{e_{i,j}} \left(v_{i+1,j} - v_{i,j+1}\right)^2 \\
&\le \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{N+M}{N+M+1}\left(\left|l_{i+\frac{1}{2},j}\right|\left(v_{i+1,j}-v_{i,j}\right)^2 + \left|l_{i,j+\frac{1}{2}}\right|\left(v_{i,j+1}-v_{i,j}\right)^2\right) \\
&\le \frac{N+M}{N+M+1}\langle Av, v\rangle.
\end{aligned}
$$

Thus, we obtain the inequalities

$$
0 \le \frac{-\langle Rv, v\rangle}{\langle Av, v\rangle} \le \frac{N+M}{N+M+1} \le \frac{a+b}{(a+b)+h}. \tag{5}
$$

In summary, we have the following.

Theorem 2. *Let* λ *be an eigenvalue of the MILU preconditioned matrix, then* $\lambda_{\min} = 1$ *and*

$$
1 \le \lambda \le \frac{a+b}{h} - 1. \tag{6}
$$

Proof. Let λ be an eigenvalue of the MILU preconditioned matrix $M^{-1}A$. From (5), we have that

$$
\frac{1}{N+M+1} \le 1 + \frac{\langle Rv, v\rangle}{\langle Av, v\rangle} \le 1, \quad \forall v \ne 0,
$$

and applying these inequalities above into (2) and (3) gives

$$
1 \le \lambda \le N+M+1 = \frac{a+b}{h} - 1, \quad \left(a = (N+1)h, b = (M+1)h\right),
$$

which shows the inequalites (6). On the other hand, the row sum property implies that 1 is an eigenvalue of $M^{-1}A$. Thus, we have $\lambda_{\min} = 1,$ and we complete the proof.

Corollary 1. *The ratio of the maximum and minimum eigenvalues of the MILU preconditioned matrix is bounded by* $O\left(h^{-1}\right)$.

Remark 1. *Our analysis deals with the two dimensional Poisson equation. It naturally extends to the three dimensional equation in a dimension-by-dimension manner.*

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