

Numerical Solution of Singularly Perturbed Two-Point Boundary Value Problem via Liouville-Green Transform

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ABSTRACT

In this paper, author describes a Liouville-Green transform to solve a singularly perturbed two-point boundary value problem with right end boundary layer in the interval $[0,1]$. They reply Liouville-Green transform into original given problem and finds the numerical solution. Then they implemented this method on two linear examples with right end boundary layer which nicely approximate the exact solution.

Keywords: Singular Perturbation; Ordinary Differential Equation; Boundary Layer; Two-Point Boundary Value Problem; Liouville-Green Transform

1. Introduction

Singular perturbation problems are of common occurrence in fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion processes, geophysics. It is a well known fact that the singularly perturbed two-point boundary value problem posses boundary or interior layers *i.e.* regions of rapid change in the solution near the end points or some interior points with width $O(1)$ as $\varepsilon \rightarrow 0$. In recent years, a large number of special methods have been developed to provide accurate numerical solutions. For details one may refer to the books of [1-5] and the references [6-11]. Many of these methods consist of: 1) dividing the problem into an inner region (boundary layer) problem and an outer region problem; 2) expressing the inner and outer solutions as asymptotic expansions; 3) equating various terms in the inner and outer expressions to determine the constants in these expressions; and 4) combining the inner and outer solutions in some fashion to obtain a uniformly valid solution. Typically, the inner region problems are obtained from the original problem by rescaling the independent variable. These techniques and their variations have been used successfully on a variety of linear and nonlinear singular perturbation problems. However, there can be difficulties in applying these methods, such as the matching of the coefficients of the inner and outer expansions. Success may depend on finding the proper scaling or the proper transformation to express the dependent and independent

variables.

A non-asymptotic method, also called boundary-value technique, has been introduced by Roberts [12-14] to solve certain classes of singular perturbation problems. He also discussed the analytical and approximate solutions of several numerical examples. To the best of our knowledge, very few asymptotic solutions were established for boundary value problems [15,16]. In this paper, author studies the second order singularly perturbed two-point boundary value problem with right end boundary layer via Liouville-Green transform and obtain asymptotic and numerical solutions. Few examples are also demonstrated for the applicability of the method.

2. Liouville-Green Transforms

Now, we discuss our method for singularly perturbed two-point boundary value problems with right-end boundary layer of the underlying interval. To be specific, we consider a class of singular perturbation problem of the form

$$\varepsilon y''(x) - f(x)y'(x) - g(x)y(x) = 0, \quad x \in [0,1] \quad (1)$$

with boundary conditions

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta \quad (2)$$

where ε is a small positive parameter ($0 < \varepsilon \ll 1$); α , β are known constants; we assume that $f(x)$, $g(x)$ are assumed to be sufficiently continuously differentiable function in $[0,1]$. Furthermore, since the coefficient of

$y'(x)$ is negative and non zero throughout the interval $[0,1]$. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 1$.

Rewrite the Equation (1) as below:

$$-\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = 0, \quad x \in [0,1] \quad (3)$$

Let the new Liouville-Green transforms $z, \phi(x), v(z)$ be

$$z = \phi(x) = \frac{1}{\varepsilon} \int f(x) dx \quad (4)$$

$$\phi(x) = \phi'(x) = \frac{1}{\varepsilon} f(x), \quad (5)$$

$$v(z) = \phi(x)y(x). \quad (6)$$

According to (6), we have

$$\frac{dy}{dx} = \frac{1}{\phi(x)} \frac{dv}{dz} z'(x) - \frac{\phi'(x)}{\phi^2(x)} v(x) = \frac{\phi'(x)}{\phi(x)} \frac{dv}{dz} - \frac{\phi'(x)}{\phi^2(x)} v(z), \quad (7)$$

$$\frac{d^2y}{dx^2} = \frac{1}{\phi(x)} \left(\phi'^2(x) \frac{d^2v}{dz^2} + \left(\phi'' - \frac{2\phi'(x)\phi'(x)}{\phi(x)} \right) \frac{dv}{dz} - \left(\frac{\phi''(x)}{\phi(x)} - \frac{2\phi'^2(x)}{\phi^2(x)} \right) v \right). \quad (8)$$

From (3), (7) and (8), we obtain

$$-\frac{\varepsilon\phi'^2}{\phi} \frac{d^2v}{dz^2} + \left(\frac{2\varepsilon\phi'\phi'}{\phi^2} - \frac{\varepsilon\phi''(x)}{\phi(x)} + f(x) \frac{\phi'(x)}{\phi(x)} \right) \frac{dv}{dz} + \left(\frac{\varepsilon\phi''(x)}{\phi^2(x)} - \frac{2\varepsilon\phi'^2(x)}{\phi^3(x)} - f(x) \frac{\phi'(x)}{\phi^2(x)} + \frac{g(x)}{\phi} \right) v(z) = 0,$$

i.e.

$$\frac{d^2v}{dz^2} + \frac{1}{\phi'^2} \left(\phi''(x) - \frac{2\phi'\phi'}{\phi} - f(x) \frac{\phi'(x)}{\varepsilon} \right) \frac{dv}{dz} - \frac{1}{\phi'^2} \left(\frac{\phi''(x)}{\phi(x)} - \frac{2\phi'^2}{\phi^2} - f(x) \frac{\phi'(x)}{\varepsilon\phi(x)} + \frac{g(x)}{\varepsilon} \right) v(z) = 0.$$

From (4), we have

$$\frac{d^2v}{dz^2} - \left(\varepsilon \frac{f'(x)}{f^2(x)} + 1 \right) \frac{dv}{dz} - \frac{1}{f^2(x)} \left(\varepsilon^2 \frac{f''(x)}{f(x)} - 2\varepsilon^2 \frac{f'^2(x)}{f^2(x)} - \varepsilon f'(x) + \varepsilon g(x) \right) v(z) = 0,$$

i.e.

$$\frac{d^2v}{dz^2} - \frac{dv}{dz} = \varepsilon \frac{f'(x)}{f^2(x)} \frac{dv}{dz} + \varepsilon \frac{1}{f^2(x)} \left(\varepsilon \frac{f''(x)}{f(x)} - 2\varepsilon \frac{f'^2(x)}{f^2(x)} - f'(x) + g(x) \right) v(z) = \varepsilon M(x) \frac{dv}{dz} + \varepsilon N(x, \varepsilon) v(z), \quad (9)$$

where

$$M(x) = \frac{f'(x)}{f^2(x)}, \quad N(x, \varepsilon) = \frac{1}{f^2(x)} \left(\varepsilon \frac{f''(x)}{f(x)} - 2\varepsilon \frac{f'^2(x)}{f^2(x)} - f'(x) + g(x) \right).$$

Since ε is a small parameter ($0 < \varepsilon \ll 1$), $\varepsilon M(x)$ and $\varepsilon N(x, \varepsilon)$ are sufficiently small on $[0,1]$. So, as $\varepsilon \rightarrow 0$, the right hand side of Equation (9) vanishes. Therefore, we have

$$\frac{d^2v}{dz^2} - \frac{dv}{dz} \approx 0. \quad (10)$$

Therefore, the approximate solutions $v(z)$ of (10) are

$$v(z) \approx C_1 + C_2 e^z, \quad (11)$$

where C_1, C_2 are two arbitrary constants. From (4)-(6), one has the asymptotic solutions of differential equations

$$y(x) = \frac{v(z)}{\phi(x)} = \varepsilon \frac{v(z)}{f(x)} \approx \frac{\varepsilon}{f(x)} \left(C_1 + C_2 e^{\frac{1}{\varepsilon} \int f(s) ds} \right). \quad (12)$$

where C_1, C_2 are two arbitrary constants.

3. Application to Two Point Boundary Value Problems

As an application, we consider the following second-order two-point boundary value problem

$$\varepsilon y''(x) - f(x)y'(x) - g(x)y(x) = 0, \quad (13)$$

$$0 < \varepsilon \ll 1, \quad 0 < x < 1,$$

Which is equivalent to

$$-\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = 0$$

$$y(0) = \alpha, \quad y(1) = \beta,$$

where α, β are constants.

Applying the boundary conditions of (13), in (12), we have

$$C_1 \frac{\varepsilon}{f(0)} + C_2 \frac{\varepsilon}{f(0)} = \alpha,$$

$$C_1 \varepsilon \left(\frac{1}{f(1)} \right) + C_2 \varepsilon \left(\frac{1}{f(1)} e^{\frac{1}{\varepsilon} \int_0^1 f(x) dx} \right) = \beta$$

One has

$$C_1 = \frac{\alpha \varepsilon \left(\frac{1}{f(1)} e^{\frac{1}{\varepsilon} \int_0^1 f(x) dx} \right) - \frac{\varepsilon \beta}{f(0)}}{|\Delta|} \tag{14}$$

$$C_2 = \frac{\frac{\varepsilon \beta}{f(0)} - \varepsilon \left(\frac{1}{f(1)} \right) \alpha}{|\Delta|} \tag{15}$$

where

$$\Delta = \frac{\varepsilon^2}{f(0)} \left(\frac{1}{f(1)} \left(e^{\frac{1}{\varepsilon} \int_0^1 f(x) dx} - 1 \right) \right) \neq 0$$

Then BVP (13) has the following asymptotic solution:

$$y(x) \approx \frac{\varepsilon}{f(x)} \left(C_1 + C_2 e^{\frac{1}{\varepsilon} \int_0^x f(s) ds} \right), \tag{16}$$

where C_1, C_2 are given by (14), (15) respectively.

Example 3.1. Consider the following singular perturbation problem

$$\varepsilon y''(x) - y'(x) = 0, x \in [0, 1] \tag{17}$$

$$\text{with } y(0) = 1 \text{ and } y(1) = 0. \tag{18}$$

Clearly this problem has a boundary layer at $x = 1$. *i.e.*; at the right-end of the underlying interval. The exact solution is given by

$$y(x) = \left(e^{(x-1)/\varepsilon} - 1 \right) / \left(e^{-1/\varepsilon} - 1 \right)$$

Comparing (17) with (13), we have

$$f(x) = 1, g(x) = 0, \alpha = 1, \beta = 0$$

$$\Delta = \varepsilon^2 \left(e^{\frac{1}{\varepsilon}} - 1 \right) \neq 0,$$

$$C_1 = \frac{\left(\frac{1}{e^{\frac{1}{\varepsilon}}} \right)}{\varepsilon \left(e^{\frac{1}{\varepsilon}} - 1 \right)}$$

$$C_2 = \frac{-1}{\varepsilon \left(e^{\frac{1}{\varepsilon}} - 1 \right)}$$

$$y(x) \approx \frac{\left(1 - e^{-\frac{(x-1)}{\varepsilon}} \right)}{\left(1 - e^{-\frac{1}{\varepsilon}} \right)} \tag{19}$$

The computational results are presented in **Tables 1** and **2** for $\varepsilon = 10^{-3}$ and 10^{-4} , respectively. **Figures 1** and **2** show our solution and exact solution for different values of x .

Example 3.2. We consider the following variable coefficient singular perturbation problem from Kevorkian and Cole [2, p. 33, Equations (2.3.26) and (2.3.27) with $\alpha = -1/2$].

Table 1. Numerical results of example 3.1 with $\varepsilon = 10^{-3}$, $h = 10^{-3}$.

x	$y(x)$	Exact solution
0.000	1.0000000	1.0000000
0.200	1.0000000	1.0000000
0.400	1.0000000	1.0000000
0.600	1.0000000	1.0000000
0.800	1.0000000	1.0000000
0.900	1.0000000	1.0000000
0.920	1.0000000	1.0000000
0.940	1.0000000	1.0000000
0.960	1.0000000	1.0000000
0.980	1.0000000	1.0000000
1.000	1.0000000	1.0000000

Table 2. Numerical results of example 3.1 with $\varepsilon = 10^{-4}$, $h = 10^{-4}$.

x	$y(x)$	Exact solution
0.0000	1.0000000	1.0000000
0.2000	1.0000000	1.0000000
0.4000	1.0000000	1.0000000
0.6000	1.0000000	1.0000000
0.8000	1.0000000	1.0000000
0.9000	1.0000000	1.0000000
0.9200	1.0000000	1.0000000
0.9400	1.0000000	1.0000000
0.9600	1.0000000	1.0000000
0.9800	1.0000000	1.0000000
1.0000	1.0000000	1.0000000

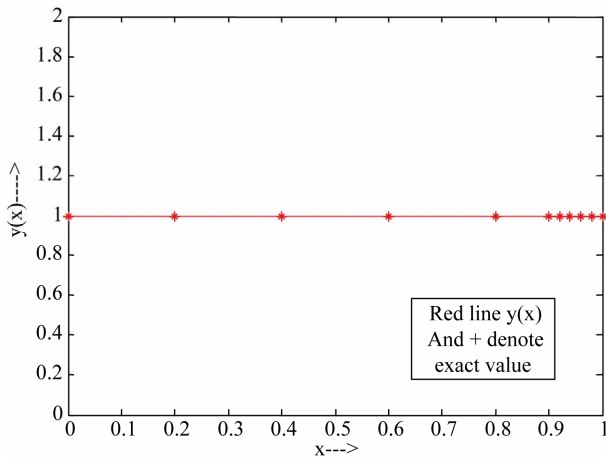


Figure 1. x versus $y(x)$ and exact solution for example 3.1 at $\varepsilon = 10^{-3}, h = 10^{-3}$.

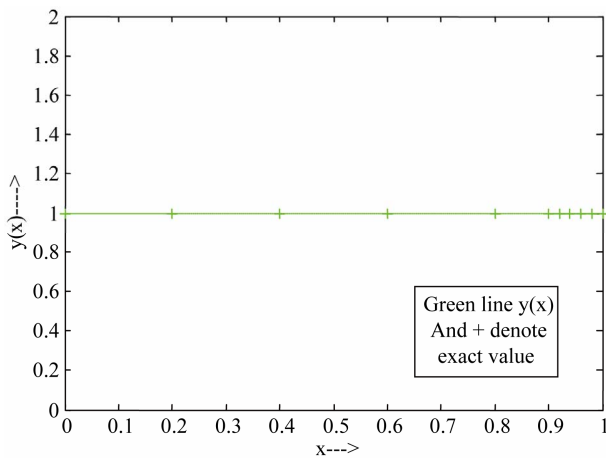


Figure 2. x versus $y(x)$ and exact solution for example 3.1 at $\varepsilon = 10^{-4}, h = 10^{-4}$.

$$\varepsilon y''(x) - \left(\frac{x}{2} - 1\right) y'(x) - \frac{1}{2} y(x) = 0, \quad x \in [0, 1] \quad (20)$$

$$\text{with } y(0) = 0 \text{ and } y(1) = 1. \quad (21)$$

He has chosen to use uniformly valid approximation (which is obtained by the method given by Nayfeh [12, p. 148, Equation (4.2.32)] as our “exact” solution;

$$y(x) = \frac{1}{(2-x)} - \frac{1}{2} e^{-(x-x^2/4)/\varepsilon}$$

The numerical results are given in **Tables 3** and **4** for $\varepsilon = 10^{-3}$ and 10^{-4} , respectively.

Comparing (20) with (13), we have

$$f(x) = \left(\frac{x}{2} - 1\right), \quad g(x) = \frac{1}{2}, \quad \alpha = 0, \quad \beta = 1.$$

$$\Delta = 2\varepsilon^2 \left(e^{\frac{-3}{4\varepsilon}} - 1 \right) \neq 0,$$

Table 3. Numerical results of example 3.2 with $\varepsilon = 10^{-3}, h = 10^{-3}$.

x	$y(x)$	Exact solution
0.000	0.0000000	0.0000000
0.200	0.5555555	0.5555556
0.400	0.6250000	0.6250000
0.600	0.7142857	0.7142857
0.800	0.8333333	0.8333333
0.900	0.9090909	0.9090909
0.920	0.9259259	0.9259259
0.940	0.9433962	0.9433962
0.960	0.9615384	0.9615384
0.980	0.9803922	0.9803922
1.000	1.0000000	1.0000000

Table 4. Numerical results of example 3.2 with $\varepsilon = 10^{-4}, h = 10^{-4}$.

x	$y(x)$	Exact solution
0.0000	0.0000000	0.0000000
0.2000	0.5555555	0.5555556
0.4000	0.6250000	0.6250000
0.6000	0.7142857	0.7142857
0.8000	0.8333333	0.8333333
0.9000	0.9090909	0.9090909
0.9200	0.9259259	0.9259259
0.9400	0.9433962	0.9433962
0.9600	0.9615384	0.9615384
0.9800	0.9803922	0.9803922
1.0000	1.0000000	1.0000000

$$C_1 = \frac{\varepsilon}{\Delta}$$

$$C_2 = \frac{-\varepsilon}{\Delta}$$

$$y(x) \approx \frac{1}{(2-x)} \frac{\left(1 - e^{\frac{(x^2-4x)}{4\varepsilon}} \right)}{\left(1 - e^{\frac{-3}{4\varepsilon}} \right)} \quad (22)$$

The computational results are presented in **Tables 3** and **4** for $\varepsilon = 10^{-3}$ and 10^{-4} , respectively. **Figures 3** and **4** show our solution and exact solution for different values of x .

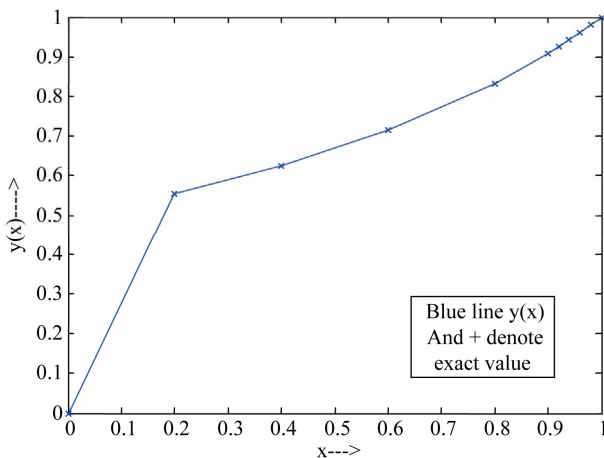


Figure 3. x versus $y(x)$ and exact solution for example 3.2 at $\varepsilon = 10^{-3}$, $h = 10^{-3}$.

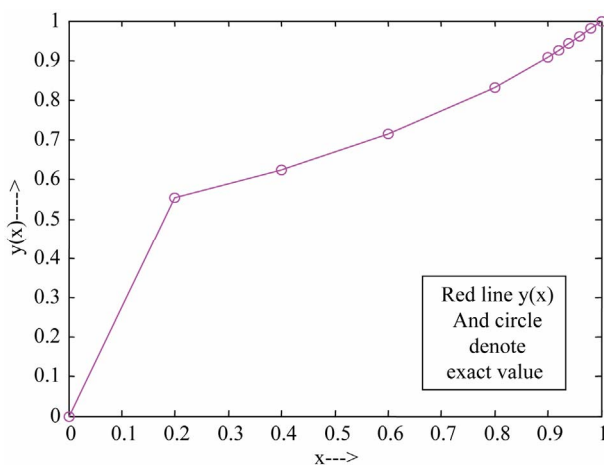


Figure 4. x versus $y(x)$ and exact solution for example 3.2 at $\varepsilon = 10^{-4}$, $h = 10^{-4}$.

4. Conclusion

In this paper, authors concerned with the numerical solutions of singularly perturbed two-point boundary value problems with right end boundary layer. Here they take the assumption that $f(x) \neq 0$ on the whole interval $[0, 1]$, i.e. the function $f(x)$ has same sign on the whole interval $[0, 1]$. Our method is good on computer implementation. But this method is not valid if the equation is changed into

$$-\varepsilon y''(x) + f(x)y'(x) + g(x)y(x) = h(x) \neq 0$$

i.e. in no-homogeneous form.

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