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Nonconforming H^1-Galerkin Mixed Finite Element Method for Pseudo-Hyperbolic Equations

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ABSTRACT

Based on H^1-Galerkin mixed finite element method with nonconforming quasi-Wilson element, a numerical approximate scheme is established for pseudo-hyperbolic equations under arbitrary quadrilateral meshes. The corresponding optimal order error estimate is derived by the interpolation technique instead of the generalized elliptic projection which is necessary for classical error estimates of finite element analysis.

Keywords: Pseudo-Hyperbolic Equation; Nonconforming; H^1-Galerkin Mixed Finite Element; Error Estimate

1. Introduction

Consider the following initial-boundary value problem of pseudo-hyperbolic equation

$$\begin{cases} u_{tt} - \nabla\cdot\left(a(X)\nabla u_t\right) - \nabla\cdot\left(a(X)\nabla u\right) + u_t \\ = f(X,t), \quad \text{in } \Omega\times(0,T], \\ u(X,t) = 0, \quad \text{on } \partial\Omega\times(0,T], \\ u(X,0) = u_0(X), u_t(X,0) = u_1(X), \quad \text{in } \Omega, \end{cases} \tag{1}$$

where $X=(x,y)$, Ω is bounded convex polygonal domain in R^2 with *Lipschitz* continuous boundary $\partial\Omega$. $a(X)$ is smooth function with bounded derivatives, $u_0(X)$, $u_1(X)$ and f are given functions, and

$$0 < a_{\min} \le a(X) \le a_{\max}, X \in \Omega,$$

for positive constants $a_{\min}$ and $a_{\max}$.

The pseudo-hyperbolic equation is a high-order partial differential system with mixed partial derivative with respect to time and space, which describe heat and mass transfer, reaction-diffusion and nerve conduction, and other physical phenomena. This model was proposed by Nagumo *et al*. [1]. Wan and Liu [2] have given some results about the asymptotic behavior of solutions for this problem. Guo and Rui [3] used two least-squares Galerkin finite element schemes to solve pseudo-hyperbolic equations.

On the other hand, H^1-Galerkin mixed finite element method (see [4]) has been under rapid progress recently since this method has the following advantages over classical mixed finite element method. The method allows the approximation spaces to be polynomial spaces with different orders without LBB consistency condition and there is no requirement of the quasi-uniform assumption on the meshes. For example, Pani [4,5] proposed an H^1-Galerkin mixed finite element procedure to deal with parabolic partial differential equations and parabolic partial integro-differential equations, respectively. Liu and Li [6,7] applied this method to deal with pseudo-hyperbolic equations and fourth-order heavy damping wave equation. Further, Shi and Wang [8] investigated this method for integro-differential equation of parabolic type with nonconforming finite elements including the ones studied in [9,10].

It is well-known that the convergence behavior of the well-known nonconforming Wilson element is much better than that of conforming bilinear element. So it is widely used in engineering computations. However, it is only convergent for rectangular and parallelogram meshes. The convergence for arbitrary quadrilateral meshes can not be ensured since it passes neither Irons Patch Test [11] nor General Patch Test [12]. In order to extend this element to arbitrary quadrilateral meshes, various improved methods have been developed in [13-24]. In particular, [19-24] generalized the results mentioned above and constructed a class of Quasi-Wilson elements which are convergent to the second order elliptic problem for narrow quadrilateral meshes [23].

In the present work, we will focus on H^1-Galerkin nonconforming mixed finite element approximation to problem (1) under arbitrary quadrilateral meshes. We firstly prove the existence and uniqueness of the solution

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for semi-discrete scheme. Then, based on a very special property of the quasi-Wilson element *i.e*. the consistency error is one order higher than interpolation error, we deduce the optimal order error estimates for semi-discrete scheme directly without using the generalized elliptic projection which is a indispensable tool in the tradition finite element methods.

This paper is arranged as follows. In Section 2, we briefly introduce the construction of nonconforming mixed finite element. In section III, we will discuss the H^1-Galerkin mixed finite element scheme for pseudo-hyperbolic equations. At last, the corresponding optimal order error estimates are obtained for semi-discrete scheme.

2. Construction of Nonconforming Mixed Finite Element

Assume $\hat{K}=[-1,1]\times[-1,1]$ to be the reference element in the $\hat{x}-\hat{y}$ plane with vertices $\hat{a}_1=(-1,-1),\hat{a}_2=(1,-1),\hat{a}_3=(1,1)$ and $\hat{a}_4=(-1,1)$.

Let $\hat{l}_1=\overline{\hat{a}_1\hat{a}_2},\ \hat{l}_2=\overline{\hat{a}_2\hat{a}_3},\ \hat{l}_3=\overline{\hat{a}_3\hat{a}_4}$ and $\hat{l}_4=\overline{\hat{a}_4\hat{a}_1}$ be the four edges of $\hat{K}$.

We define the finite elements $(\hat{K},\hat{P}^i,\hat{\Sigma}^i),(i=1,2)$ by

$$\hat{P}^1=span\{N_i(\hat{x},\hat{y}),i=1,2,3,4\},\quad \hat{\Sigma}^1=\{\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4\},$$

$$\hat{P}^2=span\{N_i(\hat{x},\hat{y}),i=1,2,3,4,\hat{\varphi}(\hat{x}),\hat{\varphi}(\hat{y})\},$$

$$\hat{\Sigma}^2=\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4,\hat{p}_5,\hat{p}_6\},$$

where $\hat{v}_i=\hat{v}(\hat{a}_i)$, $\hat{p}_i=\hat{p}(\hat{a}_i)$, $i=1,2,3,4$,

$$\hat{p}_5=\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\hat{x}^2}\mathrm{d}\hat{x}\mathrm{d}\hat{y},\ \hat{p}_6=\frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\hat{y}^2}\mathrm{d}\hat{x}\mathrm{d}\hat{y},$$

$$N_1(\hat{x},\hat{y})=\frac{1}{4}(1-\hat{x})(1-\hat{y}),\quad N_2(\hat{x},\hat{y})=\frac{1}{4}(1+\hat{x})(1-\hat{y}),$$

$$N_3(\hat{x},\hat{y})=\frac{1}{4}(1+\hat{x})(1+\hat{y}),\quad N_4(\hat{x},\hat{y})=\frac{1}{4}(1-\hat{x})(1+\hat{y}),$$

and

$$\hat{\varphi}(t)=\frac{1}{2}(t^2-1)-\frac{1}{5}(t^4-1).$$

When $\hat{\varphi}(t)=\frac{1}{8}(t^2-1)$, it is the so-called Wilson element.

The interpolations defined above are properly posed and the interpolation functions can be expressed as

$$\hat{\Pi}^1\hat{v}=\sum_{i=1}^{4}\hat{v}_iN_i(\hat{x},\hat{y})$$

and

$$\hat{\Pi}^2\hat{p}=\sum_{i=1}^{4}\hat{p}_iN_i(\hat{x},\hat{y})+\hat{p}_5\hat{\varphi}(\hat{x})+\hat{p}_6\hat{\varphi}(\hat{y}).$$

Given a convex polygonal domain $\Omega\subset R^2$, Let $\bar{\Omega}=\bigcup_{K\in\Gamma_h}K$ be a decomposition of $\bar{\Omega}$ such that Γ_h satisfies the regularity assumption [11], where K denotes a convex quadrilateral with vertices $a_i(x_i,y_i)(i=1,2,3,4)$, $h=\max_K\{h_K\}$, h_K is the diameter of the finite element K.

Then there exists a invertible mapping $F_K:\hat{K}\to K$

$$\begin{cases} x=\sum_{i=1}^{4}N_i(\hat{x},\hat{y})x_i, \\ y=\sum_{i=1}^{4}N_i(\hat{x},\hat{y})y_i. \end{cases}$$

The associated finite element space V_h and $\boldsymbol{W}_h$ are defined as

$$V_h=\{v_h;v_h|_K=\hat{v}_h\circ F_K^{-1},\hat{v}_h\in\hat{P}^1,\forall K\in\Gamma_h\}$$

and

$$\boldsymbol{W}_h=\{\boldsymbol{w}_h=(w_h^1,w_h^2);w_h^j|_K=\hat{w}_h^j\circ F_K^{-1},\hat{w}_h^j\in\hat{P}^2,\forall K\in\Gamma_h,$$
$$\text{and } w_h^j(a)=0,\forall\text{node } a\in\partial\Omega,\ j=1,2\}.$$

Then for all $v\in H^2(\Omega)$, $\boldsymbol{w}=(w_1,w_2)\in(H^2(\Omega))^2$, we define the interpolation operators Π_h^1 and Π_h^2 by

$$\Pi_h^1:H^2(\Omega)\to V_h,\Pi_h^1|_K=\Pi_K^1,\Pi_h^1v=(\hat{\Pi}^1\hat{v})\circ F_K^{-1}$$

and

$$\Pi_h^2:(H^1(\Omega))^2\to W_h,\Pi_h^2|_K=\Pi_K^2,$$

$$\Pi_h^2\boldsymbol{w}=((\hat{\Pi}^2\hat{w}_1)\circ F_K^{-1},(\hat{\Pi}^2\hat{w}_2)\circ F_K^{-1}).$$

Let $L^2(\Omega)$ be the set of square integrable functions on Ω and $(L^2(\Omega))^2$ the space of two dimensional vectors which have all components in $L^2(\Omega)$ with its norm $\|\cdot\|_0$. Let $H(div;\Omega)$ be the space of vectors in $(L^2(\Omega))^2$ which has divergence in $L^2(\Omega)$ with norm $\|\cdot\|_{H(div;\Omega)}^2=\|\cdot\|_0^2+\|\nabla\cdot\|_0^2$, $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. For our subsequent use, we also use the standard sobolve space $W^{m,p}(\Omega)$ with a norm $\|\cdot\|_{m,p}$. Especially for $p=2$, we denote $W^{m,2}(\Omega)=H^m(\Omega)$ and $\|\cdot\|_m=\|\cdot\|_{m,2}$.

Throughout this paper, C denotes a general positive constant which is independent of h.

3. Nonconforming H^1-Galerkin Mixed Finite Element Method for the Semi-Discrete Scheme

Let $\alpha = 1/a(X)$ and $\boldsymbol{p} = a(X)\nabla u$, then the corresponding weak formulation is: Find $\{u, \boldsymbol{p}\}:[0,T] \to H_0^1(\Omega) \times H(div;\Omega)$, such that

$$\begin{cases} (\nabla u, \nabla v) = (\alpha \boldsymbol{p}, \nabla v), \quad \forall\, v \in H_0^1(\Omega), \\ (\alpha \boldsymbol{p}_{tt}, \boldsymbol{w}) + (\nabla \cdot \boldsymbol{p}_t, \nabla \cdot \boldsymbol{w}) + (\nabla \cdot \boldsymbol{p}, \nabla \cdot \boldsymbol{w}) + (\alpha \boldsymbol{p}_t, \boldsymbol{w}) \\ = -(f, \nabla \cdot \boldsymbol{w}), \quad \forall \boldsymbol{w} \in H(div;\Omega), \\ u(X,0) = u_0(X), u_t(X,0) = u_1(X). \end{cases} \tag{2}$$

The corresponding semi-discrete finite element procedure is: Find $\{u_h, \boldsymbol{p}_h\}:[0,T] \to V_h \times \boldsymbol{W}_h$, such that

$$\begin{cases} (\nabla u_h, \nabla v_h) = (\alpha \boldsymbol{p}_h, \nabla v_h), \quad \forall\, v_h \in V_h, \\ (\alpha \boldsymbol{p}_{htt}, \boldsymbol{w}) + (\nabla \cdot \boldsymbol{p}_{ht}, \nabla \cdot \boldsymbol{w}_h) + (\nabla \cdot \boldsymbol{p}_h, \nabla \cdot \boldsymbol{w}_h) \\ +(\alpha \boldsymbol{p}_{ht}, \boldsymbol{w}_h) = -(f, \nabla \cdot \boldsymbol{w}_h), \quad \forall \boldsymbol{w} \in \boldsymbol{W}_h, \\ u_h(X,0) = \Pi_h^1 u_0(X), u_{ht}(X,0) = \Pi_h^1 u_1(X). \end{cases} \tag{3}$$

For all $v_h \in V_h, \boldsymbol{w}_h \in \boldsymbol{W}_h$, we define

$$\|v_h\|_h = \left(\sum_{K \in \Gamma_h} |v_h|_{1,K}^2 \right)^{\frac{1}{2}}$$

and

$$\|\boldsymbol{w}_h\|_{H(div_h;\Omega)} = \left(\sum_{K \in \Gamma_h} \|\boldsymbol{w}_h\|_0^2 + \|\nabla \cdot \boldsymbol{w}_h\|_0^2 \right)^{\frac{1}{2}}.$$

It is easy to see that $\|\cdot\|_h$ and $\|\cdot\|_{H(div_h;\Omega)}$ are norms of V_h and $\boldsymbol{W}_h$, respectively.

Theorem 1. *Problem* (3) has a unique solution.

Proof. Let $\{\phi_i\}_{i=1}^{r_1}$ and $\{\boldsymbol{\psi}_j\}_{j=1}^{r_2}$ the basis of V_h and $\boldsymbol{W}_h$. Suppose that

$$u_h = \sum_{i=1}^{r_1} h_i(t)\phi_i, \boldsymbol{p}_h = \sum_{j=1}^{r_2} g_j(t)\boldsymbol{\psi}_j, v_h = \phi_j, \boldsymbol{w}_h = \boldsymbol{\psi}_i,$$

then (3) can be written as

$$\begin{cases} (a)\; A\boldsymbol{H}(t) = B\boldsymbol{G}(t), \\ (b)\; M\dfrac{\mathrm{d}^2\boldsymbol{G}(t)}{\mathrm{d}t^2} + (M+N)\dfrac{\mathrm{d}\boldsymbol{G}(t)}{\mathrm{d}t} + N\boldsymbol{G}(t) = Q, \end{cases} \tag{4}$$

where

$$\boldsymbol{H}(t) = \left(h_1(t), \cdots, h_{r_1}(t)\right)^{\mathrm{T}},$$

$$\boldsymbol{G}(t) = \left(g_1(t), \cdots, g_{r_2}(t)\right)^{\mathrm{T}},$$

$$A = \left((\nabla\phi_i, \nabla\phi_j)\right)_{r_1 \times r_1},$$

$$B = \left((\alpha\psi_i, \nabla\phi_j)\right)_{r_1 \times r_2},$$

$$M = \left((\alpha\psi_i, \psi_j)\right)_{r_2 \times r_2},$$

$$N = \left((\nabla \cdot \psi_i, \nabla \cdot \psi_j)\right)_{r_2 \times r_2},$$

$$Q = \left(-(f, \nabla \cdot \psi_j)\right)_{1 \times r_2}.$$

Sine (4) gives a system of nonlinear ordinary differential equations (ODEs) for the vector function $\boldsymbol{H}(t)$ and $\boldsymbol{G}(t)$, by the assumptions on $a(X)$ and the theory of ODEs, it follows that $\boldsymbol{H}(t)$ and $\boldsymbol{G}(t)$ has the unique solution for $t > 0$ (see [25]). Therefore the proof is complete.

4. Error Estimates

In order to get the error estimates the following lemma which will play an important role in our analysis and can be found in [24].

Lemma 1. For all $u \in H_0^1(\Omega) \cap H^2(\Omega), \boldsymbol{\varphi} \in \boldsymbol{W}_h$, then there holds

$$\left| \sum_{K \in \Gamma_h} \int_{\partial K} u(\boldsymbol{\varphi} \cdot \boldsymbol{n})\,\mathrm{d}s \right| \le Ch|u|_2 \|\boldsymbol{\varphi}\|_0,$$

where $\boldsymbol{n}$ denotes the outward unit normal vector to ∂K.

Now, we will state the following main result of this paper.

Theorem 2. Suppose that $\{u, \boldsymbol{p}\}$ and $\{u_h, \boldsymbol{p}_h\}$ be the solutions of the (2) and (3), respectively, $u, u_t, u_{tt} \in H^2(\Omega)$, $\boldsymbol{p}, \boldsymbol{p}_t \in \left(H^2(\Omega)\right)^2$ and $\boldsymbol{p}_{tt} \in \left(H^1(\Omega)\right)^2$, then we have

$$\|u - u_h\|_h \le Ch\left(|u|_2 + \boldsymbol{p}_1 + \Phi\right) \tag{5}$$

and

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{H(div_h;\Omega)} \le Ch\left(|\boldsymbol{p}|_1 + |\boldsymbol{p}|_2 + \Phi\right), \tag{6}$$

where

$$\Phi = \left\{ \int_0^t \left(|u_t(\tau)|_2^2 + |u_{tt}(\tau)|_2^2 + |\boldsymbol{p}(\tau)|_2^2 + |\boldsymbol{p}_t(\tau)|_1^2 + |\boldsymbol{p}_t(\tau)|_2^2 + |\boldsymbol{p}_{tt}(\tau)|_1^2 \right) \mathrm{d}\tau \right\}^{\frac{1}{2}}.$$

Proof. Let $u - u_h = \left(u - \Pi_h^1 u\right) + \left(\Pi_h^1 u - u_h\right) \equiv \eta + \xi,$

$$\boldsymbol{p} - \boldsymbol{p}_h = \left(\boldsymbol{p} - \Pi_h^2 \boldsymbol{p}\right) + \left(\Pi_h^2 \boldsymbol{p} - \boldsymbol{p}_h\right) \equiv \boldsymbol{\rho} + \boldsymbol{\theta}.$$

It is easy to see that for all $v_h \in V_h, \boldsymbol{w}_h \in \boldsymbol{W}_h$, there hold the following error equations

$$\begin{cases}(a)\ (\nabla\xi,\nabla v_h)=-(\nabla\eta,\nabla v_h)+(\alpha\boldsymbol{\rho},\nabla v_h)+(\alpha\boldsymbol{\theta},\nabla v_h),\\ (b)\ (\alpha\boldsymbol{\theta}_{tt},\boldsymbol{w}_h)+(\nabla\cdot\boldsymbol{\theta}_t,\nabla\cdot\boldsymbol{w}_h)+(\nabla\cdot\boldsymbol{\theta},\nabla\cdot\boldsymbol{w}_h)\\ +(\alpha\boldsymbol{\theta}_t,\boldsymbol{w}_h)+(\boldsymbol{\theta},\boldsymbol{w}_h)\\ =(\boldsymbol{\theta},\boldsymbol{w}_h)-(\alpha\boldsymbol{\rho}_{tt},\boldsymbol{w}_h)-(\alpha\boldsymbol{\rho}_t,\boldsymbol{w}_h)-(\nabla\cdot\boldsymbol{\rho}_t,\nabla\cdot\boldsymbol{w}_h)\\ -(\nabla\cdot\boldsymbol{\rho},\nabla\cdot\boldsymbol{w}_h)+\sum_{K\in\Gamma_h}\int_{\partial K}u_t(\boldsymbol{w}_h\cdot\boldsymbol{n})\mathrm{d}s\\ +\sum_{K\in\Gamma_h}\int_{\partial K}u_{tt}(\boldsymbol{w}_h\cdot\boldsymbol{n})\mathrm{d}s.\end{cases} \tag{7}$$

Choosing $v_h=\xi$ in (7(a)) and using the Cauchy-Schwartz's inequality yields

$$\begin{aligned}\|\nabla\xi\|_0&\le C\left(\|\nabla\eta\|_0+\|\rho\|_0+\|\theta\|_0\right)\\ &\le Ch\left(|u|_2+|\boldsymbol{p}|_1\right)+C\|\theta\|_0.\end{aligned} \tag{8}$$

Further, choosing $\boldsymbol{w}_h=\boldsymbol{\theta}_t$ in (7(b)) leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\alpha^{1/2}\boldsymbol{\theta}_t\|_0^2+\|\nabla\cdot\boldsymbol{\theta}\|_0^2+\|\boldsymbol{\theta}\|_0^2\right)+\|\nabla\cdot\boldsymbol{\theta}_t\|_0^2+\|\alpha^{1/2}\boldsymbol{\theta}_t\|_0^2\\ &=(\boldsymbol{\theta},\boldsymbol{\theta}_t)-(\alpha\boldsymbol{\rho}_{tt},\boldsymbol{\theta}_t)-(\alpha\boldsymbol{\rho}_t,\boldsymbol{\theta}_t)-(\nabla\cdot\boldsymbol{\rho}_t,\nabla\cdot\boldsymbol{\theta}_t)\\ &-(\nabla\cdot\boldsymbol{\rho},\nabla\cdot\boldsymbol{\theta}_t)+\sum_{K\in\Gamma_h}\int_{\partial K}u_t(\boldsymbol{\theta}_t\cdot n)\mathrm{d}s\\ &+\sum_{K\in\Gamma_h}\int_{\partial K}u_{tt}(\boldsymbol{\theta}_t\cdot n)\mathrm{d}s\\ &\equiv\sum_{i=1}^{7}A_i.\end{aligned} \tag{9}$$

For the right side of (9), applying ε-Young's inequality and noting that $a(X)$ is a smooth function with bounded derivatives, we get

$$\begin{aligned}|A_1+A_2+A_3|&\le C\left(\|\theta\|_0^2+\|\boldsymbol{\rho}_{tt}\|_0^2+\|\boldsymbol{\rho}\|_{t0}^2\right)+\varepsilon\|\theta_t\|_0^2\\ &\le Ch^2\left(|\boldsymbol{p}_{tt}|_1^2+|\boldsymbol{p}_t|_1^2\right)+C\|\theta\|_0^2+\varepsilon\|\theta_t\|_0^2.\end{aligned} \tag{10}$$

$$\begin{aligned}|A_4+A_5|&\le C\left(\|\nabla\cdot\boldsymbol{\rho}_t\|_0^2+\|\nabla\cdot\boldsymbol{\rho}\|_0^2\right)+\varepsilon\|\nabla\cdot\boldsymbol{\theta}_t\|_0^2\\ &\le Ch^2\left(|\boldsymbol{p}_t|_2^2+|\boldsymbol{p}|_2^2\right)+\varepsilon\|\nabla\cdot\boldsymbol{\theta}_t\|_0^2.\end{aligned} \tag{11}$$

By Lemma 1 and ε-Young's inequality, we have

$$\begin{aligned}|A_6+A_7|&\le Ch\left(|u_t|_2+|u_{tt}|_2\right)\|\boldsymbol{\theta}_t\|_0\\ &\le Ch^2\left(|u_t|_2^2+|u_{tt}|_2^2\right)+\varepsilon\|\boldsymbol{\theta}_t\|_0^2.\end{aligned} \tag{12}$$

Choosing small ε and combining (9)-(12), we can derive

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\alpha^{1/2}\boldsymbol{\theta}_t\|_0^2+\|\nabla\cdot\boldsymbol{\theta}\|_0^2+\|\boldsymbol{\theta}\|_0^2\right)\\ &\le Ch^2\left(|\boldsymbol{p}_t|_2^2+|\boldsymbol{p}|_2^2+|\boldsymbol{p}_{tt}|_1^2+|\boldsymbol{p}_t|_1^2\right)\\ &+Ch^2\left(|u_t|_2^2+|u_{tt}|_2^2\right)+C|\boldsymbol{\theta}|_0^2.\end{aligned} \tag{13}$$

Integrating the both sides of (13) with respect to time from 0 to t, by Gronwall's lemma and noting $\theta(0)=0,\theta_t(0)=0$, we obtain

$$\begin{aligned}\|\boldsymbol{\theta}\|_{H(div_h;\Omega)}^2&=\|\nabla\cdot\boldsymbol{\theta}\|_0^2+\|\boldsymbol{\theta}\|_0^2\\ &\le Ch^2\int_0^t\Big(|u_t(\tau)|_2^2+|u_{tt}(\tau)|_2^2+|\boldsymbol{p}(\tau)|_2^2\\ &\quad+|\boldsymbol{p}_t(\tau)|_1^2+|\boldsymbol{p}_t(\tau)|_2^2+|\boldsymbol{p}_{tt}(\tau)|_1^2\Big)\mathrm{d}\tau\end{aligned} \tag{14}$$

together with (8), there yields

$$\begin{aligned}\|\xi\|_h&\le Ch\left(|u|_2+|\boldsymbol{p}|_1\right)\\ &+Ch\left\{\int_0^t\Big(|u_t(\tau)|_2^2+|u_{tt}(\tau)|_2^2+|\boldsymbol{p}(\tau)|_2^2\right.\\ &\left.\quad+|\boldsymbol{p}_t(\tau)|_1^2+|\boldsymbol{p}_t(\tau)|_2^2+|\boldsymbol{p}_{tt}(\tau)|_1^2\Big)\mathrm{d}\tau\right\}^{1/2}.\end{aligned} \tag{15}$$

Finally, by use of the triangle inequality, (14) and (15), we get (5) and (6). The proof is completed.

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