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An Exponential Series Method for the Solution of Free Convection Boundary Layer Flow in a Saturated Porous Medium

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Abstract

Third order nonlinear ordinary differential equation, subject to appropriate boundary conditions, arising in fluid mechanics is solved exactly using more suggestive schemes-Dirichlet series and method of stretching variables. These methods have advantages over pure numerical methods in obtaining derived quantities accurately for various values of the parameters involved at a stretch and are valid in a much larger domain compared with classical numerical schemes.

Keywords: Boundary Layer Equations, Stretching Surface, Dirichlet Series, Powell's Method, Stretching of Variables.

1. Introduction

In this article we consider the effect of blowing and suction along a vertical flat plate on free convection in air or water which are of significant interest in recent years. Earliest study on this topic was by Eichhorn (1960) who investigated the effects of both wall temperature and the blowing or suction velocity with prescribed power functions of distance from the leading edge. Eichhorn (1960) shows that the similarity solutions for the problem are possible if the exponents in the prescribed power functions are related in a particular manner. Sparrow and Celss (1961) show a perturbation method for analyzing more general problem with arbitrary values of exponents and these results were confirmed by Mabuchi (1960) by an integral method.

Our aim here is to present qualitative features of the physical problems of interest. Majority of the problems considered here are from fluid dynamics. The residual warm water discharged from a geothermal power plant is disposed of through subsurface re-injection wells which can be idealized as vertical plane surface in porous medium. The buoyancy flow past bodies immersed in a saturated porous medium have been studied by Cheng (1977, 1977). Merkin (1978) investigated the effect of uniform mass flux on the free convection boundary layer on a vertical wall in a saturated porous medium. Cheng and Minkowycz (1977) have studied similarity solution for the case of wall temperature and suction velocity varying as powers of x, the longitudinal distance. In all cases, the numerical solutions have been given for selected values of parameters involved.

The third order nonlinear ordinary differential equations over an infinite interval with suction/injection parameter f_w appear in various branches of physics and engineering and are of special interest. In very few cases, they have analytical solution. Sakiadias (1961) investigated the boundary layer flow on continuous solid surface with a constant speed. Erickson *et al.* (1966) have studied the problem of moving surface with suction or injection. Since the surface is flexible the filament may be stretched during the course of ejection and so only the surface velocity deviates without being uniform. Samuel and Hall (1973), who investigated, the similarity solution for laminar boundary layer on a continuous moving porous surface, obtain a series with exponential terms as their solutions. The heat and mass transfer on stretching sheet with suction or blowing was investigated by Gupta and Gupta (1977). Ackroyed (1978) obtains the series solution of steady two dimensional laminar boundary layer flows in fluids of constant density and constant viscosity. The stretching sheet may be considered either as an impermeable or permeable. The two dimensional steady boundary layer flow in a permeable surface with stretching velocity in a quiescent fluid in the presence of suction or injection and obtain exact solution for specific parameters and express the smallest entrainment velocity corresponding to a vanishing skin friction in a closed form was studied (Magyari and Keller, 2000).

For specific type of boundary conditions *i.e* $f'(\infty)=0$ Dirichlet series solution is more efficient and provides a uniformly valid solution throughout the boundary layer flow caused by saturated porous medium at high Raleigh numbers. Kravchenko and Yablonskii (1965) were the first to use Dirichlet series for the solution of third order nonlinear boundary value problem over infinite range. A general discussion of the convergence of the Dirichlet series may be found in (Riesz, 1957). The accuracy as well as uniqueness of the solution can be confirmed using other powerful semi-numerical schemes. Semi-numerical methods in this category require introduction of new variables, thus converting third order equations into second order equations, whose solution may be obtained by using power series. We also find the approximate analytical solution by the method of stretching variables. Sachdev *et al.* (2005) have analyzed various problems from fluid dynamics of stretching sheet using this approach and analyzed governing equations, solution obtained are more accurate compared with earlier numerical findings.

The paper is organized as follows: In Section 2 the mathematical formulation of the proposed problem with relevant boundary conditions is given, its exact solution for $A=1$ and $B=-1$ is also presented. Section 3 is devoted to semi-numerical method for the solution of the problem using Dirichlet series and in Section 4 the method of stretching variables is used. In Section 5 detailed results obtained by the novel procedures explained here are compared with the corresponding numerical solutions.

2. Mathematical Formulation

***Case I*:** The boundary layer equations of momentum and energy corresponding to flow past a vertical plate embedded in a saturated porous medium can be reduced to the form (Cheng, 1977)

$$f''-\theta'=0 \tag{1}$$

$$\theta''+\left(\frac{1+\lambda}{2}\right)f\theta'-\lambda f'\theta=0 \tag{2}$$

where the plate temperature and suction or injection velocity are given by

$$T_w=T_\infty+Ax^\lambda \text{ and } v_w=ax^n \tag{3}$$

where $n=(\lambda-1)+/2$

The relevant boundary conditions are

$$\begin{aligned}&\text{at } \eta=0: \quad \theta=1,\ \ f=f_w\\&\text{as } \eta\to\infty: \ \theta=0,\ \ f'=0\end{aligned} \tag{4}$$

and f_w is the non-dimensional form which is positive for the withdrawal of fluid (suction) and negative for the discharge of fluid (injection).

Eliminating θ from (1) and (2), we get

$$f'''+\left(\frac{1+\lambda}{2}\right)ff''-\lambda f'^2=0 \tag{5}$$

and the boundary conditions become

$$f(0)=f_w,\quad f'(0)=1,\quad f'(\infty)=0 \tag{6}$$

***Case II*:** The momentum equation for the vapour boundary layer derived by (Ackryod, 1978) is

$$f'''+3ff''-2f'^2=0 \tag{7}$$

with the boundary conditions

$$f(0)=f_w,\quad f'(0)=1,\quad f'(\infty)=0 \tag{8}$$

This problem describes the vapour boundary layer induced by the falling motion of the condensate layer on a cold vertical plate. The vapour is at saturation temperature and far from the plate at rest. Koh *et al* (1961) who first investigated the problem have shown that the condensate motion is restrained somewhat by the shear stress produced in the vapour boundary layer.

***Case III*:** We also consider the self-similar two-dimensional steady boundary layer flow induced by a permeable surface stretching with velocity $u_w=U_w(x)$ in quiescent fluid in the presence of suction or injection with velocity $v_w(x)=a\,x^{\frac{(m-1)}{2}}$. For $f_w>0$ and $m>-1$ by (Magyari and Keller, 2000)

$$f'''+ff''-\beta f'^2=0\text{, where } \beta=\frac{2m}{m+1} \tag{9}$$

satisfying boundary conditions

$$f(0)=f_w,\quad f'(0)=1,\quad f'(\infty)=0 \tag{10}$$

The equation describing the above boundary value problems can be put conveniently in more general third order nonlinear differential equation of the type

$$f'''+Aff''+\beta f'^2=0 \quad '=\frac{d}{d\eta} \tag{11}$$

satisfying boundary conditions are

$$f(0)=f_w,\quad f'(0)=1 \text{ and } f'(\infty)=0 \tag{12}$$

where A, B are constants, A is always positive and B may be positive or negative.

Under the transformation $f(\eta)=\left(\frac{1}{A}\right)^{1/2}F(\xi)$ and $\xi=\eta A^{1/2}$ Equation (11) becomes

$$F'''(\xi)+F(\xi)F''(\xi)+\beta F'^2(\xi)=0, \quad \text{where } \beta=\frac{B}{A} \tag{13}$$

and the boundary conditions (12) become

$$F(\xi)=A^{1/2}f_w,\ \ F'(0)=1,\ \ F'(\infty)=0 \tag{14}$$

For the particular case with $A=1$ and $B=1$ ($\beta=1$) integrating (13) twice with respect to ξ, using $f''(0)=-f_w$, and subjected to the boundary conditions (14), we get

$$F'+\frac{F^2}{2}=\frac{1}{2}f_w^2+1 \tag{15}$$

the Equation (15) is a Riccati type equation, whose solution is given as

$$F(\xi)=\delta\left[\frac{(f_w+\delta)e^{(\xi/2)\delta}+(f_w-\delta)e^{-(\xi/2)\delta}}{(f_w+\delta)e^{(\xi/2)\delta}-(f_w-\delta)e^{-(\xi/2)\delta}}\right] \tag{16}$$

where $\delta=\sqrt{2+f_w^2}$. For $f_w=0$ Equation (13) becomes $F(\xi)=\sqrt{2}\tanh\left(\xi/\sqrt{2}\right)$

3. Dirichlet Series Approach

We seek Dirichlet series solution for Equation (11) satisfying $f'(\infty)=0$ in the form (Kravchenko and Yablonskii, 1965)

$$f=\frac{\gamma}{A}+\gamma\sum_{i=1}^{\infty}b_ia^ie^{-i\gamma\eta} \tag{17}$$

where $\gamma>0$ and $|a|<1$. Substituting (17) into (11), we get

$$\begin{aligned}&-\sum_{i=1}^{\infty}i^3b_ia^ie^{-i\gamma\eta}+\sum_{i=1}^{\infty}i^2b_ia^ie^{-i\gamma\eta}\\&+A\sum_{i=2}^{\infty}\sum_{k=1}^{i-1}k^2b_kb_{i-k}a^ie^{-i\gamma\eta}\\&+B\sum_{i=2}^{\infty}\sum_{k=1}^{i-1}k(i-k)b_kb_{i-k}a^ie^{-i\gamma\eta}=0\end{aligned} \tag{18}$$

For $i=1$, we have $-b_1a+b_1a=0$. We assume $|b_1|=1$ and a is any arbitrary parameter. We rewrite (18) for recurrence relation to obtain coefficients as

$$b_i=\frac{1}{i^2(i-1)}\sum_{k=1}^{i-1}\left[Ak^2+Bk(i-k)\right]b_kb_{i-k} \tag{19}$$

for $i=2,3,4,\cdots$. If $|a|<1$ and $|b_1|\le 1$, then the series (17) converges absolutely for any $\gamma>0$ and $\eta=-\varepsilon$, where

$$\varepsilon=-\left(\frac{\ln|a|}{\gamma}+\delta\right)>0 \tag{20}$$

and $\delta>0$ is a sufficiently small number depending on a and γ. The series (17) converges absolutely and uniformly on the half axis $\eta>-\varepsilon$.

The series (17) contains two free parameters namely a and γ. These unknown parameters are determined from the remaining boundary conditions (12) at $\eta=0$

$$f(0)=\frac{\gamma}{A}+\gamma\sum_{i=1}^{\infty}b_ia^i=f_w \tag{21}$$

and

$$f'(0)=\gamma^2\sum_{i=1}^{\infty}(-i)b_ia^i=1 \tag{22}$$

The solution of transcendental Equations (21) and (22) yield constants a and γ. The solution of these transcendental equations is equivalent to the unconstrained minimization of the functional

$$\left[\frac{\gamma}{A}+\gamma\sum_{i=1}^{\infty}b_ia^i-f_w\right]^2+\left[\gamma^2\sum_{i=1}^{\infty}(-i)b_ia^i-1\right]^2 \tag{23}$$

We use Powell's method of conjugate directions (Press *et al* 1987) which is one of the most efficient techniques for solving unconstrained optimization problems. This helps in fixing the unknowns a and γ uniquely for different values of the parameters A, B and f_w. Alternatively, Newton method is also used to determine the unknown parameters accurately for different value of f_w.

For the shear stress at the surface for the problem (11) it is given by

$$F''(0)=\gamma\sum_{i=1}^{\infty}b_ia^i(-i\gamma)^2 \tag{24}$$

4. Method of Stretching Variables

Most of the boundary value problems over infinite intervals are not amenable in obtaining analytical solution. In such situations, it is possible to obtain approximate solution of these problems. As the governing equation is sometimes too difficult to solve exactly, one has to approach the approximate analysis. Many approximate methods are ad-hoc and often provide solutions for major engineering problems and physical insight into the problems. The approximate solution considered here to these problems, are based on the idea of stretching the variables of the flow problems. We have to choose suitable trial velocity profile f' satisfying the boundary conditions automatically and later integrate f' which will satisfy the remaining boundary conditions. Substitution this resulting function into the given equation gives the

Table 1. Comparison of the values of $f''(0)$ for $\lambda = 0$ of Equation (5) obtained by the dirichlet series method, method of stretching variable and pure numerical method for different values of f_w.

f_w	Dirichlet series			Numerical $f''(0)$	Method of stretching
	a	γ	$f''(0)$		
–4	---	---	---	–0.00305	–0.08012
–3	---	---	---	–0.01823	–0.10391
–2	---	---	---	–0.097213	–0.14549
–1	–5.81754	0.68320	–0.20404	–0.20404	–0.22871
–0.8	–4.98883	0.70035	–0.24291	–0.24291	–0.25461
–0.6	–4.26616	0.72103	–0.28633	–0.28633	–0.28493
–0.4	–3.63959	0.74565	–0.33431	–0.33431	–0.32032
–0.2	–3.09976	0.77457	–0.38682	–0.38682	–0.36129
0	–2.63759	0.80806	–0.44375	–0.44375	–0.40825
0.2	–2.24441	0.84637	–0.50499	–0.50490	–0.46129
0.4	–1.91159	0.88943	–0.57006	–0.57004	–0.52032
0.6	–1.63128	0.93732	–0.63888	–0.63888	–0.58493
0.8	–1.39596	0.98986	–0.71111	–0.71110	–0.65461
1	–1.19881	1.04683	–0.78640	–0.78640	–0.72871
2	–0.59957	1.38705	–1.19824	–1.19824	–1.14549
3	–0.33854	1.79089	–1.64747	–1.64747	–1.60391
4	–0.21181	2.23011	–2.11606	–2.11606	–2.08012

Table 2. Comparison of the values of $f''(0)$ for $\lambda = 1/3$ of Equation (5) obtained by the Dirichlet series method, Method of stretching variable and pure numerical method for different values of f_w.

f_w	Dirichlet series			Numerical $f''(0)$	Method of stretching
	A	γ	$f''(0)$		
–4	---	---	---	–0.12499	–0.15738
–3	---	---	---	–0.16640	–0.20185
–2	–7.86898	0.56238	–0.24571	–0.24372	–2.27614
–1	–3.96721	0.64512	–0.39700	–0.39700	–0.41202
–0.8	–3.37044	0.67926	–0.44153	–0.44153	–0.45136
–0.6	–2.85267	0.71814	–0.49164	–0.49164	–0.49602
–0.4	–2.40691	0.76225	–0.54760	–0.54759	–0.54654
–0.2	–2.02624	0.81201	–0.60957	–0.60958	–0.60333
0	–1.70376	0.86775	–0.67765	–0.67765	–0.66667
0.2	–1.43260	0.92965	–0.75172	–0.75172	–0.73667
0.4	–1.20609	0.99773	–0.83161	–0.83161	–0.81320
0.6	–1.01785	1.07187	–0.91701	–0.91701	–0.89602
0.8	–0.86197	1.15179	–1.00753	–1.00753	–0.98469
1	–0.73313	1.23715	–1.10272	–1.10274	–1.07869
2	–0.35388	1.72986	–1.63357	–1.63357	–1.60948
3	–0.19604	2.29516	–2.22255	–2.22255	–2.20185
4	–0.12140	2.89895	–2.84146	–2.84145	–2.82405

residual of the form $R(\xi,\alpha)$ which is called defect function. Using least squares method, the residual of the defect function can be minimized. For details see (Afzal, 1982; Ariel, 1994; Mamaloukas, 2002).

Using the transformation $f = f_w + F$ into the system (11), we get

$$F''' + A\left(f_w + F\right)F'' + BF'^2 = 0, \quad ' = \frac{d}{d\eta} \tag{25}$$

and the boundary conditions become

$$F(0) = 0, \quad F'(0) = 1, \quad F'(\infty) = 0 \tag{26}$$

We introduce a stretching parameter α for both F and η in the form

$$H(\xi) = \alpha F(\eta) \quad \text{and} \quad \xi = \alpha\eta \tag{27}$$

where $a > 0$ is an amplification factor. In view of (27), the system (25-26) are transformed to the form

$$\alpha H''' + A\left(f\alpha + H\right)H'' + BH' = 0, \quad ' = \frac{d}{d\xi} \tag{28}$$

with the boundary conditions

$$H(0) = 0, \quad H'(0) = 1, \quad H'(\infty) = 0 \tag{29}$$

We choose velocity profile for general A, B and f_W to be of the form

$$H' = \exp(-\xi) \tag{30}$$

which satisfies the derivative conditions in (29) at $\xi = 0$ and $\xi \to \infty$. Integrating (30) with respect to ξ between the limits 0 to ξ using conditions (29), we get

Table 3. Comparison of the values of $f''(0)$ for $\lambda = 1$ of Equation (5) obtained by the Dirichlet series method, method of stretching variable and pure numerical method for different values of f_w.

f_w	Dirichlet series			Numerical $f''(0)$	Method of stretching
	a	γ	$f''(0)$		
–4	–17.94427	0.23607	–0.23607	–0.2360	–0.23607
–3	–10.90833	0.30278	–0.30278	–0.3027	–0.30277
–2	–5.82843	0.41421	–0.41421	–0.4142	–0.41421
–1	–2.61803	0.61803	–0.61803	–0.6180	–0.61803
–0.8	–2.18163	0.67703	–0.67703	–0.6770	–0.67703
–0.6	–1.80642	0.74403	–0.74403	–0.7440	–0.74403
–0.4	–1.48792	0.81980	–0.81980	–0.8198	–0.81980
–0.2	–1.22099	0.90499	–0.90499	–0.9049	–0.90499
0	–1.00000	1.00000	–1.00000	–1.0000	–1.00000
0.2	–0.81900	1.10499	–1.10499	–1.104	–1.10499
0.4	–0.67208	1.21980	–1.21980	–1.219	–1.21980
0.6	–0.55358	1.34403	–1.34403	–1.344	–1.34403
0.8	–0.45837	1.47703	–1.47703	–1.477	–1.47703
1	–0.38197	1.61803	–1.61803	–1.618	–1.61803
2	–0.17157	2.41421	–2.41421	–2.414	–2.41421
3	–0.09167	3.30277	–3.30277	–3.302	–3.30278
4	–0.05573	4.23607	–4.23607	–4.236	–4.23607

Table 4. Comparison of the values of $f''(0)$ obtained by the Dirichlet series method, Method of stretching variable and pure numerical method for different values of f_w.

f_w	Dirichlet series			Numerical $f''(0)$	Method of stretching
	a	γ	$f''(0)$		
3	–0.01157	9.32272	–9.269193	–9.269193	–9.25219
2	–0.02420	6.46670	–6.389695	–6.389695	–6.36650
1	–0.07161	3.80275	–3.672835	–3.672835	–3.64087
0	–0.36212	1.80184	–1.540735	–1.540735	–1.52753
–0.2	0.51828	1.55188	–1.256436	–1.256433	–1.25671
–0.4	–0.73429	1.35370	–1.026515	–1.026417	–1.04114
–0.6	–1.02209	1.20031	–0.847661	–0.845381	–0.87295
–0.8	–1.37560	1.10092	–0.744998	–0.745377	–0.74251
–1.0	–1.65643	1.13688	–0.590908	–0.597831	–0.64087

$$H = 1 - \exp(-\xi). \tag{31}$$

Substituting (31) into (28) we get the residual defect function $R(\xi,\ \alpha)$

$$R(\xi,\ \alpha) = \exp(-2\xi)\{B + \exp(\xi)\alpha^2 - A(-1 + \exp(\xi)(1 + f_w\alpha))\} \tag{32}$$

By using the least squares method for minimization of error and using Euler-Lagrange equation which is simplified to minimization of error in the form

$$\frac{\partial}{\partial\alpha}\int_0^\infty R^2(\xi,\ \alpha)\mathrm{d}\xi = 0. \tag{33}$$

Substituting (32) into Equation (33) and solving for α, we get

$$\alpha = \frac{1}{2\sqrt{3}}\left[\sqrt{3}Af_w \pm \sqrt{4A - 8B + 3A^2 f_w^2}\right] \tag{34}$$

Thus, the final form of the solution becomes

$$f = f_w + \frac{1}{\alpha}\left[1 - \exp(-\alpha\eta)\right]. \tag{35}$$

The expression (35) gives the solution of Equation (9) for all A, B and f_w. It is striking that the Equation (35) also admits analytical solutions for $A = 1$, $B = -1$ and $f_w = 0$; $f = 1 - e^{-\eta}$, for $A = 1$, $B = 1$ and $f_w = 0$; $f = \sqrt{2}\tanh(\eta/\sqrt{2})$. It is of interest to note that, the former exact solution may also be recovered from the method of stretching variable.

Table 5. Comparison of the values of $f''(0)$ obtained by the Dirichlet series method, Method of stretching variable and pure numerical method for different values of β and f_w.

β	f_w	Dirichlet series a	γ	$f''(0)$	Numerical $f''(0)$	Method of stretching
2	1.0	–0.33757	1.58166	–1.84998	–1.84989	–1.88444
	0.5	–0.51732	1.22111	–1.54028	–1.54047	–1.56498
	0.0	–0.81027	0.90564	–1.28181	–1.28215	–1.29099
1	1.0	–0.38197	1.61803	–1.61803	–1.61803	–1.61803
	0.5	–0.60961	1.28078	–1.28078	–1.28078	–1.28078
	0.0	–1.00000	0.99999	–1.00000	–1.00000	–1.00000
–1	1.0	–0.53591	1.73207	–1.00002	–1.00009	---
	0.5	–1.00705	1.51099	–0.50747	–0.50845	---
	0.0	–1.30794	1.61320	–0.41993	---	---
–2	1.0	–0.91332	2.12093	–0.37867	---	---
	0.5	–0.88355	1.72753	–0.23406	---	---
	0.0	–0.83355	1.42219	–0.15829	---	---

5. Numerical Results

In the present paper, we have given exact analytic solution of nonlinear boundary value problem (11) and (12) in the form of Dirichlet series (17) and approximate solution by using method of stretching variable (11). The calculated values of $f''(0)$ representing the shear stress at the surface associated with different parameters A, B and f_w for different sets of values of a and γ are given in **Tables 1-5**.

The problem explained in (5) corresponds to $A=\frac{1+\lambda}{2}$, $B=-\lambda$ and for different values of f_w. An analytic solution in terms of series is obtained using Dirichlet series method and also by an approximate solution method of stretching variable. Comparison of the solution obtained is made with existing numerical solution by (Cheng, 1977) for $\lambda=0,\ \frac{1}{3},1$ and these are given in **Table 1-3**. An excellent agreement between the present computation and numerical values is achieved. Also, the exact analytical solution for $A=1$ and $B=1$ have been recovered.

The problem mentioned in (7) corresponds to $A=3$ and $B=-2$ and different values of f_w. The results obtained using Dirichlet series and method of stretching variables is given in **Table 4**. These results agree very well with the pure numerical solutions.

The problem (9) corresponds to $A=1$ and $B=-\beta=\frac{2m}{1+m}$. For specific values of $\lambda=0,\ -\frac{1}{3}$ and $-\frac{1}{2}$, for which exact analytic solution of the problem subjected to the boundary conditions (12) is given by (Magyari and Keller, 2000). For $m=1$ corresponds to $A=1$, $B=-\beta=-1$ for different f_w, an excellent agreement between Dirichlet series and method of stretching variable with exact solution is achieved and these are given in **Table 3**. Also, for $m=-\frac{1}{3}$ which corresponds to $A=1$, $B=-\beta=1$, we recover an exact analytical solution (16).

6. Conclusions

In this article, a class of nonlinear ordinary differential uations with relevant boundary conditions arising in boundary layer theory has been solved using Dirichlet series and method of stretching variables. These two methods give a simple and efficient way to solve the boundary value problems, particularly, when $f'(\infty)=0$. All the results thus obtained have been compared with that of direct numerical solution and are rather remarkable.

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