Scientific
Research

Euler, Reader of Newton: Mechanics and Algebraic Analysis

Sébastien Maronne[1,2], Marco Panza[3]
[1]Institut de Mathématiques de Toulouse, Toulouse, France
[2]Sciences Philosophie Histoire (SPHERE), Paris, France
[3]Institut d'Histoire et Philosophie des Sciences et des Techniques, Paris, France
Email: Sebastien.Maronne@math.univ-toulouse.fr, panzam10@gmail.com

We follow two of the many paths leading from Newton's to Euler's scientific productions, and give an account of Euler's role in the reception of some of Newton's ideas, as regards two major topics: mechanics and algebraic analysis. Euler contributed to a re-appropriation of Newtonian science, though transforming it in many relevant aspects. We study this re-appropriation with respect to the mentioned topics and show that it is grounded on the development of Newton's conceptions within a new conceptual frame also influenced by Descartes's views sand Leibniz's formalism.

Keywords: Isaac Newton; Leonhard Euler; Newtonian Mechanics; Classical Mechanics; 18th-Century Algebraic Analysis

Introduction

The purpose of the present paper is to follow two of the many paths leading from Newton's to Euler's scientific productions[1] and to give, at least partly, an account of Euler's role in the reception of Newton as regards two major topics: mechanics and algebraic analysis[2]. Euler contributed to a re-appropriation of Newtonian science. We will study this re-appropriation with respect to the mentioned topics and show that it is grounded on the development of Newton's ideas within a new conceptual frame also influenced by Cartesian ideas and Leibnizian formalism.

From Newtonian Geometric Mechanics to Analytic Mechanics

Euler's works on mechanics concern different domains, some of which are not considered in Newton's *Principia* (Newton, 1687). Beside his *Mechanica* (Euler, 1736)—a two-volume

---

[1]The quasi-totality of Euler's works is available online at the (Euler Archive), where one may also find many notices and references. Among other recent books devoted to Euler, cf. (Bradley & Sandifer, 2007) and (Backer, 2007).

[2]For reasons of space, we will not address in this paper some related topics like Euler's views on Newton's gravitation theory and his own celestial mechanics, particularly his lunar theory; Euler's reflections about (absolute and relative) space and time; and Euler's critical exposition of Newtonian science in his *Lettres à une princesse d'Allemagne* (Euler, 1768-1772). On Euler's celestial mechanics, cf., for instance, (Schroeder, 2007). On this matter, it is relevant to note that Euler had doubts in the 1740s about the validity of Newton's law of gravitation because of errors he observed in calculations of planetary perturbations: cf. (Euler, 1743), Kleinert's discussion of this memoir, in (Euler, OO, ser. II, vol. 31, appendix), and (Schroeder, 2007, pp. 348-379). On Euler's views on space and time, cf. also (Cassirer, 1907, Book VII, Chapter II, section II) and (Maltese, 2000). Finally, on Euler's *Lettres à une princesse d'Allemagne*, cf. (Calinger, 1976).

treatise on the motion of free or constrained punctual bodies—and a large number of papers on the same subject, Euler also much contributed to the mechanics of rigid and elastic bodies (Truesdell, 1960), the mechanics of fluids, the theory of machines and naval science. We shall limit ourselves to some of his contributions to the foundation of the mechanics of discrete systems of punctual bodies. We shall namely consider his views on the physical explanation of forces, his reformulation of the basic notions of Newtonian mechanics, and his works on the principle of least action.

The Explanation of Forces

The third book of Newton's *Principia* offers an explanation of the motion of planets around the sun and of the satellites around them. It is based on the assumption that the celestial bodies act upon each other according to an attractive force acting at a distance, the intensity of which depends on the mass of the attracting body and on its distance from the attracted one, and the effects of which are not influenced by a resistant medium. This is a special central force—a force attracting bodies along a straight line directed towards a fixed or moving centre—characterised by the following well-known equality

$$F_{C,c} = mG\frac{M}{r^2}$$

where $F_{C,c}$ is the force by which the body C attracts the body c, m and M are, respectively, the masses of c and C, r is the distance between (the centres of) them, and G is an universal constant.

In the first book of the *Principia* Newton provides a purely mathematical theory of central forces acting in absence of a resistance of the medium. In the third book, he is thus able to

give a mathematical theory of the world system by basing it only on the above assumption, that, according to him, is proved by empirical observations. Insofar as it allows the determination of the trajectories of the relevant bodies according to the mathematical theory of the first book, no supplementary hypothesis about the nature of the relevant force is necessary. In Newton's view, an hypothesis about forces is a conjecture concerning their qualitative nature and causes. His "Hypotheses non fingo"—famously claimed in the *General Scholium* of the second edition of the *Principia*—is just intended to declare that he does not venture any such conjecture, since this is not necessary to provide a satisfactory scientific explanation[3].

Apparently, this view is not shared by Euler. He seems to maintain that the notion of force cannot be primitive, and that a mathematical theory about forces cannot be separated from an account of their causes, even if this account depends more on "the province of metaphysics than of mathematics" and thus one cannot claim to undertake it with "absolute success"[4].

Euler's account—on which, cf. (Gaukroger, 1982)—is based on a Cartesian representation of the world as a plenum of matter. Here is what he writes in his 55th letter to a German Princess[5]:

> As you see nothing that impels [small bits of iron and steel] toward the loadstone, we say that the loadstone attracts them; and this phenomenon we call *attraction*. It cannot be doubted, however, that there is very subtle, though invisible, matter, which produces this effect by actually impelling the iron towards the lodestone [...]. Though this phenomenon be peculiar to the loadstone and iron, it is perfectly adapted to convey an idea of the signi-

fication of the word *attraction*, which philosophers so frequently employ. They allege then, that all bodies, in general, are endowed with a property similar to that of the loadstone, and that they mutually attract [...].

And then, in the 68th letter[6]:

> [...] as we know that the whole space which separates the heavenly bodies is filled with a subtle matter, called *ether*, it seems more reasonable to ascribe the mutual attraction of bodies to an action which the ether exercises upon them, though its manner of acting may be unknown to us, rather than to have recourse to an unintelligible property.

The Cartesian vein of Euler's account is even clearer in the *Mechanica* (Euler, 1736). It is very significant that in such a purely mathematical treatise, Euler devotes a scholium to discuss the causes and origins of forces. This is the scholium 2 of definition 10: the definition of forces, according to which, "a force [*potentia*] is the power [*vis*] that either makes a body pass from rest to motion or changes its motion."[7] This definition does not explain where the forces come from; Scholium 2 discusses the question. Euler begins by declaring that, among the real forces acting in the world, he only considers gravity. Then he argues that similar forces "are observed to exist in the magnetic and electric bodies" and adds:

> Some people think that all these [forces] arise from the motion of a somehow subtle matter; others attribute [them] to the power of attraction and repulsion of the bodies themselves. But, whatever it may be, we certainly see that forces of this kind can arise from elastic bodies and from vortices, and we shall inquire, at the appropriate occasion, whether these forces can be explained through these phenomena[8].

One could hope that Euler's Cartesian view be made more precise in a memoir presented in 1750 the title of which is quite promising: "Recherches sur l'origine des forces" (Euler, 1750)[9]. But the content of this memoir is somewhat surprising.

Euler begins by arguing that impenetrability is an essential property of bodies, and that it "comes with a force sufficient to prevent penetration"[10]. It follows—he says—that, when two

---

[3] Of course, we do not mean here that Newton had no views about the nature of forces, or never expressed them. In the same *Principia*, namely in the third of his *Regulæ Philosophandi*, opening the third book (only added in the second and third edition), he argues, for example, that inertia universally belongs to all bodies. And in his third letter to Bentley, he explicitly writes the following (we quote from (Newton, LB, pp. 25-26); a transcription of the original, kept at Trinity College Library, in Cambridge, is available online at the *Newton Project* website: www.newtonproject.sussex.ac.uk):
"That gravity should be innate, inherent and essential to Matter, so that one Body may act upon another at a Distance thro' a *Vacuum*, without the Mediation of any thing else, by and through which their Action or Force may be conveyed from one to another is to me so great an Absurdity, that I beleive no Man who has in philosophical Matters a competent Faculty of thinking can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this Agent be material or immaterial I have left to the Consideration of my Readers."
The point is that Newton does consider that the explanation of the nature of forces is neither essential to his theory of gravitation, nor *a fortiori* to his mathematical theory of motion. The last lines in the quoted passage of the third letter to Bentley is, among many others, an explicit expression of this attitude.

[4] cf. (Euler, LPAH, vol. I, p. 201). Here is Euler's original (Euler, 1768-1772 vol. I, lett. 68th, p. 265): "Il s'agit à présent d'approfondir la véritable source de ces forces attractives, ce qui appartient plutôt à la Metaphysique qu'aux Mathematiques; & je ne saurois me flatter d'y reussir aussi heureusement."

[5] cf. (Euler, LPAH, vol. I, p. 165). Here is Euler's original (Euler, 1768-1772 vol. I, lett. 55th, pp. 219-220): "Comme on ne voit rien, qui les [de petits morceaux de fer ou d'acier] pousse vers l'aimant, on dit que l'aimant les attire, & l'action même, se nomme *attraction*. On ne sauroit douter cependant qu'il n'y ait quelque matiere très subtile, quoiqu'invisible, qui produise cet effet, en poussant effectivement le fer vers l'aiman; [...] Quoique ce phénomene soit particulier à l'aimant, & au fer, il est très propre à éclaircir le terme d'attraction, dont les Philosophes modernes se servent si frequemment. Ils disent donc, qu'une propriété semblable à celle de l'aimant, convient à tous les corps, en général, & que tous les corps au monde s'attirent mutuellement".

[6] cf. (Euler, LPAH, vol. I, p. 203). Here is Euler's original (Euler, 1768-1772 vol. I, lett. 68th, p. 268): "Puisque nous savons donc que tout l'espace entre les corps célestes est rempli d'une manière subtile qu'on nomme l'éther, il semble plus raisonnable d'attribuer l'attraction mutuelle des corps, â une action que l'éther y exerce, quoique la maniere nous soit inconnue, que de recourir a une qualité inintelligible." On this same matter, cf. also the 75th letter: (Euler, 1768-1772, vol. I, pp. 297-298).

[7] cf. (Euler, 1736, p. 39): "Potentia est vis corpus vel ex quiete in motum perducens vel motum eius alterans." Here and later, we slightly modify I. Bruce's translation available at http://www.17centurymaths.com.

[8] cf. (Euler, 1736, p. 40): "Similes etiam potentiae deprehenduntur in corporibus magneticis et electricis inesse, quae certa tantum corpora attrahunt. Quas omnes a motu materiæ cuiusdam subtilis oriri alii putant, alii ipsis corporibus vim attrahendi et repellendi tribuunt. Quicquid autem sit, videmus certe ex corporibus elasticis et vorticibus huiusmodi potentias originem ducere posse, suoque loco inquiremus, num ex inde phaenomena haec potentiarum explicari possint."

[9] cf. also (Euler, 1746) and (Euler, 1765, Introduction). The chapter 2 of (Romero, 2007) presents a detailed study of (Euler, 1746) and (Euler, 1750). In (Gaukroger, 1982, pp. 134-138), an overview of the relevant parts of (Euler, 1765) is offered.

[10] cf. (Euler, 1750, art. XIX, p. 428): "Aussi-tot [...] qu'on reconnoit l'impénétrabilité des corps, on est obligé d'avoüer que l'impénétrabilité est accompagnée d'une force suffisante, pour empêcher la pénétration."

bodies meet in such a way that they could not persist in their state of motion without penetrating each other, "from the impenetrability of both of them a force arises that, by acting on them, changes [...] [this] state[11]." This being admitted, Euler shows how to derive from this only supposition the well-known mathematical laws of the shock of bodies. This he considers enough to conclude that the changes in the state of motion due to a shock of two bodies "are produced only by the forces of impenetrability"[12], so that, in this case, the origins (and cause) of forces are just the impenetrability of bodies.

One would expect Euler to go on by describing a plausible mechanical model allowing him to argue that this is also the case of any other force, namely of central forces acting at a distance. But this is not so. He limits himself to considering the case of centrifugal forces which, not basing himself on any argument, he takes to being all reducible to the case where a body is deviated from its rectilinear motion because it meets a vaulted surface (Euler, 1750, art. LI, p. 443). Finally he writes[13]:

> If it were true, as Descartes and many other philosophers have maintained, that all the changes that bodies can suffer come either from the shock of bodies or from the forces named "entrifugal", we would now have clear ideas about the origins of forces producing all these changes [...]. I even believe that Descartes' view would not be a little reinforced by those reflections, since, after having eliminated many imaginary forces with which philosophers have jumbled the first principles of physics, it is very likely that the other forces of attractions, adherence, etc. are not better established.
>
> [...] For, though nobody has been able to establish manifestly the cause of gravity and forces acting upon heavenly bodies through the shock or some centrifugal forces, we should confess that neither has anybody proved the impossibility of it. [...] Now it seems as strange to reason since it is not proved by experience that two bodies separated by a completely empty space mutually attract one another through some forces[14]. Hence, I conclude that, with the exception of forces whose spirits are perhaps able

to act upon bodies, which are probably of a quite different nature, there is no other force in the world beside those originated in the impenetrability of bodies.

Though advancing a non-Newtonian demand of explanation of the nature and causes of forces[15] and sharing both the Cartesian requirement of deriving "basic concepts of mechanics from the essence of body" (Gaukroger, 1982, p. 139), and a Cartesian conception of the world as a *plenum* of matter allowing a reduction of all forces to contact ones, Euler reaches thus a quite Newtonian (and non-Cartesian) attitude only disguised by rhetoric. His main point is finally clear, indeed: a mathematical science of motion is perfectly possible even in the ignorance of the actual causes of the forces of attraction, and the only way to ensure that there are reasons causing the forces are to show that the consideration of these reasons leads to the well-known mathematical laws of motion. These laws—rather than any possible mechanical model—are thus finally understood as the only sure expression of the reality of the universe.

The Reformulation of Newton's Mechanics Using Leibniz's Differential Calculus of and the Introduction of External Frames of Reference

As is well known, Newton's mechanics is essentially geometric. Curves are used to represent trajectories of punctual bodies and a theorem is proved ensuring that non-punctual bodies behave with respect to attractive forces as if their mass were concentrated in their centre of gravity. Instantaneous speeds are indirectly represented and measured by segments taken on the tangents of the curves-trajectories. They are taken to represent primarily the rectilinear space that a relevant point would cover in a given time, finite or infinitesimally small, if any force acting upon it ceased and the motion of this point were thus due only to its inertia. An analogous form of indirect representation and measure holds for any sort of force, or better for their accelerative punctual component. This provides a very simple way of composing forces and inertia, essentially based on the parallelogram law that is primarily conceived as holding for rectilinear uniform motions. When the consideration of time is relevant, this is typically represented and measured by appropriate geometric entities, like appropriate areas: for example the areas that are supposed to be swept in that time by a vector radius, in the case of a trajectory complying with Kepler's second law.

To solve mechanical problems, this fundamental geometric apparatus is of course not sufficient. Newton's mechanics also includes two other fundamental ingredients.

The first is a geometric method which allows to deal with punctual and/or instantaneous phenomena and to determine their macroscopic effects (like equilibrium configurations, effective trajectories, and continuously acting forces). It is provided by the method of prime and last ratios, together with a number of appropriate devices.

The second ingredient is given by a number of fundamental laws expressing some basic relation between bodies (or better their masses), their motions, and the forces acting on them and because of them. It is provided by Newton's well known laws of motion, occasionally supplemented by some principles—like the principle of maximal descent of the centre of gravity—

---

[11]cf. (Euler, 1750, art. XXV, p. 431): "[...] à la rencontre de deux corps, qui se pénétreroient s'ils continuoient à demeurer dans leur état, il nait de l'impénétrabilité de l'un et de l'autre à la fois une force qui en agissant sur les corps, change leur état."

[12]cf. (Euler, 1750, art. XLVI, p. 441): "[...] dans le choc des corps [...], il est clair que les changemens, que les corps y souffrent, ne sont produits que leurs forces d'impénétrabilité." cf. also *ibid*. art. XLIV, p. 440.

[13]cf. (Euler, 1750, arts. LVIII and LIX, pp. 446-447): "[...] s'il étoit vrai, comme Descartes et quantité d'autres Philosophes l'ont soutenu, que tous les changemens, qui arrivent aux corps, proviennent ou du choc des corps, ou des forces nommées centrifuges; nous serions à present tout à fait éclaircis sur l'origine des forces, qui opèrent tous ces changemens [...]. Je crois même que le sentiment de Descartes ne sera pas médiocrement fortifié par ces réflexions; car ayant retranché tant de forces imaginaires, dont les Philosophes ont brouillé les premiers principes de la Physique, il est très probable que les autres forces d'attraction, d'adhésion etc. ne sont pas mieux fondées. LIX. Car quoique personne n'ait encore été en état de démontrer évidemment la cause de la gravité et des forces dont les corps celestes sont sollicités par le choc ou quelque force centrifuge; il faut pourtant avouer que personne n'en a non plus démontré l'impossibilité. [...] Or que deux corps éloignés entr'eux par un espace entièrement vuide s'attirent mutuellement par quelque force, semble aussi étrange à la raison, qu'il n'est prouvé par aucune expérience. A l'exception donc des forces, dont les esprits sont peut-être capables d'agir sur les corps, lesquelles sont sans doute d'une nature tout à fait différente, je conclus qu'il n'y a point d'autres forces au monde que celles, qui tirent leur origine de l'impénétrabilité des corps."

[14]About Euler's opposition to action at distance, cf. (Wilson, 1992).

[15]But we must keep in mind that Newton's rejection of any hypotheses about the nature and the causes of the gravitational force only concerns the limited domain of the mathematical principles of natural philosophy.

which are taken to follow from them.

Though these laws are still considered as the more fundamental ingredients of classical mechanics, what we call today "Newtonian mechanics" is a quite different theory, reached through a deep transformation and reformulation of Newton's original presentation. This transformation and reformulation mainly occurred during the 18th century and they were very much of Euler's doing[16]. Giulio Maltese thus sums up the situation (Maltese, 2000, pp. 319-320):

> In fact, it was Euler who built what we now call the "Newtonian tradition" in mechanics, grounded on the laws of linear and angular momentum (which Euler was the first to consider as principles general and applicable to each part of every macroscopic system), on the concept that forces are vectors, on the idea of reference frame and of rectangular Cartesian co-ordinates, and finally, on the notion of relativity of motion.

This quotation emphasizes some basic ingredients of the Newtonian mechanics of today. We shall come back in a moment on some of them. We will then observe that the gradual emergence of these elements depends on a more basic transformation (though perhaps, not so fundamental in itself). We refer to the replacement of Newton's purely geometric forms of representation of motions, speeds and forces and of the connected method of prime and last ratios by other forms of representation and expression employing appropriate algebraic techniques enriched by the formalism of Leibnitian differential calculus.

This transformation is often described as a passage from a geometric to an analytic (understood as non-geometric) way of presenting Newton's mechanics. This is only partially true, however. Though the use of algebraic and differential formalism indeed allows the expression of the relation between the relevant mechanical quantities through equations involving the two inverse operators d and $\int$ submitted to a number of easily applicable rules of transformation, these equations are part of mechanics only if the symbols that occur in them take on a mechanical meaning. It is just the way in which this meaning is explained—and not the mere use of this formalism—that decides whether the adopted presentation is geometric or not.

For example, it is not enough to identify the punctual speed of a certain motion with the differential ratio $\dfrac{ds}{dt}$ to get a non-geometric definition of speed: whether this definition is geometric or not depends on the way in which this ratio, and namely the differential ds, are understood. If this differential is taken to be an infinitesimal difference in the length of a certain variable segment represented by an appropriate geometric diagram and indicating the direction of the speed in respect to another component of this diagram, the definition is still geometric.

This is exactly what happens in the first attempts to apply differential formalism to Newtonian mechanics, like those of Varignon, Johann Bernoulli, and Hermann[17]: the language of differential calculus is used to speak of mechanical configura-

tions represented by appropriate geometric diagrams and its rules are applied in order to get the relevant quantitative relations between the elements depicted in these diagrams (Panza, 2002). Like in Newton's *Principia*, mechanical problems are thus, in these essays, distinguished from each other according to specific features manifested by the corresponding diagrams. Hence, differences in the problems depend on differences in the diagrams.

This fragmentation of mechanics into several problems geometrically different is still particularly evident in Hermann's *Phoronomia* (Hermann, 1716), which Euler considers as the main treatise on dynamical matters written after the *Principia*. This is just what Euler wants to avoid in his *Mechanica*. Here is what he writes in the preface[18]:

> [...] what distracts the reader the most [in Hermann's *Phoronomia*] is that everything is carried out [...] with old-fashioned geometrical demonstrations [...]. Newton's *Principia Mathematica Philosophiae* are composed in a scarcely different way [...]. But what happens with all the works composed without analysis is particularly true with those which pertain to mechanics. In fact, the reader, even when he is persuaded of the truth of the things that are demonstrated, nonetheless cannot reach a sufficiently clear and distinct knowledge of them. So he is hardly able to solve the same problems with his own strengths, when they are changed just a little, if he does not research into the analysis and if he does not develop the same propositions with the analytical method. This is exactly what often happened to me, when I began to examine Newton's *Principia* and Hermann's *Phoronomia*. In fact, even though I thought that I could understand the solutions to numerous problems well enough, I could not solve problems that were slightly different. Therefore, in those years, I strove, as much as I could, to arrive at the analysis behind those synthetic methods, and to deal with those propositions in terms of analysis for my own purposes. Thanks to this procedure I perceived a remarkable improvement of my knowledge.

A major purpose of Euler's *Mechanica* is to use Leibnitian differential formalism (which is what he calls "analytic method") in order to generalise some of Newton's results. Euler aims to arrive at some general procedures which allow him to solve large families of problems. He also looks for some rules for use in appropriate circumstances to determine, in a somewhat automatic way, appropriate expressions for relevant me-

---

[16]As N. Guicciardini has remarked (Guicciardini, 1999, p. 6), "after Euler the *Principia*'s mathematical methods belong definitely to what is past and obsolete."

[17]On these essays, cf. (Aiton, 1989), (Blay, 1992), (Guicciardini, 1995), (Guicciardini, 1996), (Mazzone & Roero, 1997).

[18]A large part of this passage is quoted and translated by N. Guicciardini in (Guicciardini, 2004, p. 245). We quote his translation by adding a translation of the part he does not quote that slightly differs from Bruce's [cf. footnote 7]. Here is Euler's original (Euler, 1736, Præfatio): "[...] quod lectorem maxime distinet, omnia more veterum [...] geometricis demonstrationibus est persecutus [...]. Non multum dissimili quoque modo conscripta sunt Neutoni *Principia Mathematica Philosophiae* [...]. Sed quod omnibus scriptis, quæ sine analysi sunt composita, id potissimum Mechanicis obtingit ut Lector, etiamsi de veritate eorum, quæ proferuntur, convincatur, tamen non satis claram et distinctam eorum cognitionem assequatur, ita ut easdem quaestiones, si tantillum immutentur, proprio marte vix resolvere valeat, nisi ipse in analysin inquirat easdemque propositiones analytica methodo evolvat Idem omnino mihi, cum Neutoni *Principia* et Hermanni *Phoronomiam* perlustrare cœoepissem, usu venit, ut, quamvis plurium problematum solutiones satis percepisse mihi viderer, tamen parum tantum discrepantia problemata resolvere non potuerim. Illo igitur iam tempore, quantum potui, conatus sum analysin ex synthetica illa methodo elicere easdemque propositiones ad meam utilitatem analytice pertractare, quo negotio insigne cognitionis meae augmentum percepi".

chanical quantities.

In order to reach this aim, Euler identifies punctual speeds and accelerations with first and second differential ratios, respectively, and introduces an universal measure of a punctual speed given by the altitude from which a free falling body has to fall in order to reach such a speed.

The basic elements of Newton's mechanics appear in Euler's treatise under a new form, quite different from the original. Nevertheless, in this treatise, mechanical problems are still tackled by relying on intrinsic coordinates systems: speeds and forces are composed and decomposed according to directions that are dictated by the intrinsic nature of the problem, for example so as to calculate the total tangential and normal forces with respect to a given trajectory. This approach is quite natural, but limits the generality of possible common rules and principles.

A new fundamental change occurs when extrinsic reference frames, typically constituted by triplets of orthogonal fixed Cartesian coordinates, are introduced and when the relativity of motion is conceived to be the invariance of its laws with respect to different frames submitted to uniform retailer motions. Though this change was in fact a collective and gradual transformation (Meli, 1993) and (Maltese, 2000, p. 6), Euler played a crucial role in it. Among other important contributions connected with this change—described and discussed in (Maltese, 2000)—, it is important to consider his introduction of today's usual form of Newton's second law of motion. This is the object of a memoir presented in 1750: "Découverte d'un nouveau principe de mécanique" (Euler, 1750).

The argument that Euler offers in this memoir in order to justify the introduction of his "new principle" is so clear and apt to elucidate the crucial importance of this new achievement that it deserves to be mentioned. The starting point of this argument is the insufficiency of the tools provided both by Newton's *Principia* and by Euler's own *Mechanica* for studying the rotation of a solid body around an axis continuously changing its position with respect to the elements of the body itself. To study this motion, Euler argues, new principles are needed and they have to be deduced from the "first principle or axioms" of mechanics, which, he says, cannot but concern the rectilinear motion of punctual bodies (Euler, 1750, art. XVIII, p. 194). The problem is precisely that of formulating these axioms in the most appropriate way to allow an easy deduction of all the other principles that are needed to study the different kinds of motion of the different kinds of bodies. According to Euler, these axioms are reduced to an unique principle, and his new one is just that.

This principle is expressed by a triplet of equations that express Newton's second law with respect to the three orthogonal directions of a reference frame independent of the motions to be studied (Euler, 1750, art. XXII, p. 196):

$$2Md^2x = Pdt^2; 2Md^2y = Qdt^2; 2Md^2z = Rdt^2,$$

where M is the mass of the relevant punctual body, P, Q, and R are the total forces acting along the directions of the threes axes and 2 is a factor of normalization.

To understand the fundamental role that Euler assigns to this principle, a simple example is sufficient (Euler, 1750, art. XXIII, p. 196): from $P = Q = R = 0$, one gets, by integrating,

$$Mdx = Adt; Mdy = Bdt; Mdz = Cdt,$$

where A, B, C are integration constants. It is thus proved

that, in this case, the speed is constant in any direction so that the motion of any body on which no force acts is rectilinear and uniform, as Newton's first law asserts[19].

A New Sort of Principles: Euler's Program for Founding Newton's Mechanics on Variational Principles

Though fairly powerful, Euler's new principle only directly deals with single punctual bodies. Let's consider a system of several punctual bodies mutually attracting each other and possibly submitted to some external forces and internal constraints. In order to get the conditions of equilibrium or the equations of motions of such a system by relying on Euler's principle, a detailed and geometric analysis of all the forces operating in this system is necessary. *A fortiori*, this is also the case of any other principle dealing with single punctual bodies set in Newton's *Principia* and in Euler's *Mechanica*. Hence, the study of any particular system of several punctual bodies through these principles requires a geometrical analysis of forces that differs from system to system. Consequently, only fairly simple systems can be studied in such a way.

This is the reason why the need of a new sort of mechanical principles—directly concerned with whatever system of several punctual bodies—arose quite early. A similar principle, that would be later known as the principle of virtual velocities, was suggested by Johann Bernoulli in a letter to Varignon of January, 26th 1711 (Varignon, 1725, vol. II, pp. 174-176). But a clear statement of the difference between these two kinds of principles only appears in a memoir presented by Maupertuis in 1740[20]:

> If Sciences are grounded in certain immediately easy and clear principles, from which all their truths depend, they also include other principles, less simple indeed, and often difficult to discover, but that, once discovered, are very useful. They are to some extent the Laws that Nature follows in certain combinations of circumstances, and they teach us what it will do in similar occasions. The former principles need no proof, because they become obvious as soon as the mind examines them; the latter could not have,

---

[19]To appreciate the crucial difference between Euler's new principle and Newton's second law of motion, asserting that "A change in motion is proportional to the motive force impressed and takes place along the straight line in which the force is expressed [*Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur*]" (Newton, PMCW, p. 12); (Newton, 1687, p. 416), remark that, by supposing the motive force to be null, one can only deduce from this last law that the relevant motion is not changed, that is, it is inertial, but not that it is rectilinear uniform. To reach this conclusion, one has also to rely on Newton's first law, which just fixes the nature of inertial motion: "Every body preserves in its state of being at rest or of moving uniformly straight forward except insofar as it is compelled to change its state by forces impressed [*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare*]" (*ibid.*).

[20]cf. (Maupertuis, 1740, p. 170): "Si les Sciences sont fondées sur certains principes simples et clairs dès le premier aspect, d'où dépendent toutes les vérités qui en sont l'objet, elles ont encore d'autres principes, moins simples à la vérité, etsouvent difficiles à découvrir, mais qui étant une fois découverts, sont d'une très-grande utilité. Ceux-ci sont en quelque façon les Loix que la Nature suit dans certaines combinaisons de circonstances, et nous apprennent ce qu'elle fera dans de semblables occasions. Les premiers principes n'ont guére besoin de Démonstration, par l'évidence dont ils sont dès que l'esprit les examine; les derniers ne sçauroient avoir de Démonstration physique à la rigueur, parce qu'il est impossible de parcourir généralement tous les cas où ils ont lieu."

strictly speaking, a physical proof, since it is impossible to consider in general all cases to which they apply.

The aim of Maupertuis's memoir is to suggest a new principle of the second kind, asserting that the equilibrium of any system of n punctual bodies is obtained if an appropriate sum is maximal or minimal. This sum is

$$\sum_{i=1}^{n} M_i \left[\int P_i \, dp_i + \mathsf{L} + \int W_i \, dw_i \right]$$

where M_i is the mass of the i-th body, P_i, ... , W_i are the forces acting upon it, and p_i, ..., w_i are the distances of this body from the centres of these forces, respectively. This is the first, static, version of the principle of least action.

Maupertuis's memoir originated quite an important program concerned with the foundation of mechanics, leading—through d'Alembert, Euler, and Lagrange, among others—to Hamilton's well-known version of Newton's mechanics (Fraser, 1983), (Fraser, 1985), (Szabó, 1987), (Pulte, 1989), (Panza, 1995), (Panza, 2003).

Euler's contributions to this program were essential and concerned three major aspects:

- the elaboration of an appropriate mathematical tool for dealing with extremality conditions relative to integral forms including unknown functions;
- the generalisation of Maupertuis's principle so as to get a general principle apt to provide the equations of motion of any system of several punctual bodies and also applicable, *mutatis mutandis*, to the solution of other mechanical problems;
- the justification of such a principle.

In Euler's view, these three aspects are intimately connected. His first contribution to this matter comes from his *Methodus inveniendi* (Euler, 1744): a treatise providing the first systematisation of what is known, after Lagrange, as the calculus of variations[21]. The two appendixes to this treatise are solely devoted to enquiring the possibility of studying, respectively, the behaviour of an elastic band and the motion of an isolated body when they are submitted to forces, by relying on a general principle asserting that "absolutely nothing happens in the world, in which a condition of *maximum* or *minimum* does not reveal itself"[22]. Euler's main aim is not to find new results concerned with these problems, but to show how some already known results can be derived from a condition of maximum or minimum for an integral form. The particular nature of this condition in the cases considered is taken to clarify the way in which a principle, which is analogous to Maupertuis's one, can be stated in these cases and then, if possible, generalised. This same approach also governs Euler's other works on the principle of least action: cf. in particular (Euler, 1748), (Euler, 1748), (Euler, 1751), (Euler, 1751), and (Euler, 1751).

Euler's and Maupertuis's approaches are contrastive. Maupertuis is mainly interested in looking for metaphysical and theological arguments (Maupertuis, 1744), (Maupertuis, 1746), (Maupertuis, 1750), (Maupertuis, 1756). Indeed, he aims to support his claim to have found the very quantity in which Nature is thrifty, and thus the real final cause acting in it. Euler is looking for mathematical invariances of the form

$$\int Z \, dz = Max / Min$$

emerging in different conditions and from which already known results regarding different mechanical problems can be drawn.

Euler's main idea is thus that of looking for an appropriate mathematical way to state a new principle that, being in agreement with several results obtained through Newton's original method of analysis of forces, could be generalised so as to get a principle of a new sort, namely a general variational principle.

This research constituted a major event in the history of mechanics for it allowed to pass from a geometrical-based study of a particular concrete system to an analytical treatment of any sort of system based on an unique and general equation. In our view, these are the most fundamental origins of analytical mechanics (Panza, 2002). Lagrange's first general formulation of the principle of least action (Lagrange, 1761) essentially depends on the results obtained by Euler in this way.

Algebraic Analysis

Among the many well-known differences between Newton's and Leibniz's approaches to calculus, a fairly relevant one deals with their opposite conceptions about its relation with the whole corpus of mathematics. Whereas Leibniz often stressed the novelty of his differential calculus, notably because of its special concern with infinity. Newton always conceived his results on tangents, quadratures, punctual speeds and connected topics as natural extensions of previous mathematics.

Newton's first research on these matters was explicitly based on the framework of Descartes' geometry and geometrical algebra provided in *La Géométrie* (Descartes, 1637). It mainly dates back to the years 1664-1666 (Panza, 2005), but culminates with the composition of the *De analysis* in 1669 (Newton, MWP, vol. II, pp. 206-247) and of the *De methodis* in 1671 (Newton, MWP, vol. III, pp. 32-353), where the new theory of fluxions is exposed.

Later, Newton famously changed his mind about the respective merits of Descartes' new way of making geometry and the classical (usually considered as synthetic) approach, especially identified with the style of Apollonius' *Conics* (Galuzzi, 1990; Guicciardini, 2004), and based the *Principia* on the method of first and ultimate ratios, which he took to be perfectly compatible with this last approach[23]. Finally, in the more mature presentation of the theory of fluxions, the *De quadratura curvarum* (Newton, 1704)[24], Newton stresses explicitly[25]:

[...] To institute analysis in this way and to investigate the first or last ratios of nascent or vanishing finites is in harmony with the geometry of the ancients, and I wanted to show that in the method of fluxions there should be no need to introduce infinitely small figures into geometry.

The *De methodis* begins as follows[26]:

---

[21]On Euler's version of the calculus of variations, cf. (Fraser, 1994). C. Fraser has also devoted many works to the history of the calculus of variation. A general survey of his result is offered in (Fraser, 2003).

[22]cf. (Euler, 1744, p. 245): "nihil omnino in mundo contingit, in quo non maximi minimive ratio quæ piam eluceat."

[23]In the concluding lemma of the section I of book I of the *Principia*, Newton claims that he proved the lemmas to which his method pertains "in order to avoid the tedium of working out lengthy proofs by *reductio ad absurdum* in the manner of the Ancient geometers" (Newton, PMCW, p. 441).

[24]See also (Newton, MWP, Vol. VIII, pp. 92-168).

[25]cf. (Newton, MWP, VIII, p. 129): "[...] Analysin sic instituere, & finiterum nascentium vel evanescentium rationes primas vel ultimas investigare, consonum est Geometriæ Veterum: & volui ostendere quo in Methodo Fluxionum non opus sit figuras infinite parvas in Geometriam introducere."

[26]cf. (Newton, 1670-1671, pp. 32-33): "Animadvertenti plerosque Geometras [...] Analyticæ excolendæ plurimum incumbere, et ejus ope tot tantasque difficultates superasse [...]: placuit sequentia quibus campi analytici terminos expandere juxta ac curvarum doctrinam promovere possem [...]."

Observing that the majority of geometers [...] now for the most part apply themselves to the cultivation of analysis and with its aid have overcome so many formidable difficulties [...] I found it not amiss [...] to draw up the following short tract in which I might at once widen the boundaries of the field of analysis and advance the doctrine of curves.

In the early-modern age, the term "analysis" and its cognates were used in mathematics in different, though strictly connected, senses. Two of them were dominant in the middle of 17th century. Analysis, in the first sense, refers to the first part of a twofold method—the method of analysis and synthesis—paradigmatically expounded in the 7th book of Pappus *Mathematical Collection* (Pappus, CMH). In the second sense, it refers to a new domain of mathematics the introduction of which was typically ascribed to Viète, who had explicitly identified it with a "new algebra" (Viete, 1591).

In our view, this discipline should be considered more as a family of techniques for making both arithmetic and geometry (Panza, 2007a), than as a separate theory somehow opposed to arithmetic and geometry. In the previous passage, Newton is undoubtedly referring to this discipline and he is claiming that his treatise aims to extend it so as to make it appropriate for studying curves.

However, this extension crucially depends not only on the addition of new techniques, based on Descartes' algebraic formalism, but also on a new conception about quantities, according to which mathematics should deal not only with particular sorts of quantities, such as numbers, segments, etc., but also with quantities purely conceived, that is, with fluents (Panza, 2012).

Hence, in the *De methodis*, the extended "field of analysis" no longer presents itself as a family of powerful techniques, but it rather takes the form of a new theory dealing with quantities purely conceived. These quantities are supposed to belong to a net of operational relations expressed through Descartes' algebraic formalism appropriately extended so as to include infinitary expressions like series.

Euler's Theory of Functions

Newton's later opposition to Descartes' way of doing geometry and the independence of the mathematical method of first and ultimate ratios from the analytic formalism of the theory of fluxions—in the presence of the well-known Newton-Leibniz priority quarrel and its consequences—lead, in the 18th century, to a polarisation between two ways of understanding calculus. A Newtonian way, based on a classic conception of geometry, conceives fluxions as ratios of vanishing quantities. A Leibnitian way, based on the introduction of an appropriate new formalism, deals with infinitesimals.

Maclaurin's *Treatise of fluxions* (Maclaurin, 1742) is usually pointed out as the major example of the Newtonian view. "Fluxional 'computations' are not presented as a blind manipulation of symbols, but rather as meaningful language that could always be translated into the terminology of [...] [a] kinematic-geometric model" (Guicciardini, 2004, pp. 239-240).

In contrast, Euler's trilogy composed by the *Introductio in analysin infinitorum*, the *Institutiones calculi differentialis*, and the *Institutionum calculi integralis* (Euler, 1748), (Euler, 1755), (Euler, 1768) is indicated as the major example of the Leibnitian view.

Nonetheless the two traditions were not as opposed, and the respective scientific communities were not as separated as it has been too often claimed. A clear example of this—which is mainly relevant here—is provided by Euler's approach. Though there is no doubt that the theory expounded by Euler in the *Institutiones* and in the *Institutionum* uses Leibniz's differential and integral formalism, some of the basic conceptions it is founded on derive from Newton's views.

Some of these conceptions are strictly internal to the organisation of the theory. An example is provided by Euler's idea that the main objects of differential calculus are not differentials of variables quantities but differential ratios of functions conceived as ratios of vanishing differences (Ferraro, 2004). As he writes in the preface of the *Institutiones*[27]:

Differential calculus [...] is a method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment [...] Therefore, differential calculus is concerned not so much with vanishing increments, which indeed are nothing, but with their mutual ratio and proportion. Since these ratios are expressed as finite quantities, we must think of calculus as being concerned with finite quantities.

In this way, Euler unclothes Newton's notion of prime or ultimate ratio of its classically geometric apparel and transfers it to a purely formal domain using the language of Leibniz's differential calculus.

Another strictly connected example comes from Euler's definition of integrals as anti-differentials and of the integral calculus as the "method" to be applied for passing "from a certain relation among differentials to the relation of their quantities", that is, in the simplest case (Euler, 1768), definitions 2 and 1, respectively, from

$$\frac{dy}{dx} = z$$

to

$$y = f(x) = \int z\,dx.$$

These definitions—which contrast with Leibniz's conception of the integral as a sum of differentials—are instead clearly in agreement with the second problem of Newton's *De methodis*: "when an equation involving the fluxions of quantities is exhibited, to determine the relation of the quantities one to an- other"[28].

The closeness of Euler's and Newton's views in both those examples depends on a more fundamental concern: the idea that both differential and integral calculus are part of a more general theory of functions (Fraser, 1989).

This theory is exposed by Euler in the first volume of the *In-*

---

[27]We slightly modify Blanton's translation (Euler, ICDB , p. vii). Here is Euler's original (Euler, 1755, p. VIII): "[...] calculi Differentialis [...] est methodus determinandi rationem incrementorum evanescentium, quæ functiones qæ cunque accipiunt, dum quantitati variabili, cuius sunt functiones, incrementum evanescens tribuitur [...]. Calculus igitur differentialis non tam in his ipsis incrementis evanescentibus, quippe quæ sunt nulla, exquirendis, quam in eorum ratione ac proportione mutua scrutanda occupatur: et cun hæ rationes finitis quantitabus exprimantur, etiam hic calculus circa quantitates finitas versari est censendus."

[28]cf. (Newton, 1670-1671, pp. 82-83): "Exposita Æquatione fluxiones quantitatum involvente, invenire relationem quantitatum inter se". On Newton's notion of primitive, cf. (Panza, 2005, pp. 284-293, 323-325 and 439-460).

troductio (Euler, 1748). According to him, it is not merely a mathematical theory among others. It is rather the fundamental framework of the whole of mathematics. Differential calculus is thus not conceived by Euler as a separated theory characterised by its special concern with infinity, as in Leibniz's conception, but rather as a crucial part of an unitary building the foundations of which consist of a theory of functions (Panza, 1992).

As a matter of fact, this theory comes in turn from a large and ordered development of the results that Newton had presented in his *De methodis* before attaching the two main problems of the theory of fluxions, and that provided for him the base on which its extended "field of analysis" was grounded. A function is identified with an expression indicating the operational relations about two or more quantities and expressing a quantity purely conceived (Panza, 2007b). And the fundamental part of the theory concerns the power series expansions of functions.

Though the language and the formalism that are used in Euler's trilogy openly come back to the Leibnitian tradition, such a trilogy should thus be viewed, for many and fundamen- tal reasons, as a realisation of the unification program that Newton had foreseen in the *De methodis*, a realisation that re- lies, moreover, on the basic idea underlying Newton's method of prime and last ratios.

The Classification of Cubics and Algebraic Curves

The second volume of the *Introductio* (Euler, 1748) is devoted to algebraic curves: the curves expressed by a polynomial equation in two variables when referred to a system of rectilinear coordinates. Euler relies on some results obtained in the first volume to show that algebraic curves can be studied and classified without making use of calculus: as a matter of fact, this marks the birth of algebraic geometry.

The problem of the classification of curves is quite ancient (Rashed, 2005). However, in his *Géométrie* (Descartes, 1637), Descartes returns to it in a new form: he concentrates only on algebraic curves (that he calls "geometric", whereas he recommends to reject from geometry other curves, termed "mechanical") and bases his classification on the degree of the corresponding equation (Bos, 2001, pp. 356-357), (Rashed, 2005, pp. 32-50). This is, in fact, quite a broad classification, since equations of the same degree can express curves which look very different from each other. The classification of (non-degenerate) conics (the algebraic curves whose equations are the irreducible ones of degree 2) is well known: they split up into ellipses (including circles), parabolas and hyperbolas. But what about curves expressed by equations of higher degrees?

Newton answers the question for cubics (the algebraic curves whose equations are the non-reducible ones of degree 3), in a tract appeared in 1704 as an appendix to the *Opticks* (Newton, 1704), but the different stages of composition of which presumably date back to 1667-1695 (Newton, MWP, vol. VII, p. 565-655): the *Enumeratio Linearum Tertii Ordinis* (Newton, 1704).

In the second volume of the *Introductio*, Euler tackles the same problem using a quite different method, and shows that Newton's classification is incomplete (Euler, 1748, vol. II, Ch. 9). He also provides an analogous classification of quartics (the algebraic curves whose equations are the non-reducible ones of degree 4), and explains how the method used for classifying cubics and quartics can, in principle, be applied to algebraic curves of any order (Euler, 1748, Vol. II, ch. 11 and 12-14, respectively).

Whereas Newton's classification of cubics is based on their figure in a limited (*i.e.* finite) region of the plane and depends on the occurrences of points or line singularities as for instance nodes, cusps, double tangents, Euler suggests classifying algebraic curves of any order by relying on the number and on the nature of their infinite branches. Here is what he writes[29]:

> Hence, we reduced all third order lines to *sixteen species*, in which, therefore, all those of the *seventy-two* species in which Newton divided the third order lines are contained[30]. It is not odd, in fact, that there is such a difference between our classification and Newton's, since we obtained the difference of species only from the nature of branches going to infinity, while Newton considered also the shape of curves within a bounded region, and established the different species on the basis of their diversity. Although this criterion may seem arbitrary, however, by following his criteria Newton could have derived many more species, whereas using my method I am able to draw neither more nor less species.

Euler's last remark alludes to much more than what it says. He rejects Newton's criterion because of the impossibility of applying it as the order of curves increases, since so great a variety of shapes arise, as witnessed by the mere case of cubics. Indeed, a potentially general criterion has to deal with some properties of curves that can be systematically and as exhaustively as possible explored in any order, like those of infinite branches, according to Euler.

This task could be difficult, however, if these curves were studied through their equations taken as such, since the complexity of a polynomial equation in two variables increases very quickly with its order. Insofar as a polynomial equation in two variables cannot be transformed by changing its global degree so that it continues to express the same curve, Euler shows how to determine the number and nature of infinite branches of a curve by considering its equation for some appropriate transformations that lower its degree in one variable, namely "both by choosing the most convenient axis and the most apt inclination of the coordinates", and attributing to a variable a convenient value[31].

---

[29]We slightly modify Blanton's translation (Euler, IAIB, Vol. II, p. 147). Here is Euler's original (cf. Euler, 1748, Vol. II, 236, p. 123): "Omnes ergo Lineas tertii ordinis reduximus ad Sedeciem Species, in quibus propterea omnes illæ Species Septuaginta dua, in quas Newtonus Lineas tertii ordinis divisit, continentur. Quod vero inter hanc nostram divisionem ac Newtonianam tantum intercedat discrimen mirum non est; hic enim tantum ex ramorum in infinitum excurrentium indole Specierum diversitatem desumsimus, cum Newtonus quoque ad statum Curvarum in spatio finito spec- tasset, atque ex hujus varietate diversas Species constituisset. Quanquam autem hæc divisionis ratio arbitraria videtur, tamen Newtonus suam tandem rationem sequens multo plures Species producere potuisset, cum equidem mea methodo utens neque plures neque pauciores Species eruere queam."

[30]This is the reason why Euler prefers to use the term "genre" instead of "species" in order to indicate his classes of curves. The latter terms is reserved to distinguish curves of a given genre according to their shape in a limited region of the plane (Euler, 1748, vol. II, § 238, p. 126).

[31]Once again, we slightly modify Blanton's translation (Euler, IAIB, Vol. II, p. 176). Here is Euler's original (Euler, 1748, Vol. II, 272, p. 150): "Negotium autem hoc per reductionem æ quationis ad formam simpliciorem, dum et Axis commodissimus, et inclinatio Coordinatarum aptissima assumitur, valde sublevari potest: tum etiam, quia perinde est, utra Coordinatarum pro Abscissa accipiatur, labor maxime diminuetur, si ea Coordinatarum, cujus paucissimæ dimensiones in æ quatione occurrunt, pro Applicata assumatur."

Conclusion

Euler's results of which we have given an account, both in the case of foundation of mechanics and in that of algebraic analysis, depend on the effort to carry out or to extend a Newtonian program. But in both cases, this is done by relying on Cartesian and Leibnitian conceptions and tools. C. A. Truesdell has summed up the situation about mechanics by saying that Euler inaugurated the tradition of Newtonian mechanics because he "put most of mechanics in their modern form"[32]. *Mutatis mutandis*, the same also holds for Euler's theory of functions. In both cases the following question naturally arises: what does remain, then, of Newton's conceptions in Euler's theories? This is too difficult a question to hope to offer a complete answer in a single paper. We merely hope to have provided some elements for such an answer.

Acknowledgements

REFERENCES

Aiton, E. J. (1989). The contributions of Isaac Newton, Johann Bernoulli and Jakob Hermann to the inverse problem of central forces. In H.-J. Hess (Ed.), Der ausbau des calculus durch leibniz und die brüder Bernoulli, Studia Leibnitiana, Sonderhefte (pp. 48-58). Stuttgart: Steiner-Verlag.

Backer, R. (Ed.) (2007). Euler reconsidered. Tercentenary essays. Heber City: Kendrick Press.

Bartoloni-Meli, D. (1993). The emergence of reference frames and the transformation of mechanics in the Enlightenment. Historical Studies in the Physical Sciences, 23, 301-335.

Blay, M. (1992). La naissance de la mécanique analytique. Paris: PUF.

Bos, H. J. M. (2001). Redefining geometrical exactness. Descartes' transformation of the early modern concept of construction. Sources and studies in the history of mathematics and physical sciences. New York: Springer.

Bradley R., & Sandifer, E. (2007). Leonhard Euler. Life, work and legacy. Amsterdam: Elsevier.

Calinger, R. (1976). Euler's letters to a princess of Germany as an expression of his mature scientific outlook. Archive for History of Exact Sciences, 150, 211-233.

Cassirer, E. (1907). Das erkenntnisproblem in der philosophie und wissenschaft der neueren zeit, vol. II. Berlin: Bruno Cassirer.

Descartes, R. (1637). La géométrie. In Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences. Plus la dioptrique. Les météores & la géométrie qui sont des essais de cette méthode (pp. 297-413). Leyde: I. Maire.

Euler, L. (1736). Mechanica, sive motus scientia analytice exposita. 2 vols. Petropoli: Ex Typographia Academiae Scientiarum. In (Euler, OO, Series II, 1-2).

Euler, L. (1743). De causa gravitatis [publ. anonymously]. Miscellanea Berolinensia, 7, 360-370. In (Euler, OO, Series II, 31, 373-378).

Euler, L. (1744). Methodus inveniendi lineas curvas maxime minimive proprietate gaudentes. Lausanæ et Genevæ: M. M. Bousquet et Soc. In (Euler, OO, Series I, 24).

Euler, L. (1746). De la force de percussion et de sa véritable mesure. Mémoires de l'Académie des Sciences de Berlin, 1, 21-53. In (Euler, OO, Series II, 8, 27-53).

Euler, L. (1748). Introductio in analysin infinitorum. Lausanæ: Apud Marcum-Michælem Bousquet & Socios. In (Euler, OO, Series I, 8-9).

Euler, L. (1748). Recherches sur les plus grands et les plus petits qui se trouvent dans les actions des forces. Histoire de l'Académie Royale

des Sciences et des Belles Lettres [de Berlin], 4, 149-188. In (Euler, OO, Series I, 5, 1-37).

Euler, L. (1748). Réflexions sur quelques loix générales de la nature qui s'observent dans les effets des forces quelconques. Histoire de l'Académie Royale des Sciences et des Belles Lettres [de Berlin], 4, 183-218. In (Euler, OO, Series I, 5, 38-63).

Euler, L. (1750). Recherche sur l'origine des forces. Mémoires de l'Académie des Sciences de Berlin, 6, 419-447. In (Euler, OO, Series II, 5, 109-131).

Euler, L. (1750). Découverte d'un nouveau principe de mécanique. Mémoires de l'Académie des sciences de Berlin, 6, 185-217. In (Euler, OO, Series II, 5, 81-110).

Euler, L. (1751). Harmonie entre les principes generaux de repos et de mouvement de M. de Maupertuis. Histoire de l'Académie Royale des Sciences et des Belles Lettres [de Berlin], 7, 169-198. In (Euler, OO, Series I, 5, 152-172).

Euler, L. (1751). Essay d'une démonstration métaphysique du principe générale de l'équilibre. Histoire de l'Académie Royale des Sciences et des Belles Lettres [de Berlin], 7, 246-254. In (Euler, OO, Series II, 5, 250-256).

Euler, L. (1751). Sur le principe de la moindre action. Histoire de l'Académie Royale des Sciences et des Belles Lettres [de Berlin], 7, 199-218. In (Euler, OO, Series I, 5, 179-193).

Euler, L. (1755). Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum. Impensis Academia Imperialis Scientiarum Petropolitanæ. Berolini: ex officina Michælis, Reprint (1787), Ticini: typographeo P. Galeatii [we refer to this reprint]. Also in (Euler, OO, Series I, 10).

Euler, L. (1765). Theoria motus corporum solidorum seu rigidorum [...]. Rostochii et Gryphiswaldiæ: litteris et impensis A. F. Röse. In (Euler, OO, Series II, 3).

Euler, L. (1768). Institutionum Calculi Integralis, 3 vols. Petropoli: Impensis Academia Imperialis Scientiarum. In (Euler, OO, Series 1, 11-13).

Euler, L. (1768-1772). Lettres à une princesse d'Allemagne sur divers sujets de physique & de philosophie (3 vols.). Saint-Petersbourg: Imprimerie de l'Académie Impériale des Sciences. In (Euler, OO, Series III, 11-12).

Euler, L (LPAH). Letters of Euler on different subjects in natural philosophy. Adressed to a German Princess (2 vols.). Translation of (Euler, 1768-1772) by H. Hunter. 3rd Edition. Edinburgh: Printed for W. and C. Tait [...].

Euler, L. (IAIB). Introduction to analysis of the infinite. New York, Berlin: Springer Verlag. English Translation of (Euler, 1748) by J. D. Blanton, 2 vols.

Euler, L. (ICDB). Foundations of differential calculus. New York, Berlin: Springer Verlag. English translation of (Euler, 1755, Vol. I) by J. D. Blanton.

Euler, L. (OO). Leonhardi Euleri Opera Omnia (76 vols). Leipzig, Berlin, Basel: Societas Scientiarum Naturalium Helveticæ.

Euler, L. The Euler archive. The works of Leonhard Euler online. http://math.darmouth.edu/euler

Ferraro, G. (2004). Differentials and differential coefficients in the eulerian foundations of the calculus. Historia Mathematica, 31, 34-61.

Fraser, C. G. (1983). J. L. Lagrange's eraly contributions to the principles and methods of mechanics. Archive for History of Exact Sciences, 28, 197-241.

Fraser, C. G. (1985). D'Alembert principle: The original formulation and application in Jean d'Alembert's *Traité de Dynamique* (1743). Centaurus, 28, 31-61 and 145-159.

Fraser, C. G. (1989). The calculus as algebraic analysis: Some observations on mathematical analysis in the 18th century. Archive for History of Exact Sciences, 390, 317-335.

Fraser, C. G. (1994). The origins of Euler's variational calculus. Archive for History of Exact Sciences, 470, 103-141.

Fraser, C. G. (2003). The calculus of variations: A historical survey. In H. N. Jahnke (Ed.), A history of analysis (pp. 355-384). New York: American Mathematical Society and London Mathematical Society.

Galuzzi, M. (1990). I *marginalia* di Newton alla seconda edizione latina della *Geometria* di Descartes e i problemi ad essi collegati. In G. Belgioioso (Ed.), Descartes, Il metodo e i saggi. Atti del Convegno

---

[32]cf. (Truesdell, 1968, p. 106). See also (Truesdell, 1970) and (Maglo, 2003, p. 139) that resumes Truesdell's views on this matter neatly and clearly.

per il 350e anniversario della pubblicazione del Discours de la Méthode e degli Essais (2 vols) (pp. 387-417). Firenze: Armando Paoletti.

Gaukroger, S. (1982). The metaphysics of impenetrability: Euler's conception of force. The British Journal for the History of Science, 15, 132-154.

Guicciardini, N. (1995). Johann Bernoulli, John Keill and the inverse problem of central forces. Annals of Science, 52, 537-575.

Guicciardini, N. (1996). An episode in the history of dynamics: Jakob Hermann's proof (1716-1717) of Proposition 1, Book 1, of Newton's *Principia*. Historia Mathematica, 23, 167-181.

Guicciardini, N. (1999). Reading the Principia: The debate on Newton's mathematical methods for natural philosophy from 1687 to 1736. Cambridge: Cambridge University Press.

Guicciardini, N. (2004a). Dot-age: Newton's mathematical legacy in the eighteenth century. Early Science and Medicine, 9, 218-256.

Guicciardini, N. (2004b). Geometry and mechanics in the preface to Newton's *Principia*: A criticism of Descartes' *Géométrie*. Graduate Faculty Philosophy Journal, 25, 119-159.

Hermann, J. (1716). Phoronomia, sive de viribus et motibus corporum solidorum et fluidorum libri duo. Amstelædami: Apud R. & G. Wetstenios H. FF.

Lagrange, J. L. (1761). Applications de la méthode exposée dans le mémoire précédent à la solution de différents problèmes de dynamique. Mélanges de Philosophie et de Mathématiques de la Société Royale de Turin, 2, 196-298.

Maclaurin, C. (1742). A treatise of fluxions: In two books. Edinburgh: T.W. and T. Ruddimans.

Maglo, K. (2003). The reception of Newton's Gravitational Theory by Huygens, Varignon, and Maupertuis: How normal science may be revolutionary. Perspectives on Science, 11, 135-169.

Maltese, G. (2000). On the relativity of motion in Leonhard Euler's science. Archive for History of Exact Sciences, 54, 319-348.

Maupertuis, P. L. M. (1740). Loi du repos des corps. Histoire de l'Académie Royale des Sciences [de Paris], Mémoires de Mathématiques et Physique, 170-176. In (Euler, OO, Ser. II, 5, 268-273).

Maupertuis, P. L. M. (1744). Accord de différentes loix de la nature qui avoient jusqu'ici paru incompatibles. Histoire de l'Académie Royale des Sciences de Paris, 417-426. In (Euler, OO, Ser. II, 5, 274-281).

Maupertuis, P. L. M. (1746). Les Loix du mouvement et du repos déduites d'un principe métaphysique. Histoire de l'Académie Royale des Sciences et des Belles Lettres de Berlin, 267-294. In (Euler, OO, Ser. II, 5, 282-302).

Maupertuis, P. L. M. (1750). Essay de Cosmologie. s. n., s. l.

Maupertuis, P. L. M. (1756). Examen philosophique de la preuve de l'existence de dieu employée dans l'essai de cosmologie. Histoire de l'Académie Royale des Sciences et des Belles Lettres [de Berlin], 12, 389-424.

Mazzone, S., & Roero, C. S. (1997). Jacob Hermann and the diffusion of the Leibnizian Calculus in Italy. Biblioteca di "Nuncius": Studi e testi, 26. Firenze: Olchski.

Newton, I. (1670-1671). De methodis serierum et fluxionum. In (Newton, MWP, vol. 3, chapter 1, pp. 32-254).

Newton, I. (1687). Philosophiæ naturalis Principia Mathematica. Londini: Iussu Societatis regiae ac Typis Josephi Streater. [2nd ed. (1713), Cantabrigiæ; 3rd ed. (1726), Londinis: apud G. & J. Innys].

Newton, I. (1704a). Opticks or a treatise of the reflexions, refractions, inflexions and colours of light. Also two treatises of the species and magnitude of curvilinear figures. London: Printed for S. Smith and B. Walford.

Newton, I. (1704b). Enumeratio linearum tertii ordinis. In (Newton, 1704a, 138-162 of the second numbering).

Newton, I. (1704c). Tractatus de quadratura curvarum. In (Newton, 1704a, 163-211 of the second numbering).

Newton, I. (1756). Four letters from Sir Isaac Newton to doctor Bentley containing some arguments in proof of a deity. London: Printed for R. and J. Dodsley.

Newton, I. (MWP). The mathematical papers of Isaac Newton (8 vol.). Edited by D.T. Whiteside. Cambridge: Cambridge University Press.

Newton, I. (PMCW). The Principia. Mathematical Principles of Natural Philosophy. A New Translation by I. Bernard Cohen and Anne Whitman, assisted by Julia Budenz, preceded by a Guide to Newton's *Principia* by I. Bernard Cohen. Berkeley, Los Angeles, London: University of California Press.

Panza, M. (1992). La forma della quantità. Analisi algebrica e analisi superiore: Il problema dell'unità della matematica nel secondo dell'illuminismo (2 vols.). Number 38-39 in Cahiers d'histoire et de philosophie des sciences. Nouvelle série. Paris: Société française d'histoire des sciences.

Panza, M. (1995). De la nature épargnante aux forces généreuses. Le principe de moindre action entre mathématique et métaphysique: Maupertuis et Euler (1740-1751). Revue d'Histoire des Sciences, 48, 435-520.

Panza, M. (2002). Mathematisation of the science of motion and the birth of analytical mechanics: A historiographical note. In P. Cerrai, P. Freguglia, & C. Pellegrini (Eds.), The Application of Mathematics to the Sciences of Nature. Critical moments and Aspects (pp. 253-271). New York: Kluwer/Plenum publishers.

Panza, M. (2003). The origins of analytic mechanics in the 18th century. In H. N. Jahnke (Ed.), A History of Analysis (pp. 137-153). New York: American Mathematical Society and London Mathematical Society.

Panza, M. (2005). Newton et les origines de l'analyse: 1664-1666. Paris: Librairie Albert Blanchard.

Panza, M. (2007a). What is new and what is old in Viète's *analysis restituita* and *algebra nova*, and where do they come from? Some reflections on the relations between algebra and analysis before Viète. Revue d'Histoire des mathématiques, 13, 85-153.

Panza, M. (2007b). Euler's *Introductio in analysin infinitorum* and the program of algebraic analysis: quantities, functions and numerical partitions. In R. Backer (Ed.), Euler reconsidered. Tercentenary essays (pp. 119-166). Heber City: Kendrick Press.

Panza, M. (2012). From velocities to fluxions. In A. Janiak, & E. Schliesser (Eds.), Interpreting Newton: Critical essays (pp. 219-254). Cambridge: Cambridge University Press.

Pappus (CMH). Pappi Alexandrini Collectionis [...]. Edited with Latin translation and commentary by F. Hultsch. 3 vols. Berolini: Weidmann.

Pulte, H. (1989). Das prinzip der kleinsten wirkung und die kraftkonzeptionen der rationalen mechanik. Volume 19 of Studia Leibnitiana. Stuttgart: F. Steiner Verlag.

Rashed, R. (2005). Les premières classifications des courbes. Physis, 42, 1-64.

Romero, A. (2007). La Mécanique d'Euler: Prolégomènes à la pensée physique des milieux continus [...]. Ph.D. Thesis, Paris: Université Paris Diderot—Paris 7.

Schroeder, P. (2007). La loi de la gravitation universelle. Newton, Euler et Laplace. Paris: Springer.

Szabó, I. (1987). Geschichte der mechanischen prinzipien. Basel, Boston, Stutgart: Birkhäuser.

Truesdell, C. A. (1960). The rational mechanics of flexible or elastic bodies, 1638-1788. Bâle: Birkhäuser. In (Euler, OO, Ser. II, 11, 2).

Truesdell, C. A. (1968). Essays in the history of mechanics. Berlin: Springer.

Truesdell, C. A. (1970). Reactions of late baroque mechanics to success, conjecture, error, and failure in Newton's *Principia*. In Robert Palter, (Ed.), The annus mirabilis of sir Isaac Newton 1666-1966 (pp. 192-232). Cambridge, Mass.: MIT Press.

Varignon, P. (1725). Nouvelle mécanique ou statique. 2 vols. Paris: C. Jombert.

Viète, F. (1591). In artem analiticem isagoge. Turonis: J. Mettayer.

Wilson, C. (1992). Euler on action at a distance and fundamental equations in continuum mechanics. In P. M. Harman, & A. E. Shapiro, (Eds.), The investigation of difficult things (pp. 399-420). Cambridge: Cambridge University Press.