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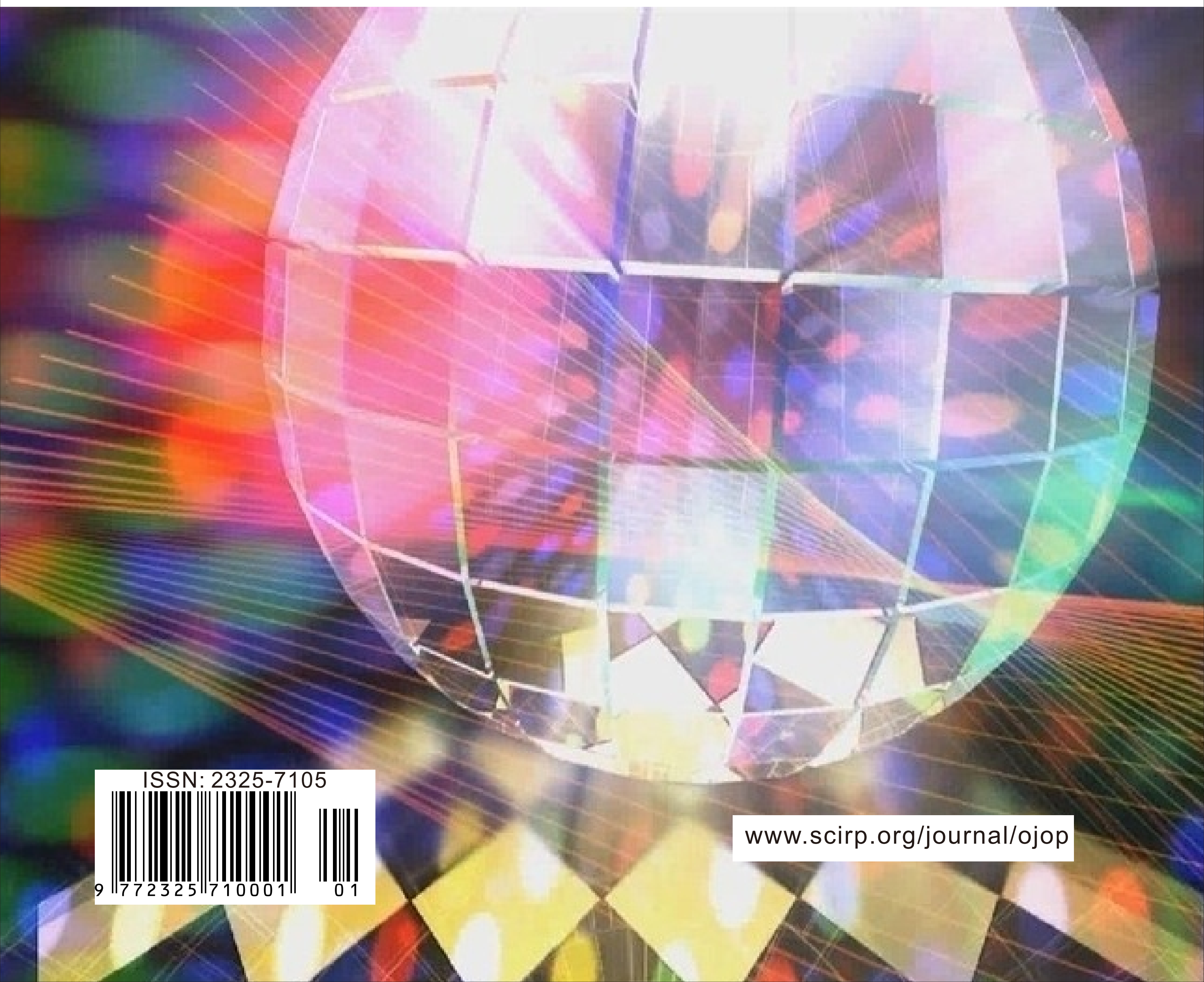

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A New Approach for Solving Linear Fractional Programming Problems with Duality Concept

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Abstract

Most of the current methods for solving linear fractional programming (LFP) problems depend on the simplex type method. In this paper, we present a new approach for solving linear fractional programming problem in which the objective function is a linear fractional function, while constraint functions are in the form of linear inequalities. This approach does not depend on the simplex type method. Here first we transform this LFP problem into linear programming (LP) problem and hence solve this problem algebraically using the concept of duality. Two simple examples to illustrate our algorithm are given. And also we compare this approach with other available methods for solving LFP problems.

Keywords

Linear Fractional Programming, Linear Programming, Duality

1. Introduction

The linear fractional programming (LFP) problem has attracted the interest of many researches due to its application in many important fields such as production planning, financial and corporate planning, health care and hospital planning.

Several methods were suggested for solving LFP problem such as the variable transformation method introduced by Charnes and Cooper [1] and the updated objective function method introduced by Bitran and Novaes [2]. The first method transforms the LFP problem into an equivalent linear programming problem and uses the variable transformation $y = tx, t \geq 0$ in such a way that $dt + \beta = \gamma$ where $\gamma \neq 0$ is a specified number and transform LFP to an LP problem. And the second method solves a sequence of linear programming problems depending on updating the local gradient of the fractional objective function at successive points. But to solve this sequence of problems, sometimes may need much iteration. Also some aspects concerning duality and sensitivity analysis in linear fractional program were discussed by Bitran and Magnant [3] and Singh [4], in his paper made a useful study about the optimality condition in fractional programming. Assuming the positivity of denominator of the objective function of LFP over the feasible region, Swarup [5] extended the well-known simplex method to solve the LFP. This process cannot continue infinitely, since there is only a finite number of basis and in non-degenerate case, no basis can ever be repeated, since F is increased at every step and the same basis cannot yield two different values of F. While at the same time the maximum value of the objective function occurs at of the basic feasible solution. Recently, Tantawy [6] has suggested a feasible direction approach and the main idea behind this method for solving LFP problems is to move through the feasible region via a sequence of points in the direction that improves the objective function. Tantawy [7] also proposed a duality approach to solve a linear fractional programming problem. Tantawy [8] develops another technique for solving LFP which can be used for sensitivity analysis. Effati and Pakdaman [9] propose a method for solving interval-valued linear fractional programming problem. A method for solving multi objective linear plus linear fractional programming problem based on Taylor series approximation is proposed by Pramanik *et al.* [10]. Tantawy and Sallam [11] also propose a new method for solving linear programming problems.

In this paper, our main intent is to develop an approach for solving linear fractional programming problem which does not depend on the simplex type method because method based on vertex information may have difficulties as the problem size increases; this method may prove to be less sensitive to problem size. In this paper, first of all, a linear fractional programming problem is transformed into linear programming problem by choosing an initial feasible point and hence solves this problem algebraically using the concept of duality.

2. Definition and Method of Solving LFP

A linear fractional programming problem occurs when a linear fractional function is to be maximized and the problem can be formulated mathematically as follows:

Maximize $F(x)=\dfrac{c^{\mathrm{T}}x+\gamma}{d^{T}x+\beta}$.

Subject to,

$$x\in X=\{x:Ax\le b\},x\ge 0 \tag{1}$$

where c, d and $x\in\mathbb{R}^{n}$, A is an $(m+n)n$ matrix, $b\in\mathbb{R}^{m+n}$ and γ and β are scalars.

We point out that the nonnegative conditions are included in the set of constraints and that $d^{\mathrm{T}}x+\beta>0$ has to be satisfied over the compact set X.

To transform the LFP problem into LP problem, we choose a feasible point x^{*} of the compact set X. Then

$$F^{*}=F(x^{*})=\frac{c^{\mathrm{T}}x^{*}+\gamma}{d^{\mathrm{T}}x^{*}+\beta} \tag{2}$$

is a given constant vector computed at a given feasible point x^*. Thus the level curve of objective function for (1) can be written as

$$\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)x = \beta F^* - \gamma$$

Hence the linear programming problem is as follows:

Maximize $\varphi(x) = \left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)x$

Subject to,

$$x \in X = \{x : Ax \le b\}, x \ge 0 \tag{3}$$

Proposition

If x^* solves the LFP problem (1) with objective function values F^* then x^* solves the LP problem defined by (3) with objective function value $\varphi^* = \beta F^* - \gamma$.

Now rewrite the LP problem (3) in the form

Maximize $H(x) = C^{\mathrm{T}} x$

Subject to,

$$x \in X = \{x : Ax \le b\} \tag{4}$$

where, C^{T} is a matrix whose row is represented by $\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)$ and $C, x \in \mathbb{R}^n$, A is a $(m+n)\times n$ matrix, $b \in \mathbb{R}^{m+n}$. we point out that the nonnegative conditions are included in the set of constraints.

Now consider the dual problem for the linear program (4) in the form

Minimize $w = u^{\mathrm{T}} b$

Subject to,

$$u^{\mathrm{T}} A = C^{\mathrm{T}}, u \ge 0 \tag{5}$$

Since the set of constraints of this dual problem is written in the matrix form hence we can multiply both side by a matrix $T = \left(T_1 | T_2\right)$, where $T_1 = C\left(C^{\mathrm{T}} C\right)^{-1}$ and the columns of the matrix T_2 constitute the bases of $\{x : C^{\mathrm{T}} x = 0\}$.

Thus this implies

$$u^{\mathrm{T}} A T_1 = 1,\ u^{\mathrm{T}} A T_2 = 0 \ \text{ and } \ u \ge 0. \tag{6}$$

If we define $l \times (m+n)$ matrix P of nonnegative entries such that $PAT_2 = 0$, then (6) can be written as

$$v^{\mathrm{T}} G = 1, v \ge 0 \tag{7}$$

where $G = PAT_1$ and $v^{\mathrm{T}} P = u^{\mathrm{T}}$, Equation (7) will play an important role for finding the optimal solution of the LP problem (4). Using the Equation (7) the equivalent LP problem of (5) can be written as

Minimize $w = v^{\mathrm{T}} g$

Subject to,

$$v^{\mathrm{T}} G = 1,\ v \ge 0 \tag{8}$$

with $G = PAT_1, g = Pb, v^{\mathrm{T}} P = u^{\mathrm{T}}$, the linear programming (8) has the dual programming problem in just one unknown Z in the form.

Maximize Z

Subject to,

$$GZ \le g,\ Z \ge 0 \tag{9}$$

Note: The set of constraints of the above linear programming problem will give the maximum value Z^* and also will define only one active constraint for this optimal value. We have to note that from the complementary slackness theorem the corresponding dual variable will be positive and the remaining dual variables will be zeros for the corresponding non active constraints.

3. Algorithm for Solving LFP Problems

The method for solving LFP problems summarize as follows:

- Step 1: Select a feasible point x^* and using Equation (2) to compute F^*.
- Step 2: Find the level curve of objective function
$$\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)x = \beta F^* - \gamma$$
Hence find the LP problem (2) which can be rewritten as (3).
- Step 3: Compute $T_1 = C\left(C^{\mathrm{T}}C\right)^{-1}$, and the matrix T_2 as the bases of $\left\{x : C^{\mathrm{T}}x = 0\right\}$.
- Step 4: Find the matrix P of nonnegative entries such that $PAT_2 = 0$ and hence compute $G = PAT_1,\ g = Pb$.
- Step 5: Find the LP problem (8) and dual of this LP (9). Use the LP (9) to find the optimal value Z^* and also determine the corresponding active constraints and use the constraint of (8) to compute v^{T}.
- Step 6: Find the dual variables $u^{\mathrm{T}} = v^{\mathrm{T}}P$, for each positive variable $u_i, i = 1, 2, \cdots, m$ find the corresponding active set of constraint of the matrix A.
- Step 7: Solve a $n \times n$ system of linear equations for these set of active constraints (a subset from a $m + n$ constraints) to get the optimal solution of LP problem (4) and hence for the LFP problem (1).

4. Computational Process

Choose x^* in such a way that

$$x^* \in X = \left\{x^* : Ax^* \le b\right\}$$

$$F^* \leftarrow F\left(x^*\right) \leftarrow \frac{c^{\mathrm{T}}x^* + \gamma}{d^{\mathrm{T}}x^* + \beta}$$

$$d^{\mathrm{T}}x^* + \beta > 0.$$

The level curve is $\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)x = \beta F^* - \gamma$.

Then $\varphi(x) \leftarrow \left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)x$ or $H(x) \leftarrow C^{\mathrm{T}}x$; $C^{\mathrm{T}} \leftarrow \left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right)$;

$$T_1 \leftarrow C\left(C^{\mathrm{T}}C\right)^{-1};\ T_2 \leftarrow \left\{x : C^{\mathrm{T}}x = 0\right\};$$

Find P such that $PAT_2 = 0$.

Compute $G \leftarrow PAT_1, g \leftarrow Pb$;

Formulate, Maximize Z

Subject to, $GZ \le g,\ \ Z \ge 0$.

Find Z^* and corresponding active constraint and compute v^{T} for $v^{\mathrm{T}}G = 1$;

Then $u^{\mathrm{T}} \leftarrow v^{\mathrm{T}}P$; hence find x^{T} from corresponding $n \times n$ active constraints satisfied by positive u^{T};

Compute H^* and F^*.

5. Numerical Examples

Here we illustrate two examples to demonstrate our method.

Example 1: Consider the linear fractional programming (LFP) problem

Maximize $F(x) = \dfrac{x_2 + 1}{x_1 + 3}$

Subject to,

$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1 + 2x_2 \le 1$$
$$x_1 \le 5$$
$$x_1, x_2 \ge 0.$$

Solution:

Step 1: Let $x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $F^* = \dfrac{1+1}{1+3} = \dfrac{1}{2}$ and hence we have

$$\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right) x = \left[\begin{pmatrix} 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{2}x_1 + x_2$$

Step 2: Therefore we have the following LP problem

Maximize $H(x) = -\dfrac{1}{2}x_1 + x_2$

Subject to,

$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1 + 2x_2 \le 1$$
$$x_1 \le 5$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

Dual problem for this LP problem is

Minimize $w(x) = u_1 + 2u_2 + 7u_3 + 5u_4$

Subject to,

$$-u_1 + u_3 + u_4 - u_5 = \frac{1}{2}$$
$$u_1 + u_2 + 2u_3 - u_6 = 1$$
$$u_1, u_2, u_3, u_4, u_5, u_6 \ge 0$$

Step 3: Compute $T_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \left[\begin{pmatrix} -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} \right]^{-1} = \frac{4}{5} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{4}{5} \end{pmatrix}$.

And the matrix $T_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Step 4: Compute nonnegative matrix P such that $PAT_2 = 0$,

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}.$$

Also compute $G = PAT_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$,

$$g = Pb = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 2 \\ 7 \\ 5 \\ 7 \end{pmatrix}$$

Step 5: We get the LP problem of the form

Maximize Z

Subject to,

$$2Z \le 3$$
$$0Z \le 6$$
$$0Z \le 2$$
$$2Z \le 7$$
$$0Z \le 5$$
$$0Z \le 7$$

For this LP problem we get that the first constraint is the only active constraint and this active constraint shows that the maximum optimal value is $Z^* = \frac{3}{2}$. Corresponding this active constraint of (8), we get the dual variables $v^{\mathrm{T}} = \left(\frac{1}{2}, 0, 0, 0, 0, 0\right)$.

Step 6: Compute $u^{\mathrm{T}} = v^{\mathrm{T}} P = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$ with objective value $w^* = \frac{3}{2}$.

This indicates that in the original set of constraints the first and the second constraints are the only active constraints.

Step 7: Solve the system of linear equations

$$-x_1 + x_2 = 1$$
$$x_2 = 2$$

We get the optimal solution $x^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ of the LP problem with objective value $H^* = \frac{3}{2}$.

Finally we get our desired optimal solution of the given LFP problem is $x^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with the optimal value $F^* = \frac{3}{4}$.

Example 2: Consider the linear fractional programming (LFP) problem

Maximize $F(x) = \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}$

Subject to,

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

Solution:

Step 1: Let $x^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $F^* = \frac{5+3}{5+2+1} = 1$ and hence we have

$$\left(c^{\mathrm{T}} - F^* d^{\mathrm{T}}\right) x = \left[\begin{pmatrix} 5 & 3 \end{pmatrix} - 1 * \begin{pmatrix} 5 & 2 \end{pmatrix}\right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2$$

Step 2: Therefore we have the following LP problem

Maximize $H(x) = x_2$

Subject to,

$$3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$-x_1 \le 0$$
$$-x_2 \le 0$$

Dual problem for this LP problem is

Minimize $w(x) = 15u_1 + 10u_2$

Subject to,

$$3u_1 + 5u_2 - u_3 = 0$$
$$5u_1 + 2u_2 - u_4 = 1$$
$$u_1, u_2, u_3, u_4 \ge 0$$

Step 3: Compute $T_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

And the matrix $T_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Step 4: Compute nonnegative matrix P such that $PAT_2 = 0$,

$$P = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Also compute $G = PAT_1 = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 5 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \\ -1 \end{pmatrix}$,

$$g = Pb = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 15 \\ 10 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \\ 10 \\ 0 \end{pmatrix}$$

Step 5: We get the LP problem of the form

Maximize Z

Subject to,

$$5Z \le 15$$

$$2Z \le 10$$

$$Z \le 10$$

$$-Z \le 0$$

For this LP problem we get that the first constraint is the only active constraint and this active constraint shows that the maximum optimal value is $Z^* = 3$. Corresponding to this active constraint of (8), we get the dual variables $v^{\mathrm{T}} = \left(\frac{1}{5}, 0, 0, 0, 0, 0\right)$.

Step 6: Compute $u^{\mathrm{T}} = v^{\mathrm{T}} P = \left(\frac{1}{5}, 0, \frac{3}{5}, 0, 0, 0\right)$ with objective value $w^* = 3$. This indicates that in the original set of constraints the first and the third constraints are the only active constraints.

Step 7: Solve the system of linear equations

$$-3x_1 + 5x_2 = 15$$

$$x_1 = 0$$

We get the optimal solution $x^* = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ of the LP problem with objective value $H^* = 3$.

Finally we get our desired optimal solution of the given LFP problem is $x^* = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ with the optimal value $F^* = \frac{9}{7}$.

Table 1. Results of existing and our methods for **Example 1** and **Example 2.**

	Bitran and Novea	Swarup	Tantawy	Our Method
Example 1	3 iterations with lots of calculations	3 iterations with clumsy calculations	2 iterations	1 iterations with simple calculations
Example 2	3 iterations	3 iterations	2 iterations	1 iterations

Now different methods can be compared with our method and all the methods give the same results for Example 1 and Example 2. Table 1 shows the results of number of iterations that are required for our method and the existing methods for these Examples.

6. Comparison

In this Section, we find that our method is better than any other available method. The reason can be given as follows:

- Any type of LFP problem can be solved by this method.
- The LFP problem can be transformed into LP problem easily with initial guess.
- In this method, problems are solved by algebraically with duality concept. So that it's computational steps are so easy from other methods.
- The final result converges quickly in this method.
- In some cases of numerator and denominator, other existing methods are failed but our method is able to solve any kind of problem easily.

7. Conclusion

In this paper, we give an approach for solving linear fractional programming problems. The proposed method differs from the earlier methods as it is based upon solving the problem algebraically using the concept of duality. This method does not depend on the simplex type method which searches along the boundary from one feasible vertex to an adjacent vertex until the optimal solution is found. In some certain problems, the number of vertices is quite large, hence the simplex method would be prohibitively expensive in computer time if any substantial fraction of the vertices had to be evaluated. But our proposed method appears simple to solve any linear fractional programming problem of any size.

References

[1] Charnes, A. and Cooper, W.W. (1962) Programming with Fractional Functions. *Naval Research Logistic Quarterly*, **9**, 181-186. https://doi.org/10.1002/nav.3800090303

[2] Bitran, G.R. and Novaes, A.G. (1973) Linear Programming with a Fractional Objective Functions. *Operations Research*, **21**, 22-29. https://doi.org/10.1287/opre.21.1.22

[3] Bitran, G.R. and Magnanti, T.L. (1976) Duality and Sensitivity Analysis with Fractional Objective Function. *Journal of Operation Research*, **24**, 675-699. https://doi.org/10.1287/opre.24.4.675

[4] Sing, H.C. (1981) Optimality Condition in Fractional Programming. *Journal of Optimization Theory and Applications*, **33**, 287-294. https://doi.org/10.1007/BF00935552

[5] Swarup, K. (1964) Linear Fractional Programming. *Operation Research*, **13**, 1029-1036. https://doi.org/10.1287/opre.13.6.1029

[6] Tantawy, S.F. (2008) A New Procedure for Solving Linear Fractional Programming Problems. *Mathematical and Computer Modelling*, **48**, 969-973. https://doi.org/10.1016/j.mcm.2007.12.007

[7] Tantawy, S.F. (2014) A New Concept of Duality for Linear Fractional Programming Problems. *International Journal of Engineering and Innovative Technology*, **3**, 147-149.

[8] Effati, S. and Pakdaman, M. (2012) Solving the Interval-Valued Linear Fractional Programming Problem. *American Journal of Computational Mathematics*, **5**, 51-55. https://doi.org/10.4236/ajcm.2012.21006

[9] Pramanik, S., Dey, P.P. and Giri, B.C. (2011) Multi-Objective Linear Plus Linear Fractional Programming Problem Based on Taylor Series Approximation. *International Journal of Computer Applications*, **32**, 61-68.

[10] Tantawy, S.F. (2008) An Iterative Method for Solving Linear Fractional Programming (LFP) Problem with Sensitivity Analysis. *Mathematical and Computational Mathematics*, **13**, 147-151.

[11] Tantawy, S.F. and Sallam, R.H. (2013) A New Method for Solving Linear Fractional Programming Problems. *International Journal of Recent Scientific Research*, **4**, 623-625.

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An Alternative Approach to the Solution of Multi-Objective Geometric Programming Problems

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Abstract

The aim of this study is to present an alternative approach for solving the multi-objective posynomial geometric programming problems. The proposed approach minimizes the weighted objective function comes from multi-objective geometric programming problem subject to constraints which constructed by using Kuhn-Tucker Conditions. A new nonlinear problem formed by this approach is solved iteratively. The solution of this approach gives the Pareto optimal solution for the multi-objective posynomial geometric programming problem. To demonstrate the performance of this approach, a problem which was solved with a weighted mean method by Ojha and Biswal (2010) is used. The comparison of solutions between two methods shows that similar results are obtained. In this manner, the proposed approach can be used as an alternative of weighted mean method.

Keywords

Multi Objective Geometric Programming, Kuhn-Tucker Conditions, Taylor Series Expansion, Numerical Method, Weighted Mean Method

1. Introduction

Geometric Programming Problem (GPP) is a special type of nonlinear programming that often used in the applications for production planning, personal allocation, distribution, risk managements, chemical process designs and other engineer design situations. GPP is a special technique that is developed in order to find the optimum values of posynomial and signomial functions. In the classical optimization technique, a system of nonlinear equations is generally faced after taking partial derivatives for each variable and equalizing them to zero.

Since the objective function and the constraints in the GPPs will be in posynomial or signomial structures, the solution of the system of nonlinear equations obtained by the classic optimization technique will be very difficult. The solution to the GPP follows the opposite method with respect to the classical optimization technique and it depends on the technique of first finding the weight values and calculating the optimum value for the objective function, then finding the values of the decision variables.

GPP has been known and used in various fields since 1960. GPP started to be modeling as part of nonlinear optimization by Zener [1] in 1961 and Duffin, Peterson and Zener [2] in 1967 and particular algorithms were used when trying to solve GPP. After that many important studies were done in various fields: communication systems [3], engineering design [4] [5] [6], resource allocation [7], circuit design [8], project management [9] and inventory management [10].

When there are multiple objectives in the GPP, the problem is defined as the Multi-Objective Geometric Programming Problem (MOGPP). In general, there are two types (namely fuzzy GPP and weighted mean method) of solving approaches are exist in the literature. The studies deal with fuzzy GPP method can be given as Nasseri and Alizadeh [11], Islam [12], Liu [5], Biswal [13], Verma [14] and Yousef [23]. Besides, to solve the multi-objective optimization problem, another and the simplest way is using the weighted mean method. The weighted mean method is also used and applied for the solution of the MOGPP by Ojha and Biswall [15].

Numerical approximations are widely used to solve the Multi-objective programming problems. One of the numerical approximations is the Taylor series expansion which is also given as a solution method in this study. Toksarı [16] and Güzel and Sivri [17] have used Taylor series to solve the multi-objective linear fractional programming problem and have given examples.

In this study, a numerical approach to solve the multi-objective posynomial geometric programming problems is proposed. This numerical approach minimizes the weighted objective function subject to Kuhn-Tucker Conditions expanded the first order Taylor series expansion about any arbitrary initial feasible solution. The same process is continued iteratively until the desired accuracy is achieved. The solution obtained at the end of the iterative processing gives the pareto optimal solution to solve the multi-objective posynomial geometric programming problem. When the results obtained are compared to the results of the weighted mean method [15] used to solve the multi-objective posynomial geometric programming problems, the same results are found.

In the next section of this study, MOGPP, weighted method for MOGPP and dual form of MOGPP are respectively mathematically explained. In the third section, the model that we suggest depending on the Kuhn-Tucker Conditions and first order Taylor Series expansion will be clarified. Then, the results obtained by weighted mean method and the results obtained by the approach that we suggest will be compared for a numeric example. In the last section, conclusion and comments will be included.

2. Multi-Objective Geometric Programming Problem

2.1. Standard Geometric Programming Problem

Let $x_1, x_2, \cdots, x_n$ show n real positive variables and $X = (x_1, x_2, \cdots, x_n)$ a vector with components x_i. A real valued function f of x, with the form,

$$f(x) = Cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \tag{1}$$

where $C > 0$ and $a_i \in R$. The function is named a monomial function. A sum of one or more monomial functions is named a posynomial function. The term "posynomial" is meant to suggest a combination of "positive" and "polynomial". A posynomial function of the term,

$$f(x) = \sum_{k=1}^{K} C_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}} \tag{2}$$

where $C_k > 0$ and $a_{ik} \in R$.

GPP is a problem with generalized posynomial objective and inequality constraints, and monomial equality constraints. Standard form of a GPP can be written as

$$\left.\begin{aligned} &\min_x f_o(x) \\ &\text{subject to} \\ &f_i(x) \le 1, \quad i = 1, \cdots, m \\ &h_j(x) = 1, \quad j = 1, \cdots, p \end{aligned}\right\} \tag{3}$$

where $f_0, f_1, \cdots, f_m$ are posynomials and $h_0, h_1, \cdots, h_p$ are monomials.

GPP in standard form is not a convex optimization problem. GP is a nonlinear, nonconvex optimization problem that can be logarithmic transformed into a nonlinear, convex problem.

Assuming for simplicity that the generalized posynomials involved are ordinary posynomials, it can express a GPP clearly, in the so-called standard form:

$$\left.\begin{aligned} &\min_x \sum_{k=1}^{K_0} c_{k0} x^{a_0} \\ &\text{subject to} \\ &\sum_{k=1}^{K_i} c_{ki} x^{a_{ki}} \le 1, \quad i = 1, \cdots, m \\ &g_j x^{f_j} = 1, \quad j = 1, \cdots, p \end{aligned}\right\} \tag{4}$$

where $a_0, a_1, \cdots, a_m$ and $c_0, c_1, \cdots, c_m$ are vectors in R^n and $c_i > 0,\ i = 1, 2, \cdots, m,\ g > 0$ are vectors with positive components.

Most of these posynomial type GPP's have zero or positive degrees of difficulty. Parameters of GPP, except for exponents, are all positive and called posynomial problems. GPP's with some negative parameters are also called signomial problems.

The degree of difficulty is defined as the number of terms minus the number of variables minus one, and is equal to the dimension of the dual problem. If the degree of difficulty is zero, the problem can be solved analytically. If the degree of difficulty is positive, then the dual feasible region must be searched to maximize the dual objective, and if the degree of difficulty is negative, the dual constraints may be inconsistent [15].

GPP in standard form is not a convex optimization problem. GPP is a nonlinear, nonconvex optimization problem that can be logarithmic transformed into a nonlinear, convex problem.

2.2. Multi-Objective Geometric Programming Problem

General form of multi objective GPP, where p is the number of objective functions which are minimized and n is the number of positive decision variables, is defined as:

$$\left.\begin{aligned} &\min g_{k0}(x) = \sum_{t=1}^{T_{k0}} C_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}}, \quad k = 1, 2, \cdots, p \\ &\text{subject to} \\ &g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} \le 1, \qquad i = 1, 2, \cdots, m \\ &x_j > 0, \qquad\qquad\qquad\qquad j = 1, 2, \cdots, n \end{aligned}\right\} \tag{5}$$

where d_{itj} and a_{k0tj} are real numbers for all *i, k, t, j* and C_{k0t} for all k and t are positive real numbers, $x \in X, x \in R^n$ and $g_{0k} : R^n \to R$, $k = 1, 2, \cdots, p$. The number of terms in the k^{th} objective function is T_{k0}, and the number of terms in the i^{tk} constraint is T_i. X is the set of constraints, considered as non-empty compact feasible region. When all of the C constants are positive, the function is called a posynomial. When at least one of them is negative, it is called a signomial [18] [25]. The model in this study consists only of posynomials. The degree of difficulty is found by subtracting the number of variables in the primal problem plus one from the number of terms in the primal problem. If the degree of difficulty is zero, only one solution will be achieved since the number of equations given under the normality and orthagonality conditions will be equal to the number of unknown terms. When the degree of difficulty is below zero, the dual constraints may be inconsistent. And when the degree of difficulty is above zero, in order to maximize the dual objective, the dual feasible region must be searched [18] [25].

Definition 1 $x^* \in X$ is a pare to optimal solution of MOGPP (5) if there does not exist another feasible solution $\overline{x} \in X$ such that $g_{0k}(\overline{x}) \le g_{0k}(x^*), k = 1, 2, \cdots, p$ and $g_{0j}(\overline{x}) < g_{0j}(x^*)$ at least one j.

Definition 2 $x^* \in X$ is a weakly pare to optimal solution of MOGPP (5) if there does not exist another feasible solution $\overline{x} \in X$ such that $g_{0k}(\overline{x}) < g_{0k}(x^*), k = 1, 2, \cdots, p$.

3. The Weighting Method to the Multi-Objective Geometric Programming Problem

General form of multi objective optimization problem can be mathematically stated as:

$$\left.\begin{array}{l}\text{Minimize}\left\{f_1(x), f_2(x), \cdots f_p(x)\right\}, p \geq 2 \\ \text{subject to} \\ x \in X,\end{array}\right\} \quad (6)$$

where $x \in R^n$ and $f_i : R^n \to R,\ i = 1, 2, \cdots, p$. X is the set of constraints, considered as non-empty compact feasible region.

A multi-objective problem is often solved by combining its multiple objectives into one single-objective scalar function. This approach is in general known as the weighted-sum or scalarization method. In more detail, the weighted-sum method minimizes a positively weighted convex sum of the objectives, that is,

$$\left.\begin{array}{l}\text{Min}\sum_{i=1}^{p} w_i f_i(x) \\ \sum_{i=1}^{p} w_i = 1, \\ w_i > 0,\ i = 1, 2, \cdots, p \\ x \in X\end{array}\right\} \quad (7)$$

that represents a new optimization problem with a single objective function. We denote the above minimization problem with $P_X(w)$.

The following result by Geoffrion [19] states a necessary and sufficient condition in the case of convexity as follows: If the solution set $x \in X$ is convex and the p objectives $f_i(x)$ are convex on X, x^* is a strict Pareto optimum if and only if it exists $w \in R^n$, such that x^* is an optimal solution of problem $P_X(w)$. If the convexity hypothesis does not hold, then only the necessary condition remains valid, *i.e.*, the optimal solutions of $P_X(w)$ are strict Pareto optimal [20].

In order to the above MOGPP defined in problem (5) consider the following procedure of the weighting method, a new minimization type objective function $Z(\mu)$ may be defined as:

$$\left.\begin{array}{l}\min Z_\mu(x) = \sum_{k=1}^{p} \mu_k g_{ko}(x) = \sum_{k=1}^{p} \mu_k \left(\sum_{t=1}^{T_{k0}} C_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}} \right) = \sum_{k=1}^{p} \sum_{t=1}^{T_{k0}} \mu_k C_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}} \\ \text{subject to} \\ \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_i^{d_{itj}} \leq 1, \qquad i = 1, 2, \cdots, m \\ x_j > 0, \qquad\qquad j = 1, 2, \cdots, n \\ \text{where} \\ \sum_{k=1}^{p} \mu_k = 1, \mu_k > 0, \quad k = 1, 2, \cdots, p\end{array}\right\} \quad (8)$$

4. The Kuhn-Tucker Theorem

The basic mathematical programming problem is that of choosing values of variables so as to minimize a function of those variables subject to m inequality constraints:

$$\left.\begin{array}{ll} \min g_0(x) & \\ \text{subject to} & \\ g_i(x) \le 0, & i = 1, 2, \cdots, m \\ x_j > 0, & j = 1, 2, \cdots, m \end{array}\right\} \tag{9}$$

This problem is a generalization of the classical optimization problem, since equality constraints are a special case of inequality constraints. By m additional variables, called slack variables, $y_i \left(i = 1, 2, \cdots, m\right)$, the mathematical programming problem (9) can be rewritten as a classical optimization problem:

$$\left.\begin{array}{ll} \min g_0(x) & \\ \text{subject to} & \\ g_i(x) + y_i^2 = 0, & i = 1, 2, \cdots, m \\ x_j > 0, & j = 1, 2, \cdots, m \end{array}\right\} \tag{10}$$

The solution to problem (10) is then analogous to the Lagrange theorem for classical optimization problems. The Lagrange theory for a classical optimization problem can be extended to problem (10) by the following theorem.

Theorem 4.1 Assume that $g_k(x), (k = 1, 2, \cdots, m)$ are all differentiable. If the function $g_0(x)$ attains at point x^0 a local minimum subject to the set $K = \left\{x \middle| g_i(x) \le 0, i = 1, 2, \cdots, m\right\}$, then there exists a vector of Lagrange multipliers u^0 such that the following conditions are satisfied:

$$\left.\begin{array}{ll} \dfrac{\partial g_0\left(x^0\right)}{\partial x_j} + \sum_{i=1}^{m} u_i^0 \dfrac{\partial g_i\left(x^0\right)}{\partial x_j} = 0, & j = 1, 2, \cdots, n \\ g_i\left(x^0\right) \le 0, & i = 1, 2, \cdots, m \\ u_i^0 g_i\left(x^0\right) = 0, & i = 1, 2, \cdots, m \\ u_i^0 \ge 0, & i = 1, 2, \cdots, m \end{array}\right\} \tag{11}$$

The conditions (11) are necessary conditions for a local minimum of problem. The conditions (11) are called the Kuhn-Tucker conditions.

For proof of theorem, the Lagrange function can be defined as:

$$L(x, y, u) = g_0(x) + \sum_{i=1}^{m} u_i \left(g_0(x) + y_i^2\right) = 0 \tag{12}$$

The necessary conditions for its local minimum are

$$\frac{\partial L(x, y, u)}{\partial x_j} = \frac{\partial g_0\left(x^0\right)}{\partial x_j} + \sum_{i=1}^{m} u_i^0 \frac{\partial\left(\left(g_i\left(x^0\right) + y_i^2\right)\right)}{\partial x_j} = 0,\ j = 1, 2, \cdots, n \tag{13}$$

$$\frac{\partial L(x, y, u)}{\partial y_i} = 2u_i^0 y_i^0 = 0, \quad i = 1, 2, \cdots, m \tag{14}$$

$$\frac{\partial L(x, y, u)}{\partial u_j} = g_i\left(x^0\right) + \left(y_i^0\right)^2 = 0, \quad i = 1, 2, \cdots, m \tag{15}$$

The conditions (11) are obtained from the conditions (12)-(15) [24].

When there are inequalities constraints in nonlinear optimization problems, Kuhn-Tucker Conditions can be used which are based on Lagrange multipliers. The Kuhn-Tucker Conditions satisfy the necessary and sufficient conditions for a local optimum point to be a global optimum point [21] [22].

5. Proposed Method to Solve MOGPP

The multi-objective geometric problem (5) as a single objective function using the weighting method can be rewritten as follows:

$$\left.\begin{array}{ll} \min Z_\mu(x) = \sum_{k=1}^{p}\sum_{t=1}^{T_{k0}} \mu_k C_{k0t} \prod_{j=1}^{n} x_j^{a_{k0tj}} & \\ \text{subject to} & \\ g_i(x_i) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} - 1 \le 0, & i = 1,2,\cdots,m \\ x_j > 0 & j = 1,2,\cdots,n \\ \text{where} & \\ \sum_{k=1}^{p} \mu_k = 1,\ \mu_k > 0, & k = 1,2,\cdots,p \end{array}\right\} \quad (16)$$

The above problem (16) may be slightly modified by introducing new variables y_i, whose values is transformed into single objective GPP as:

$$\left.\begin{array}{l} \min Z_\mu(x) = \sum_{k=1}^{p}\sum_{t=1}^{T_{k0}} \mu_k C_{kot} \prod_{j=1}^{n} x_j^{a_{k0tj}} \\ \text{subject to} \\ \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} + y_i^2 - 1 = 0, \quad i = 1,2,\cdots,m \end{array}\right\} \quad (17)$$

Assume that $Z_\mu(x)$ and $\sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} + y_i^2 - 1 = 0\,(i = 1,2,\cdots,m)$ are all differentiable. The new function is formed by introducing m multipliers u_i for $(i = 1,2,\cdots,m)$ to problem (17) according to theorem 4.1 can be defined as

$$\left.\begin{array}{ll} L(x,y,u) = Z_\mu(x) + \sum_{i=1}^{m} u_i \left(\sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} + y_i^2 - 1 \right), & i = 1,2,\cdots,m \\ \text{The necessary conditions for its local minimum are} & \\ \frac{\partial Z_\mu(x^0)}{\partial x_j} + \sum_{i=1}^{m} u_i^0 \frac{\partial \left(\sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} (x^0)_j^{d_{itj}} - 1 \right)}{\partial x_j} = 0, & j = 1,2,\cdots,n \\ u_i^0 \left(\sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} (x^0)_j^{d_{itj}} - 1 \right) = 0, & i = 1,2,\cdots,m \\ \left(\sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{d_{itj}} - 1 \right) \le 0, & i = 1,2,\cdots,m \\ u_i \ge 0, & i = 1,2,\cdots,m \end{array}\right\} \quad (18)$$

where at the point x^0, the objective function $Z_\mu(x^0)$ attaints a local minimum according to theorem (4.1). The optimization problem to minimize the objective function $Z_\mu(x^0)$ subject to conditions (18) can be rewritten as follows:

$$
\left.
\begin{array}{ll}
\min Z_\mu(x) & \\
\text{subject to} & \\
\dfrac{\partial Z_\mu(x)}{\partial x_j}+\sum_{i=1}^{m}u_i^0\dfrac{\partial\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}(x)_j^{d_{itj}}-1\right)}{\partial x_j}=0, & j=1,2,\cdots,n\\
u_i^0\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}(x)_j^{d_{itj}}-1\right)=0, & i=1,2,\cdots,m\\
\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}x_j^{d_{itj}}-1\right)\le 0, & i=1,2,\cdots,m\\
u_i\ge 0, & i=1,2,\cdots,m
\end{array}
\right\}
\tag{19}
$$

Since the necessary conditions (17) are also the sufficient conditions for a minimum problem if the objective function of the geometric programming problem (19) is convex. Therefore, optimal solution of the problem (19) gives the solution of the problem (16).

The above problem (19) is nonlinear problem since both the objective function and the constraints are nonlinear. We will use the Taylor theorem for the linearization to the problem (19). Let be both the objective function and the constraints have differentiable. Then they are expanded using the Taylor theorem about any arbitrary initial feasible solution $x^0 \in R^n$ and any arbitrary initial feasible values $u^0 \in R^m$ to problem (19). Thus, the problem (19) as the linear approximation problem can be rewritten as follows:

$$
\left.
\begin{array}{ll}
\min Z_\mu\left(x^0\right)+\nabla\left(Z_\mu\left(x^0\right)\right)\left(x-x^0\right) & \\
\text{subject to} & \\
\left(\dfrac{\partial Z_\mu\left(x^0\right)}{\partial x_j}+\sum_{i=1}^{m}u_i^0\dfrac{\partial\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)}{\partial x_j}\right) & \\
+\nabla\left(\dfrac{\partial Z_\mu\left(x^0\right)}{\partial x_j}+\sum_{i=1}^{m}u_i^0\dfrac{\partial\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)}{\partial x_j}\right)\left(x-x^0\right)=0, & j=1,2,\cdots,n\\
u_i^0\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)+\nabla\left(u_i^0\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)\right)\left(x-x^0\right)=0, & i=1,2,\cdots,m\\
\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)+\nabla\left(\left(\sum_{t=1}^{T_i}C_{it}\prod_{j=1}^{n}\left(x^0\right)_j^{d_{itj}}-1\right)\right)\left(x-x^0\right)\le 0, & i=1,2,\cdots,m\\
u_i\ge 0, & i=1,2,\cdots,m
\end{array}
\right\}
\tag{20}
$$

The linear approximation problem is solved, giving an optimal solution x^1 and u^1, a new linear programming problem is derived from the solution x^1

and u^1. Linear approximation problem is solved, giving an optimal solution x^2 and u^2. The following steps are involved from the initial step till reaching the desired optimal solution or until $\left|x^{i+1}-x^i\right|$ is as close to zero as possible iteratively. The optimal solution x^{i+1} is taken as the pare to optimal solution for MOGPP since solution x^{i+1} is better than x^i.

The steps for the proposed solution algorithm are given below:

Step 1: Formulate the given MOGPP is as a single objective GP using the weighting method.

Step 2: Construct the constraints for the new problem from Kuhn-Tucker conditions.

Step 3: Set the nonlinear model taking the single objective function in step 1 and the constraints in step 2 to MOGPP.

Step 4: t value denotes the iteration or step number of the proposed iterative approach and x^t and u^t denote the vector parameter assigned to the vector of objective function and constraints in step 1. Take the initial solution $t=0$, x^0 and u^0, arbitrarily.

Step 5: Expanded both the objective function and constraints of the problem obtained in step 3 using first order Taylor polynomial series about x^t and u^t in the feasible region of problem. Reduced the problem obtained in step 3 to a linear programming problem.

Step 6: Solve the problem in step 5. Calculate to the approximate solution x^{t+1} and u^{t+1}

Step 7: For eps > 0 and eps as close to 0 as possible, if $\left|x^{t+1}-x^t\right|<eps$ is taken as the pareto optimal solution to MOGPP and the values for the objective functions are calculated. Else, take $t=t+1,\ x^t=x^{t+1};\ u^t=u^{t+1}$, go back to step 5.

Numerical example

To illustrate the proposed model we consider the following problem which is also used in [15].

Find x_1, x_2, x_3, x_4

$$\min g_{10}(x)=4x_1+10x_2+4x_3+2x_4$$

$$\max g_{20}(x)=x_1x_2x_3$$

subject to

$$\frac{x_1^2}{x_4^2}+\frac{x_2^2}{x_4^2}\le 1$$

$$\frac{100}{x_1x_2x_3}\le 1$$

$$x_1, x_2, x_3, x_4>0$$

The primal problem above can be written as below:

$$\min g_{10}(x)=4x_1+10x_2+4x_3+2x_4$$

$$\min g'_{20}(x)=x_1^{-1}x_2^{-1}x_3^{-1}$$

subject to

$$x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \le 1$$

$$100 x_1^{-1} x_2^{-1} x_3^{-1} \le 1$$

$$x_1, x_2, x_3, x_4 > 0$$

Using the weights w_1 and w_2, the primal problem is written as below:

$$Z(x) = w_1 (4x_1 + 10x_2 + 4x_3 + 2x_4) + w_2 \left(x_1^{-1} x_2^{-1} x_3^{-1} \right)$$

subject to

$$x_1^2 x_4^{-2} + x_2^2 x_4^{-2} \le 1$$

$$100 x_1^{-1} x_2^{-1} x_3^{-1} \le 1$$

$$x_1, x_2, x_3, x_4 > 0$$

where $w_1 + w_2 = 1, w_1, w_2 > 0$

In this problem, the primal term number is 8, primal variable number is 4 and thus the degree of difficulty is 3.

The dual problem corresponding to the last primal problem is given below:

$$\max_w V(w) = \left(\frac{4w_1}{w_{01}}\right)^{w_{01}} \left(\frac{10w_1}{w_{02}}\right)^{w_{02}} \left(\frac{4w_1}{w_{03}}\right)^{w_{03}} \left(\frac{2w_1}{w_{04}}\right)^{w_{04}} \left(\frac{w_2}{w_{05}}\right)^{w_{05}} \left(\frac{1}{w_{11}}\right)^{w_{11}} \left(\frac{1}{w_{12}}\right)^{w_{12}} (w_{11} + w_{12})^{(w_{11}+w_{12})} 100^{w_{21}}$$

subject to

$$w_{01} + w_{02} + w_{03} + w_{04} + w_{05} = 1$$

$$w_{01} - w_{05} + 2w_{11} - w_{21} = 0$$

$$w_{02} - w_{05} - 2w_{12} - w_{21} = 0$$

$$w_{03} - w_{05} - w_{21} = 0$$

$$w_{04} - 2w_{11} - 2w_{12} = 0$$

$$w_1 + w_2 = 1$$

$$w_{01}, w_{02}, w_{03}, w_{04}, w_{05}, w_{11}, w_{12}, w_{21} \ge 0$$

$$w_1, w_2 > 0$$

$$g_{10} = 87.98776 \quad \text{and} \quad g_{20} = 0.01$$

Problem 1 will now be solved using the proposed model. The value interval for w_1 and w_2 will be between 0.1 and 0.9. For the weights $w_1 = 0.5$, $w_2 = 0.5$ the given geometric problem from the Problem 1 is written as

$$\min Z_w(x) = 2x_1 + 5x_2 + 2x_3 + x_4 + \frac{1}{2x_1 x_2 x_3},$$

subject to

$$1 - x_1^2 x_4^{-2} - x_2^2 x_4^{-2} - y_1^2 = 0$$

$$1 - 100 x_1^{-1} x_2^{-1} x_3^{-1} - y_2^2 = 0$$

$$x_1, x_2, x_3, x_4 > 0$$

$$w_1 + w_2 = 1, w_1, w_2 > 0.$$

Then, the above problem according to the Kuhn-Tucker Conditions can be formulated as in Model 1 as follows:

$$\begin{aligned}&h_w\left(x_1,x_2,x_3,x_4,\gamma_1,\gamma_2\right)=2x_1+5x_2+2x_3+x_4+\frac{1}{2x_1x_2x_3}\\&-\gamma_1\left(1-x_1^2x_4^{-2}-x_2^2x_4^{-2}-y_1^2\right)-\gamma_2\left(1-100x_1^{-1}x_2^{-1}x_3^{-1}-y_2^2\right)\end{aligned} \tag{21}$$

From Equation (21) the problem is written as follows:

$$\min Z_w\left(x\right)=2x_1+5x_2+2x_3+x_4+\frac{1}{2x_1x_2x_3}$$

subject to

$$-\frac{0.9}{x_1^2x_2x_3}+\frac{2\gamma_1x_1}{x_4^2}-\frac{100\gamma_2}{x_1^2x_2x_3}+0.4=0,$$

$$-\frac{0.9}{x_1x_2^2x_3}+\frac{2\gamma_1x_1}{x_4^2}-\frac{100\gamma_2}{x_1x_2^2x_3}+0.1=0,$$

$$-\frac{0.9}{x_1x_2x_3^2}-\frac{100\gamma_2}{x_1x_2x_3^2}+0.4=0,$$

$$-\gamma_1\left(\frac{2x_1^2}{x_4^3}+\frac{2x_2^2}{x_4^3}\right)+0.2=0,$$

$$\gamma_1\left(-\frac{x_1^2}{x_4^2}-\frac{x_2^2}{x_4^2}+1\right)\geq 0,$$

$$\gamma_2\left(-\frac{100}{x_1x_2x_3}+1\right)\geq 0\,,$$

$$x_1,x_2,x_3,x_4,\gamma_1,\gamma_2>0.$$

To linearize the nonlinear objective function with the nonlinear constraints in the above problem, we use the first order Taylor polynomial series at any initial feasible point

$$X\left(0\right)=\left(x_1=5,x_2=3,x_3=7,x_4=6,\gamma_1=2,\gamma_2=10\right)$$

as follows:

$$\min Z_w\left(x\right)\approx 1.999047619x_1+4.998413x_2+1.99932x_3+x_4+0.01905\,,$$

subject to

$$0.8734x_1+0.63524x_2+0.272245x_3-0.1852x_4+0.277778\gamma_1-0.1905\gamma_2-5.0683=0,$$

$$\begin{aligned}&0.63524x_1+2.2286x_2+0.45374x_3-0.11111x_4+0.166667\gamma_1\\&-0.11376\gamma_2-7.37302=0,\end{aligned}$$

$$0.272245x_1+0.453742x_2+0.3889x_3-0.1361\gamma_2-3.4456=0\,,$$

$$-0.1852x_1-0.11111x_2+0.3148x_4-0.3148\gamma_1+0.37037=0\,,$$

$$-\frac{5}{9}x_1-\frac{1}{3}x_2+\frac{17}{27}x_4+\frac{1}{18}\gamma_1\geq 0,$$

$$\frac{40}{21}x_1+\frac{200}{63}x_2+\frac{200}{147}x_3+\frac{1}{21}\gamma_2-\frac{200}{7}\geq 0,$$

$$x_1, x_2, x_3, x_4, \gamma_1, \gamma_2 > 0.$$

The solution of the above problem is

$$X(1) = (x_1 = 5.14076376, x_2 = 2.4703573, x_3 = 7.523979, \\ x_4 = 5.5904741, \gamma_1 = 2.8711, \gamma_2 = 14.7081),$$

$$\min Z_w(X(1)) = 43.27685778 \text{ and } g_{10} = 86.5435 \text{ and } g_{20} = 0.0104656.$$

When the same procedure is applied to point $X(1)$, the solution $X(2)$ is obtained. If the same iteration continues for the weights $w_1 = 0.5$, $w_2 = 0.5$, the calculated solution points $X(2), X(3), X(4), X(5), X(6)$ and the corresponding objective function values g_{10} and g_{20} are given in **Table 1**. As seen in **Table 1**, the absolute value of the difference between the points X(5) and X(5) is reduced enough to a smaller value, and the iteration is terminated. One of the points $X(5)$ or $X(6)$ can be assumed the par to optimal solution point of the given MOGPP for the weights $w_1 = 0.5$, $w_2 = 0.5$.

By considering different values of w_1 and w_2, the corresponding solutions of the problem applying the taylor approach in each iteration are given in **Table 2**.

6. Result and Conclusion

In this study, we proposed an alternative approach to the approximate pare to solution of MOGPP based on the weighting method. In this model, MOGPP has been reduced to a sequential linear programming problem and the Pareto optimal solution of MOGPP has been calculated approximately in an easier and more speedy way. Besides in GP problems and MOGPP the solution becomes more difficult when the degree of difficulty is a positive number whereas such a difficulty does not exist in the developed model. The solution for the problem given in the example by the weighted mean method is shown in **Table 3** and the

Table 1. The corresponding iteration solution for $w_1 = 0.5$ and $w_2 = 0.5$, using the Taylor series approach.

	Variables							
s	x_1	x_2	x_3	x_4	γ_1	γ_2	g_{10}	g_{20}
0	5	3	7	6	2	10	88	0.009524
1	5.140764	2.470357	7.523979	5.590474	2.871	14.708	86.543491	0.010466
2	5.091219	2.661165	7.349591	5.737520	2.8738	14.686	87.849933	0.010043
3	5.084131	2.682310	7.332497	5.748260	2.874	14.65986	87.986130	0.010001
4	5.084056	2.682555	7.332314	5.748367	2.874	14.65962	87.987763	0.010000
5	5.084056	2.682555	7.332314	5.748367	2.874	14.65962	87.987464	0.010000
6	5.084056	2.682555	7.332314	5.748367	2.874	14.65962	87.987764	0.010000

Table 2. The solution from the numerical approach method.

Variables								
w_1	w_2	x_1	x_2	x_3	x_4	g_{10}	g_{20}	s
0.1	0.9	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5
0.2	0.8	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5
0.3	0.7	5.084056	2.682555	7.332314	5.748367	87.987762	0.01000	5
0.4	0.6	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5
0.5	0.5	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	6
0.6	0.4	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	4
0.7	0.3	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5
0.8	0.2	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5
0.9	0.1	5.084056	2.682555	7.332314	5.748367	87.987764	0.01000	5

Table 3. Primal solutions [15].

Variables						
w_1	w_2	x_1	x_2	x_3	x_4	Z
0.1	0.9	5.084055	2.682555	7.332315	5.748367	8.08776
0.2	0.8	5.084055	2.682555	7.332315	5.748367	8.08776
0.3	0.7	5.084055	2.682555	7.332315	5.748367	8.08776
0.4	0.6	5.084055	2.682555	7.332315	5.748367	8.08776
0.5	0.5	5.084055	2.682555	7.332315	5.748367	8.08776

solution by the model that we developed is shown in **Table 2** and the results are almost the same. For this reason, proposed method can be used as an alternative of weighted mean method.

References

[1] Zener, C. (1961) A Mathematical Aid in Optimizing Engineering Design. *Proceedings of National Academy of Sciences*, **47**, 537-539. https://doi.org/10.1073/pnas.47.4.537

[2] Duffin, R.J., Peterson E.L. and Zener, C. (1967) Geometric Programming: Theory and Application. John Wiley and Sons, New York.

[3] Chiang, M. (2005) Geometric Programming for Communication Systems. Now Publishers Inc., Boston.

[4] Choi, J.C. and Bricker, D.L. (1996) Effectiveness of a Geometric Programming Algorithm for Optimization of Machining Economics Models. *Computers & Operations Research*, **23**, 957-961. https://doi.org/10.1016/0305-0548(96)00008-1

[5] Liu, S.T. (2007) Geometric Programming with Fuzzy Parameters in Engineering Optimization. *International Journal of Approximate Reasoning*, **46**, 484-498. https://doi.org/10.1016/j.ijar.2007.01.004

[6] Ny, J.L. and Pappas, G.J. (2010) Geometric Programming and Mechanism Design for Air Traffic Conflict Resolution. *Proceedings of the* 2010 *American Control*

Conference, Baltimore, 30 June-2 July 2010, 3069-3074.

[7] Elmaghraby, S.E. and Morgan, C.D. (2007) Resource Allocation in Activity Networks under Stochastic Conditions: A Geometric Programming-Sample Path Optimization Approach. *Tijdschrift voor Economie en Management*, **52**, 367-389.

[8] Chu, C. and Wong, D.F. (2001) VLSI Circuit Performance Optimization by Geometric Programming. *Annals of Operations Research*, **105**, 37-60. https://doi.org/10.1023/A:1013345330079

[9] Scott, C.H. and Jefferson, T.R. (1995) Allocation of Resources in Project Management. *International Journal of System Science*, **26**, 413-420. https://doi.org/10.1080/00207729508929042

[10] Jung, H. and Klein, C.M. (2001) Optimal Inventory Policies under Decreasing Cost Functions via Geometric Programming. *European Journal of Operational Research*, **132**, 628-642. https://doi.org/10.1016/S0377-2217(00)00168-5

[11] Nasseri, S.H. and Alizadeh, Z. (2014) Optimized Solution of a Two-Bar Truss Nonlinear Problem Using Fuzzy Geometric Programming. *Journal Nonlinear Analysis and Application*, **2014**, 1-9. https://doi.org/10.5899/2014/jnaa-00230

[12] Islam, S. (2010) Multi-Objective Geometric Programming Problem and Its Applications. *Yugoslav Journal of Operations Research*, **20**, 213-227. https://doi.org/10.2298/YJOR1002213I

[13] Biswal, M.P. (1992) Fuzzy Programming Technique to Solve Multi-Objective Geometric Programming Problems. *Fuzzy Sets and Systems*, **51**, 67-71. https://doi.org/10.1016/0165-0114(92)90076-G

[14] Verma, R.K. (1990) Fuzzy Geometric Programming with Several Objective Functions. *Fuzzy Sets and Systems*, **35**, 115-120. https://doi.org/10.1016/0165-0114(90)90024-Z

[15] Ojha, A.K. and Biswal, K.K. (2010) Multi-Objective Geometric Programming Problem with Weighted Mean Method. *International Journal of Computer Science and Information Security*, **7**, 82-86.

[16] Toksarı, M.D. (2008) Taylor Series Approach to Fuzzy Multiobjective Linear Fractional Programming. *Information Sciences*, **178**, 1189-1204. https://doi.org/10.1016/j.ins.2007.06.010

[17] Güzel, N. and Sivri, M. (2005) Taylor Series Solution of Multi Objective Linear Fractional Programming Problem. *Trakya University Journal of Science*, **6**, 91-98.

[18] Boyd, S., Kim, S.J., Vandenberghe, L. and Hassibi, A. (2007) A Tutorial on Geometric Programming. *Optimization and Engineering*, **8**, 67-127. https://doi.org/10.1007/s11081-007-9001-7

[19] Geoffrion, A.M. (1968) Proper Efficiency and the Theory of Vector Maximization. *Journal of Mathematical Analysis and Applications*, **22**, 618-630. https://doi.org/10.1016/0022-247X(68)90201-1

[20] Caramia, M. and Dell'Olmo, P. (2008) Multi-Objective Management in Freight Logistics. Springer, London.

[21] Bazaraa, S., Sherali, H.D. and Shetty, C.M. (2006) Nonlinear Programming Theory and Algorithms. 3rd Edition, John Wiley and Sons, New York. https://doi.org/10.1002/0471787779

[22] Kwak, N.K. (1973) Mathematical Programming with Business Applications. McGraw-Hill, New York.

[23] Yousef, S., Badra, N. and Abu-El Yazied, T.G. (2009) Geometric Programming Problems with Fuzzy Parameters and Its Application to Crane Load Sway. *World Applied Science Journal,* **7**, 94-101.

[24] Luptacik, M. (2010) Mathematical Optimization and Economic Analysis. Springer, New York. https://doi.org/10.1007/978-0-387-89552-9

[25] Ota, R.R. and Ojha, A.K. (2015) A Comparative Study on Optimization Techniques for Solving Multi-Objective Geometric Programming Problems. *Applied Mathematical Sciences*, **9**, 1077-1085. https://doi.org/10.12988/ams.2015.4121029

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The Cost Functional and Its Gradient in Optimal Boundary Control Problem for Parabolic Systems

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Abstract

The problems of optimal control (OCPs) related to PDEs are a very active area of research. These problems deal with the processes of mechanical engineering, heat aeronautics, physics, hydro and gas dynamics, the physics of plasma and other real life problems. In this paper, we deal with a class of the constrained OCP for parabolic systems. It is converted to new unconstrained OCP by adding a penalty function to the cost functional. The existence solution of the considering system of parabolic optimal control problem (POCP) is introduced. In this way, the uniqueness theorem for the solving POCP is introduced. Therefore, a theorem for the sufficient differentiability conditions has been proved.

Keywords

Constrained Optimal Control Problems, Necessary Optimality Conditions Parabolic System, Adjoint Problem, Exterior Penalty Function Method, Existence and Uniqueness Theorems

1. Introduction

Many researches in recent years have been devoted to the studies of optimal control problems for a distributed parameter system. Optimal control is widely applied in aerospace, physics, chemistry, biology, engineering, economics and other areas of science and has received considerable attention of researchers.

The optimal boundary control problem for parabolic systems is relevant in mathematical description of several physical processes including chemical reactions, semiconductor theory, nuclear reactor dynamics, population dynamics [1] and [2]. The partial differential equations involved in these problems include elliptic equations, parabolic equations and hyperbolic equations [3] [4].

Optimization can be of constrained or unconstrained problems. The presence of constraints in a nonlinear programming creates more problems while finding the minimum as compared to unconstrained ones. Several situations can be identified depending on the effect of constraints on the objective function. The simplest situation is when the constraints do not have any influence on the minimum point. Here the constrained minimum of the problem is the same as the unconstrained minimum, *i.e.*, the constraints do not have any influence on the objective function. For simple optimization problems it may be possible to determine, beforehand, whether or not the constraints have any influence on the minimum point. However, in most of the practical problems, it will be extremely difficult to identify it. Thus one has to proceed with general assumption that the constraints will have some influence on the optimum point. The minimum of a nonlinear programming problem will not be, in general, an extreme point of the feasible region and may not even be on the boundary. Also the problem may have local minima even if the corresponding unconstrained problem is not having local minima. Furthermore, none of the local minima may correspond to the global minimum of the unconstrained problem. All these characteristics are direct consequences of the introduction of constraints and hence we should to have general algorithms to overcome these kinds of minimization problems [5] [6] [7] [8] [9].

The algorithms for minimization are iterative procedures that require starting values of the design variable x. If the objective function has several local minima, the initial choice of x determines which of these will be computed. There is no guaranteed way of finding the global optimal point. One suggested procedure is to make several computer runs using different starting points and pick the best Rao [10]. The majority of available methods are designed for unconstrained optimization, where no restrictions are placed on the de-sign variables. In these problems the minima, if they exist are stationary points (points where gradient vector of the objective function vanishes). There are also special algorithms for constrained optimization problems, but they are not easily accessible due to their complexity and specialization.

All of the many methods available for the solution of a constrained nonlinear programming problem can be classified into two broad categories, namely, the direct methods and the indirect methods approach. In the direct methods the constraints are handled in an explicit manner whereas in the most of the indirect methods, the constrained problem is solved as a sequence of unconstrained minimization problems or as a single unconstrained minimization problem. Here we are concerned on the indirect methods of solving constrained optimization problems. A large number of methods and their variations are available in the literature for solving constrained optimization problems using indirect methods. As is frequently the case with nonlinear problems, there is no single method that is clearly better than the others. Each method has its own strengths and weaknesses. The quest for a general method that works effectively for all types of problems continues. Sequential transformation methods are the oldest methods also known as Sequential Un-Constrained Minimization Techniques (SUMT) based upon the work of Fiacco and McCormick, 1968. They are still among the most popular ones for some cases of problems, although there are some modifications that are more often used. These methods help us to remove a set of complicating constraints of an optimization problem and give us a frame work to exploit any available methods for unconstrained optimization problems to solve, perhaps, approximately. [5] [6] [7] [8] [9]. However, this is not without a cost. In fact, this transforms the problem into a problem of non-smooth (in most cases) optimization which has to be solved iteratively. The sequential transformation method is also called the classical approach and is perhaps the simplest to implement. Basically, there are two alternative approaches. The first is called the exterior penalty function method (commonly called penalty method), in which a penalty term is added to the objective function for any violation of constraints. This method generates a sequence of infeasible points, hence its name, whose limit is an optimal solution to the original problem. The second method is called interior penalty function method (commonly called barrier method), in which a barrier term that prevents the points generated from leaving the feasible region is added to the objective function. The method generates a sequence of feasible points whose limit is an optimal solution to the original problem. Luenberger [11] illustrated that penalty and barrier function methods are procedures for approximating constrained optimization problems by unconstrained problems.

In the meanings of constrained conditions, these optimal control problems can be divided into control con-strained problems and state constrained problems. In each of the branches referred above, there are many excellent works and also many difficulties to be solved.

The rest of this paper is organized as follows. In Section 2, the proposed system of optimal control problem with respect to a parabolic equation is offered. Section 3 describes the analysis of existence and uniqueness of the solution of the POCP. In Section 4, the variation of the functional and its gradient is presented. Section 5 describes Lipschitz continuity of the gradient cost functional. Finally, conclusions are presented in Section 6.

2. Problem Statement

Consider the following POCP process be described in:

$$\Omega_T = \{(x,t): 0 < x < \theta \, , \, 0 < t < T\}:$$

$$\frac{\partial \varphi(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(\mu(x)\frac{\partial \varphi(x,t)}{\partial x}\right) + v_o(x,t), \tag{1}$$

with the initial and the boundary conditions:

$$\varphi(x,t) = v_1(x), \quad 0 < x < \theta, t = 0 \tag{2}$$

$$\frac{\partial \varphi(x,t)}{\partial x} = 0, \quad x = 0, 0 < t \le T,$$
$$\frac{\partial \varphi(x,t)}{\partial x}\mu(x) = \tau\left[v_2(t) - \varphi(x,t)\right], \quad x = \theta, 0 < t \le T \tag{3}$$

where the solution of the problem (1-3) is $\varphi(x,t)$, since, $\mu(x) > 0$, $\mu(x) \in L_\infty[0,\theta]$, the coefficient of convection τ is positive constant-sometimes τ is called coefficient of heat transfer. The admissible controls is a set $V = V_\circ \times V_1 \times V_2$ defined as

$$V = \left\{v = \left(v_o(x,t), v_1(x), v_2(t)\right) : v_o(x,t) \in L_2(\Omega_T), v_1(x) \in L_2(0,\theta), v_2(t) \in L_2(0,T)\right\}.$$

Many physical and engineering settings have the mathematical model (1-3), in particular in hydrology, material sciences, heat transfer and transport problems [12]. In the case of heat transfer, the Robin condition physically is realized as follows. Let the surface $x = \theta$ of the rod be exposed to air or other fluid with temperature. Then $\varphi(\theta,t) - v_2(t)$ is the temperature difference at $x = \theta$ between the rod and its surroundings. According to Newton's law of cooling, the rate at which heat is transferred from the rod to the fluid is proportional to the difference in the temperature between the rod and the fluid, *i.e.*

$$\frac{\partial \varphi(\theta,t)}{\partial x}\mu(\theta) = \tau\left[-v_2(t) - \varphi(\theta,t)\right] \tag{4}$$

The purpose is to find the optimal control $\varphi(x,t)$ that minimizes the following cost functional:

$$J_\alpha(v) = \gamma\int_0^\theta\left[q(x) - \varphi(x,T)\right]^2 \mathrm{d}x + \alpha\int_0^T\left[v_2(t) - w(t)\right]^2 \mathrm{d}t \tag{5}$$

and

$$r_1 \le \varphi(x,t) \le r_2 \tag{6}$$

where γ, α are given positive numbers, $q(x)$ is given function from $L_2[0,\theta]$, $w(t)$ is given function from $L_2[0,T]$ with T is a fixed time. Penalty function methods are the most popular constraint handling methods among users. Two main branches of penalty method have been proposed in the literature: Exterior and Interior which is also called the barrier method. The basic idea in penalty method is to eliminate some or all constraints and add to the objective function a penalty term which prescribes a high cost to infeasible points. Associated with this method is a parameter A_τ, which determines the severity of penalty and as a consequence the extent to which the resulting unconstrained problem approximates the original constrained problem. We restrict attention to the polynomial order-even penalty function. The constrained optimal control problem (5-6) is converted to unconstrained optimal control problem by adding a penalty function [13] to the cost functional (5), yielding the modified function:

$$\psi_{\alpha,\tau}(v,q_\tau):$$
$$\psi_{\alpha,\tau}(v,q_\tau) \equiv \psi(v) = J_\alpha(v) + P_\tau(v), \tag{7}$$

where $P_\tau(v) = A_\tau \int_0^\theta \int_0^T \left[\left(s_1(\varphi) + s_2(\varphi) \right) \right] \mathrm{d}x\mathrm{d}t$, $s_1(\varphi) = \left[\max\left\{ r_1 - \varphi(x,t;w); 0 \right\} \right]^2$,

$s_2(\varphi) = \left[\max\left\{ \varphi(x,t;w) - r_2; 0 \right\} \right]^2$ and $A_\tau > 0$, $\tau = 0,1,2,\cdots$, $\lim_{\tau\to\infty} A_\tau = +\infty$.

3. Well-Posedness of System

This section present the concept of the weak solution of the system (1-3) and the existence solution. Let a function $\varphi \in L_2(\Omega_T)$ of the weak solution of the problem, and satisfies the following integral, for all $\kappa(x,t) \in L_2(\Omega_T)$:

$$\begin{aligned}
&\iint_{\Omega_T} \mu(x) \frac{\partial \kappa(x,t)}{\partial x} \frac{\partial \varphi(x,t)}{\partial x} \mathrm{d}x\mathrm{d}t - \iint_{\Omega_T} \varphi(x,t) \frac{\partial \kappa(x,t)}{\partial t} \mathrm{d}x\mathrm{d}t \\
&+\int_0^\theta \varphi(x,T)\, \kappa(x,T)\mathrm{d}x - \int_0^\theta v_1(x)\, \kappa(x,0)\mathrm{d}x - \tau \int_0^T \left[\varphi(\theta,t) - v_2(t) \right] \kappa(\theta,t)\mathrm{d}t \\
&= \iint_{\Omega_T} v_0(x,t)\, \kappa(x,t)\, \mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{8}$$

The weak solution $\varphi \in L_2(\Omega_T)$ of the direct problem exists and unique under the above conditions with respect to the given data [14] [15]. According to [12], the solution of the optimal control problem can be defined as a solution of the minimization problem for the cost functional $J_\alpha(v)$ under condition (6), given by (5):

$$\psi_\alpha(v_*) = \inf_{v\in V} \psi_\alpha(v) \tag{9}$$

Theorem 1:

Under the above conditions, the optimal control problem has an optimal solution (φ, v) in $L_2(\Omega_T) \times V$.

Proof: when $\psi_\alpha(v_*) = 0$, the solution $v_* \in V$ is a strict solution of systems (1-3) and (5-6), where $v_* \in V$ satisfies the equation of functional, $\varphi(x, t = T; v) = q(x)$, $x \in (0, \theta)$. In parabolic problems and according to the theory of weak solution, can prove that the sequence $\left\{ v^{(n)} \right\} \subset V$ weakly converges to the function $v \in V$, so that the traces sequence $\left\{ \varphi\left(x, T; v^{(n)} \right) \right\}$ of corresponding solutions of system (1-3) converges to the solution $\{\varphi(x,T;v)\}$ in $\left\| L_2(\Omega_T) \right\|$, hence, when $n \to \infty$ then $\psi_\alpha\left(v^{(n)} \right) \to \psi_\alpha(v)$ [16]. Therefore the functional $\psi_\alpha(v)$ is weakly continuous on V, and the non-empty set of solutions $V_* = \left\{ v \in V : \psi_\alpha(v_*) = (\psi_\alpha)_* = \inf \psi_\alpha(v) \right\}$ for the minimization problem (5-6) [17].

4. The Variation of the Functional and Its Gradient

The main objective here, the proof of Theorem 2 (found in tail of this section) which requires the following two lemmas; lemma 1 and lemma 2. Let the first variation of the cost functional $\psi_\alpha(v)$ of the cost functional (7) as follows:

$$\Delta \psi_\alpha(v) = \psi_\alpha(v + \Delta v) - \psi_\alpha(v) \tag{10}$$

therefore,

$$
\begin{aligned}
\Delta\psi_\alpha(v) &= 2\gamma\int_0^\theta \left[\varphi(x,T;v)-q(x)\right]\Delta\varphi(x,T;v)\mathrm{d}x \\
&+\gamma\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2\mathrm{d}x \\
&+2\alpha\int_0^T\left[v_2(t)-w(t)\right]\Delta v_2(t)\,dt+\alpha\int_0^T\left[\Delta v_2(t)\right]^2\mathrm{d}t \\
&+A_\tau\iint_{\Omega_T}\left[\left(s_1'(\varphi)+s_2'(\varphi)\right)\right]\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{11}
$$

where

$$\Delta\varphi(x,t;v)=\varphi(x,t,v+\Delta v)-\varphi(x,t,v)\in L_2(\Omega_T),$$

$$s_1(\Delta\varphi)=s_1\left(\varphi(x,t,v+\Delta v)-\varphi(x,t,v)\right),$$

$$s_2(\Delta\varphi)=s_2\left(\varphi(x,t,v+\Delta v)-\varphi(x,t,v)\right),$$

$$v+\Delta v=\left\{v_0(x,t)+\Delta v_0(x,t),v_1(x)+\Delta v_1(x),v_2(t)+\Delta v_2(t)\right\}\in V.$$

Therefore the function $\Delta\varphi=\Delta\varphi(x,t;v)$ is the solution of the following system:

$$
\begin{cases}
\dfrac{\partial\Delta\varphi(x,t)}{\partial t}=\dfrac{\partial}{\partial x}\left(\mu(x)\dfrac{\partial\Delta\varphi(x,t)}{\partial x}\right)+\Delta v_o(x,t); & (x,t)\in\Omega_T \\
\dfrac{\partial\Delta\varphi(x,t)}{\partial x}=0; & x=0 \\
\Delta\varphi(x,t)=\Delta v_1(x); & t=0,0<x<\theta \\
\mu(x)\dfrac{\partial\Delta\varphi(x,t)}{\partial x}=\tau\left[\Delta v_2(t)-\Delta\varphi(x,t)\right]; & x=\theta,0<t\le T
\end{cases}
\tag{12}
$$

Lemma 1:

If the direct system (1-3) have the corresponding solution $\varphi=\varphi(x,t;v)\in L_2(\Omega_T)$ and $\rho(x,t;v)\in L_2(\Omega_T)$ is the solution of the adjoint parabolic problem [18]:

$$
\begin{cases}
\dfrac{\partial\rho(x,t)}{\partial t}=-\dfrac{\partial}{\partial x}\left(\mu(x)\dfrac{\partial\rho(x,t)}{\partial x}\right)-A_\tau\left[\left(s_1'(\varphi)+s_2'(\varphi)\right)\right], & (x,t)\in\Omega_T \\
\dfrac{\partial\rho(x,t)}{\partial x}=0, x=0 \\
\rho(x,t)=2\gamma\left[\varphi(x,t;v)-q(x)\right], & t=T,0<x<\theta \\
-\mu(x)\dfrac{\partial\rho(x,t)}{\partial x}=\tau\rho(x,t), & x=\theta,0<t\le T
\end{cases}
\tag{13}
$$

then the following integral identity holds for all elements
$v=\left\{v_0(x,t),v_1(x),v_2(x)\right\}$ and
$v+\Delta v=\left\{v_0(x,t)+\Delta v_0(x,t),v_1(x)+\Delta v_1(x),v_2(t)+\Delta v_2(t)\right\}\in V$:

$$
\begin{aligned}
2\gamma\int_0^\theta \left[\varphi(x,T;v)-q(x)\right]\Delta\varphi(x,T;v)\mathrm{d}x &= \iint_{\Omega_T}\rho(x,t;v)\Delta v_0(x,t)\mathrm{d}x\mathrm{d}t \\
&+\int_0^\theta\rho(x,0;v)\Delta v_1(x)\mathrm{d}x \\
&+\tau\int_0^T\rho(\theta,t;v)\Delta v_2(t)\mathrm{d}t \\
&-A_\tau\iint_{\Omega_T}\left[\left(s_1'(\varphi)+s_2'(\varphi)\right)\right]\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t
\end{aligned}
\tag{14}
$$

Proof: At $t=T$ with the condition in (13) to transform the left-hand side of (14) as follows:

$$\begin{aligned}
&2\gamma\int_0^{\theta}\left[\varphi(x,T;v)-q(x)\right]\Delta\varphi(x,T;v)\mathrm{d}x\\
&=\int_0^{\theta}\rho(x,T;v)\Delta\varphi(x,T;v)\mathrm{d}x\\
&=\iint\limits_{\Omega_T}\left[\rho(x,t;v)\Delta\varphi(x,t;v)\right]_t\mathrm{d}x\mathrm{d}t\\
&=\iint\limits_{\Omega_T}\left[\rho_t(x,t;v)\Delta\varphi(x,t;v)+\rho(x,t;v)\Delta\varphi_t(x,t;v)\right]\mathrm{d}x\mathrm{d}t\\
&=\iint\limits_{\Omega_T}\rho(x,t;v)\left(\mu(x)\Delta\varphi_x(x,t;v)\right)_x\mathrm{d}x\mathrm{d}t-\iint\limits_{\Omega_T}\left(\mu(x)\rho_x(x,t;v)\right)_x\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t\\
&+\iint\limits_{\Omega_T}\Delta v_0(x,t)\rho(x,t;v)\mathrm{d}x\mathrm{d}t-A_\tau\iint\limits_{\Omega_T}\left[\left(s_1'(\varphi)+s_2'(\varphi)\right)\right]\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t\\
&=\int_0^{T}\left[\rho(x,t;v)\mu(x)\Delta\varphi_x(x,t;v)-\mu(x)\rho_x(x,t;v)\Delta\varphi(x,t;v)\right]_{x=0}^{x=\theta}\mathrm{d}t\\
&+\iint\limits_{\Omega_T}\Delta v_0(x,t)\rho(x,t;v)\mathrm{d}x\mathrm{d}t-A_\tau\iint\limits_{\Omega_T}\left[\left(s_1'(\varphi)+s_2'(\varphi)\right)\right]\Delta\phi(x,t;v)\mathrm{d}x\mathrm{d}t
\end{aligned}$$

At the boundary conditions in (13) and (14) for the functions $\rho(x,t;v)$ and $\Delta\varphi(x,t;v)$; we obtain (14). Corresponding to the inverse problem in system (1-3) and (5-6), the parabolic problem (13) define as an adjoint problem. By backward one of the Equation (13), the "final condition" at $t=T$ it is a well-posed initial boundary-value problem under a time reversal. The first variation of the cost functional $\psi_\alpha(v)$ obtain by using integral identity in (14) on the right-hand side of Equation (11):

$$\begin{aligned}
\Delta\psi_\alpha(v)&=\iint\limits_{\Omega_T}\Delta v_0(x,t)\rho(x,t;v)\mathrm{d}x\mathrm{d}t+\tau\int_0^{T}\rho(\theta,t;v)\Delta v_2(t)\mathrm{d}t\\
&+\int_0^{\theta}\rho(x,0;v)\Delta v_1(x)\mathrm{d}x+2\alpha\int_0^{T}\left[v_2(t)-w(t)\right]\Delta v_2(t)\mathrm{d}t\\
&+\alpha\int_0^{T}\left[\Delta v_2(t)\right]^2\mathrm{d}t+\gamma\int_0^{\theta}\left[\Delta\varphi(x,T;v)\right]^2\mathrm{d}x
\end{aligned}\tag{15}$$

Using the definition of the Fréchet-differential and the above the scalar product definition in V, transform the right-hand side of (15) need into the following expression:

$$\Delta\psi_\alpha(v)=\left\langle\psi_\alpha'(v),\Delta v\right\rangle_V+\alpha\int_0^{T}\left[\Delta v_2(t)\right]^2\mathrm{d}t+\gamma\int_0^{\theta}\left[\Delta\varphi(x,T;v)\right]^2\mathrm{d}x\tag{16}$$

Now we need to show that the last two terms on the right-hand side of (15) are of order $O\left(\|v\|_V^p\right)$, with $p\geq 1$.

Lemma 2:

If the parabolic problem (12) have the solution $\Delta\varphi=\Delta\varphi(x,t;v)\in L_2(\Omega_T)$, $v\in V$, then the following inequality holds:

$$\ell\int_0^{\theta}\left[\Delta\varphi(x,T;v)\right]^2\mathrm{d}x\leq c_0\|\Delta v\|_V^2,\quad\forall\,\Delta v\in V\tag{17}$$

where $\|\Delta v\|_V$ is the norm $L_2(\Omega_T)-$norm of the function $\Delta v\in V$,

$\|\Delta v\|_V=\left[\iint\limits_{\Omega_T}\left|\Delta v_0(x,t)\right|^2\mathrm{d}x\mathrm{d}t+\int_0^{\theta}\left|\Delta v_1(x)\right|^2\mathrm{d}x+\int_0^{T}\left|\Delta v_2(t)\right|^2\mathrm{d}t\right]^{1/2}$ is the norm

$L_2(\Omega_T)$ of the function $\Delta v \in V$, and the constants c_0, $\ell > 0$ are defined as follows:

$$c_0 = \max\{1,\tau\} > 0, \mu_* = \min_{0\le x\le\theta} \mu(x) > 0,$$
$$\ell = \min\{\mu_*/\theta^2, \quad 2\tau/(\tau+2\theta)\} > 0 \tag{18}$$

Proof:

Multiplying the Equation (12) by $\Delta\varphi$, then integrating the result on Ω_T, $(\mu(x)\Delta\phi_x)_x \Delta\varphi = (\mu(x)\Delta\varphi_x \Delta\varphi)_x - \mu(x)(\Delta\varphi_x)^2$, $\Delta\phi\varphi\frac{\partial\Delta\varphi}{\partial t} = \frac{\partial\Delta\varphi^2}{2\partial t}$.

We obtain energy identity after applying the initial and boundary conditions as the following:

$$\int_0^\theta [\Delta\varphi(x,T;v)]^2 \mathrm{d}x + 2\tau\int_0^T [\Delta\varphi(\theta,t;v)]^2 \mathrm{d}t + 2\iint_{\Omega_T} \mu(x)(\Delta\varphi_x)^2 \mathrm{d}x\mathrm{d}t$$
$$= 2\iint_{\Omega_T} \Delta v_0(x,t)\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t + 2\tau\int_0^T \Delta\varphi(\theta,t;v)\Delta v_2(t)\mathrm{d}t + \int_0^\theta [\Delta v_1(x)]^2 \mathrm{d}x \tag{19}$$

We use the ℓ-inequality $\alpha\gamma \le ((\alpha^2\ell/2) + (\gamma^2/2\ell), \forall\alpha,\gamma \in R$, $\forall\ell > 0$ for the solution $\Delta\varphi = \Delta\varphi(x,t;v)$ of the parabolic problem (19). Then for all $\ell > 0$ we have:

$$2\iint_{\Omega_T} \Delta v_0(x,t)\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t + 2\tau\int_0^T \Delta\varphi(\theta,t;v)\Delta v_2(t)\,\mathrm{d}t + \int_0^\theta [\Delta v_1(x)]^2 \mathrm{d}x$$
$$\le \ell\iint_{\Omega_T} [\Delta\varphi(x,t;v)]^2 \mathrm{d}x\mathrm{d}t + \frac{1}{\ell}\iint_{\Omega_T} [\Delta v_0(x,t)]^2 \mathrm{d}x\mathrm{d}t + \tau\ell\int_0^T [\Delta\varphi(\theta,t;v)]^2 \mathrm{d}t \tag{20}$$
$$+ \frac{\tau}{\ell}\int_0^T [\Delta v_2(t)]^2 \mathrm{d}t + \int_0^\theta [\Delta v_1(x)]^2 \mathrm{d}x$$

Applying the Cauchy inequality to estimate the term $[\Delta\varphi(x,t)]^2$:

$$[\Delta\varphi(x,t)]^2 = \left[\int_x^\theta \Delta\varphi_\kappa(\kappa,t;v)\mathrm{d}\kappa - \Delta\varphi(\theta,t;v)\right]^2$$
$$\le 2\left(\int_x^\theta \Delta\varphi_\kappa(\kappa,t;v)\mathrm{d}\kappa\right)^2 + 2(\Delta\varphi(\theta,t;v))^2$$
$$\le 2\theta\int_0^\theta [\Delta\varphi_x(x,t;v)]^2 \mathrm{d}x + 2(\Delta\varphi(\theta,t;v))^2$$

By integrating the both sides of above inequality on Ω_T, we obtain:

$$\iint_{\Omega_T} [\Delta\varphi(x,t;v)]^2 \mathrm{d}x\mathrm{d}t \le 2\theta^2 \iint_{\Omega_T} [\Delta\varphi_x(x,t;v)]^2 \mathrm{d}x\mathrm{d}t + 2\theta\int_0^T [\Delta\varphi(\theta,t;v)]^2 \mathrm{d}t \tag{21}$$

and use this estimate on the right-hand side of (20):

$$2\iint_{\Omega_T} \Delta v_0(x,t)\Delta\varphi(x,t;v)\mathrm{d}x\mathrm{d}t + 2\tau\int_0^T \Delta\varphi(\theta,t;v)\Delta v_2(t)\mathrm{d}t + \int_0^\theta [\Delta v_1(x)]^2 \mathrm{d}x$$
$$\le 2\theta^2\ell\iint_{\Omega_T} [\Delta\varphi_x(x,t;v)]^2 \mathrm{d}x\mathrm{d}t + (2\theta\ell + \tau\ell)\int_0^T [\Delta\varphi(\theta,t;v)]^2 \mathrm{d}t$$
$$+ \frac{1}{\ell}\iint_{\Omega_T} [\Delta v_0(x,t)]^2 \mathrm{d}x\mathrm{d}t + \frac{\tau}{\ell}\int_0^T [\Delta v_2(t)]^2 \mathrm{d}t + \int_0^\theta [\Delta v_1(x)]^2 \mathrm{d}x$$

From (19) with above inequality, we obtain:

$$c_1 \iint_{\Omega_T} \left[\Delta\varphi_x\left(x,t;v\right)\right]^2 \mathrm{d}x\mathrm{d}t + c_2\int_0^T \left[\Delta\varphi\left(\theta,t;v\right)\right]^2 \mathrm{d}t + \int_0^\theta \left[\Delta\varphi\left(x,T;v\right)\right]^2 \mathrm{d}x$$
$$\le \frac{1}{\ell}\iint_{\Omega_T}\left[\Delta v_0\left(x,t\right)\right]^2 \mathrm{d}x\mathrm{d}t + \frac{\tau}{\ell}\int_0^T \left[\Delta v_2\left(t\right)\right]^2 \mathrm{d}t + \int_0^\theta \left[\Delta v_1\left(x\right)\right]^2 \mathrm{d}x \tag{22}$$

where $c_1 = 2\left(\mu_* - \theta^2\ell\right)$ and $c_2 = 2\tau - 2\theta\ell - \tau\ell$, we get bound (18) with $\ell > 0$ for estimate (22):

$$\int_0^\theta \left[\Delta\varphi\left(x,T;v\right)\right]^2 \mathrm{d}x \le \frac{1}{\ell}\iint_{\Omega_T}\left[\Delta v_0\left(x,t\right)\right]^2 \mathrm{d}x\mathrm{d}t + \frac{\tau}{\ell}\int_0^T \left[\Delta v_2\left(t\right)\right]^2 \mathrm{d}t + \int_0^\theta \left[\Delta v_1\left(x\right)\right]^2 \mathrm{d}x$$

Hence, the last integral (15) is bounded by $O\left(\left\|\Delta v\right\|_V^2\right)$ and using Fréchet-differential definition at $v \in V$.

$$\Delta\psi_\alpha\left(v\right) = \left\langle \psi_\alpha'\left(v\right), \Delta v\right\rangle_V + \alpha\int_0^T \left[\Delta v_2\left(t\right)\right]^2 \mathrm{d}t + \gamma\int_0^\theta \left[\Delta\varphi\left(x,T;v\right)\right]^2 \mathrm{d}x$$

we obtain the following theorem:

Theorem 2:

The cost functional $\psi_\alpha\left(v\right) \in C^{1,1}\left(V\right)$ is Fréchet-differentiable in the considered problem hold, and Fréchet derivative at $v \in V$ of $\psi_\alpha\left(v\right)$ can be defined by the solution $\rho \in W_2^{1,0}\left(\Omega_T\right)$ of the adjoint problem (13) as follows:

$$\psi_\alpha'\left(v\right) = \left\{ \rho\left(x,t;v\right), \rho\left(x,0;v\right); \tau\rho\left(\theta,t;v\right)\right\} \tag{23}$$

5. The Continuity of Gradient Functional

In this section, by helping the gradient of cost functional $\psi_\alpha\left(v\right)$ we prove the Lipschitz continuity of $\psi_\alpha'\left(v\right)$. The minimization problem (9) need an estimation of the iteration parameter $\alpha_\tau > 0$ beginning with the initial iteration $v^{(0)} \in V$:

$$v^{(n+1)} = v^{(n)} - \alpha_n \psi_\alpha'\left(v^{(n)}\right), \quad n = 0,1,2,\cdots \tag{24}$$

In many situation estimations of determine the parameter α_τ in various gradient methods is a difficult problem [19]. However, for arbitrary parameters $\delta_0, \delta_1 > 0$, the parameter α_n can be estimated via the Lipschitz constant in the case of Lipschitz continuity of the gradient $\psi_\alpha'\left(v\right)$ as follows:

$$0 < \delta_0 \le \alpha_n \le 2/\left(L + 2\delta_1\right), \tag{25}$$

Lemma 3:

The functional $\psi_\alpha\left(v\right)$ is of Hölder class $C^{1,1}\left(V\right)$ under the conditions of Theorem 2 and

$$\left\|\psi_\alpha'\left(v+\Delta v\right) - \psi_\alpha'\left(v\right)\right\|_V \le L\left\|\Delta v\right\|_V + \left[2c_4 A_\tau \iint_{\Omega_T}\left[\left(\Delta s_1'\left(\varphi\right) + \Delta s_2'\left(\varphi\right)\right)\right]\mathrm{d}x\mathrm{d}t\right]^{1/2} \tag{26}$$

where

$$\left\|\psi_\alpha'\left(v+\Delta v\right) - \psi_\alpha'\left(v\right)\right\|_V^2 = \iint_{\Omega_T}\left(\Delta\rho\left(x,t;v\right)\right)^2 \mathrm{d}x\mathrm{d}t + \tau^2\int_0^T \left(\Delta\rho\left(\theta,t;v\right)\right)^2 \mathrm{d}t$$
$$+ c_4\int_0^\theta \left(\Delta\rho\left(x,0;v\right)\right)^2 \mathrm{d}x \tag{27}$$

where $c_4 = \theta^2/\mu_* + \theta/\tau + \tau/2$ and for parameters c_0, $\ell > 0$, the Lipschitz constant is defined in (22) as follows:

$$L = 2\gamma\sqrt{c_0 c_4/\ell} > 0 \tag{28}$$

Proof: Let the following backward parabolic problem

$$\begin{cases} \dfrac{\partial \Delta\rho(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}\left(\mu(x)\dfrac{\partial \Delta\rho(x,t)}{\partial x}\right) - A_\tau\left[\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right], (x,t) \in \Omega_T \\ \Delta\rho(x,T) = 2\gamma\Delta\varphi(x,T;v) \quad x \in (0,\theta) \\ \dfrac{\partial \Delta\rho(0,t)}{\partial x} = 0, -\mu(\theta)\dfrac{\partial \Delta\rho(\theta,t)}{\partial x} = \tau\Delta\rho(\theta,t), t \in (0,T] \end{cases} \tag{29}$$

has the solution $\Delta\rho(x,t;v) = \rho(x,t;v+\Delta v) - \rho(x,t;v) \in W_2^{1,0}(\Omega_T)$. Therefore, using the initial and boundary conditions after multiplying both sides of Equation (29) by $\Delta\rho(x,t;v)$, and integrating on Ω_T as in the proof of Lemma 2, we can get the following energy identity:

$$\begin{aligned} &\iint_{\Omega_T} \mu(x)\left[\Delta\rho_x(x,t;v)\right]^2 \mathrm{d}x\mathrm{d}t + \tau\int_0^T \left[\Delta\rho(\theta,t;v)\right]^2 \mathrm{d}t + \frac{1}{2}\int_0^\theta \left[\Delta\rho(x,0;v)\right]^2 \mathrm{d}x \\ &= 2\gamma\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2 \mathrm{d}x + A_\tau\iint_{\Omega_T}\left[\left(\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{30}$$

implies the following two inequalities:

$$\begin{aligned} &\mu_*\iint_{\Omega_T}\left[\Delta\rho_x(x,t;v)\right]^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^\theta \left[\Delta\rho(x,0;v)\right]^2 \mathrm{d}x \\ &\le 2\gamma\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2 \mathrm{d}x + A_\tau\iint_{\Omega_T}\left[\left(\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\tau\int_0^T \left[\Delta\rho(\theta,t;v)\right]^2 \mathrm{d}t + \frac{1}{2}\int_0^\theta \left[\Delta\rho(x,0;v)\right]^2 \mathrm{d}x \\ &\le 2\gamma\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2 \mathrm{d}x + A_\tau\iint_{\Omega_T}\left[\left(\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{32}$$

Multiplying the first and the second inequality by $2\theta^2/\mu_*$ and $2\theta/\tau$, correspondingly, summing up them, and then using the inequality (20) we obtain:

$$\begin{aligned} &\iint_{\Omega_T}\left[\Delta\rho(x,t;v)\right]^2 \mathrm{d}x\mathrm{d}t + \left(\theta^2/\mu_* + \theta/\tau\right)\int_0^\theta \left[\Delta\rho(x,0;v)\right]^2 \mathrm{d}x \\ &\le 4\gamma^2\left(\frac{\theta^2}{\mu_*} + \frac{\theta}{\tau}\right)\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2 \mathrm{d}x + 2A_\tau\left(\frac{\theta^2}{\mu_*} + \frac{\theta}{\tau}\right)\iint_{\Omega_T}\left[\left(\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{33}$$

Computing of the second integral on the right-hand side of (26) by the same term. From the energy identity (30) we can obtain the following:

$$\begin{aligned} &\tau^2\int_0^T \left[\Delta\rho(\theta,t;v)\right]^2 \mathrm{d}t + \frac{\tau}{2}\int_0^\theta \left[\Delta\rho(x,0;v)\right]^2 \mathrm{d}x \\ &\le 2\gamma^2\tau\int_0^\theta \left[\Delta\varphi(x,T;v)\right]^2 \mathrm{d}x + A_\tau\iint_{\Omega_T}\left[\left(\Delta s_1'(\varphi) + \Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{34}$$

This, with the last estimate, concludes

$$\begin{aligned}&\frac{1}{c_4}\iint\limits_{\Omega_T}\left[\Delta\rho(x,t;v)\right]^2\mathrm{d}x\mathrm{d}t+\frac{\tau^2}{c_4}\int_0^T\left[\Delta\rho(\theta,t;v)\right]^2\mathrm{d}t+\int_0^\theta\left[\Delta\rho(x,0;v)\right]^2\mathrm{d}x\\&\le 4\gamma^2\int_0^\theta\left[\Delta\varphi(x,T;v)\right]^2\mathrm{d}x+2A_\tau\iint\limits_{\Omega_T}\left[\left(\Delta s_1'(\varphi)+\Delta s_2'(\varphi)\right)\right]\mathrm{d}x\mathrm{d}t\end{aligned}\tag{35}$$

where $c_4=\theta^2/\mu_*+\theta/\tau+\tau/2$, using this in (27) and taking into account Lemma 2 we obtain (26) with the Lipschitz constant L in (28).

6. Conclusion

In this paper, we studied a class of the constrained OCP for parabolic systems. The existence and uniqueness of the system is introduced. In this way, the uniqueness theorem for the solving POCP is introduced. Therefore, a theorem for the sufficient differentiability conditions has been proved. By using the exterior penalty function method, the constrained problem is converted to new unconstrained OCP. The common techniques of constructing the gradient of the cost functional using the solving of the adjoint problem is investigated.

References

[1] Henry, D. (1981) Geometric Theory of Semi Linear Parabolic Equations. *Lecture Notes in Mathematics*, Vol. 840, Springer-Verlag, Berlin. https://doi.org/10.1007/BFb0089647

[2] Liu, H. and Yang, J. (2014) Optimal Control of Semi Linear Parabolic Systems with State Constraint. *Journal of Mathematical Analysis and Applications*, **417**, 787-803. https://doi.org/10.1016/j.jmaa.2014.03.070

[3] Guliyev, H.F. and Jabbarova, K.H. (2011) On an Optimal Control Problem for Weakly Nonlinear Hyperbolic Equations. *Acta Mathematica Hungarica*, **131**, 197-207. https://doi.org/10.1007/s10474-011-0074-6

[4] Tagiev, R.K. (2009) Optimal Coefficient Control in Parabolic Systems. *Differential Equations*, **45**, 1526-1535. https://doi.org/10.1134/S0012266109100164

[5] Berhe, H.W. (2012) Penalty Function Methods Using Matrix Laboratory (MATLAB). *African Journal of Mathematics and Computer Science Research*, **5**, 209-246.

[6] De Sampaio, R.J.B., Wollmann, R.R.G., Yuan, J.Y. and Favaretto, F. (2014) Using Penalty in Mathematical Decomposition for Production-Planning to Accommodate Clearing Function Constraints of Capacity, Optimization and Control Techniques and Applications. *Springer Proceedings in Mathematics & Statistics*, **86**, 137-152. http://www.springer.com/series/10533

[7] Farag, M.H., Nofal, T.A., El-Sayed, M.A. and Al-Baqmi, N.M. (2015) Computation of Optimal Controls for a Class of Distributed Parameter Systems. *Journal of Computer and Mathematical Sciences*, **6**, 439-448.

[8] Farag, M.H., Nofal, T.A., El-Sayed, M.A. and AL-Qarni, T.M. (2015) An Optimization Problem with Controls in the Coefficients of Parabolic Equations. *British Journal of Mathematics & Computer Science*, **6**, 532-543.

[9] Razzaghi, M. and Marzban, H.R. (2002) Optimal Control of Singular Systems via Piecewise Linear Polynomial Functions. *Mathematical Methods in the Applied Sciences*, **25**, 399-408. https://doi.org/10.1002/mma.297

[10] Rao, S.S. (2009) Engineering Optimization: Theory and Practice. 4th Edition, John Wiley & Sons, Inc. https://doi.org/10.1002/9780470549124

[11] Luenberger, D.G. (1974) A Combined Penalty Function and Gradient Projection

Method for Nonlinear Programming. *Journal of Optimization Theory and Applications*, **14**, 477-495. https://doi.org/10.1007/BF00932843

[12] Hasanov, A. (2007) Simultaneous Determination of Source Terms in a Linear Parabolic Problem from the Final over Determination Weak Solution Approach. *Journal of Mathematical Analysis and Application*, **330**, 766-779. https://doi.org/10.1016/j.jmaa.2006.08.018

[13] Aliay, M., Golalikhanib, M., Mohsen, M. and Zhuang, J. (2012) A Computational Study on Different Penalty Approaches for Solving Constrained Global Optimization Problems with the Electromagnetism-Like Method. *Optimization*, **63**, 403-419. https://doi.org/10.1080/02331934.2012.655691

[14] Ladyzhenskaya, O.A. (1985) The Boundary Value Problems in Mathematical Physics. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4757-4317-3

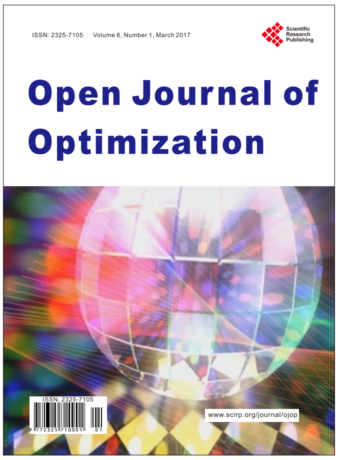
[15] Ladyzhenskaya, O.A., Solonnikov, V.A. and Ural'tseva, N.N. (1976) Linear and Quasilinear Parabolic Equations. Nauka, Moscow.

[16] Mikhailov, V.P. (1983) Partial Differential Equations. Nauka, Moscow.

[17] Gelashvili, K. (2011) The Existence of Optimal Control on the Basis of Weierstrass's Theorem. *Journal of Mathematical Sciences*, **177**, 373-382. https://doi.org/10.1007/s10958-011-0463-y

[18] Lions, J.L. (1968) Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Dunod Gauthier-Villars, Paris.

[19] Koppel, L.B. (1984) Introduction to Control Theory with Applications to Process Control. Prentice-Hall, Englewood Cliffs.



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