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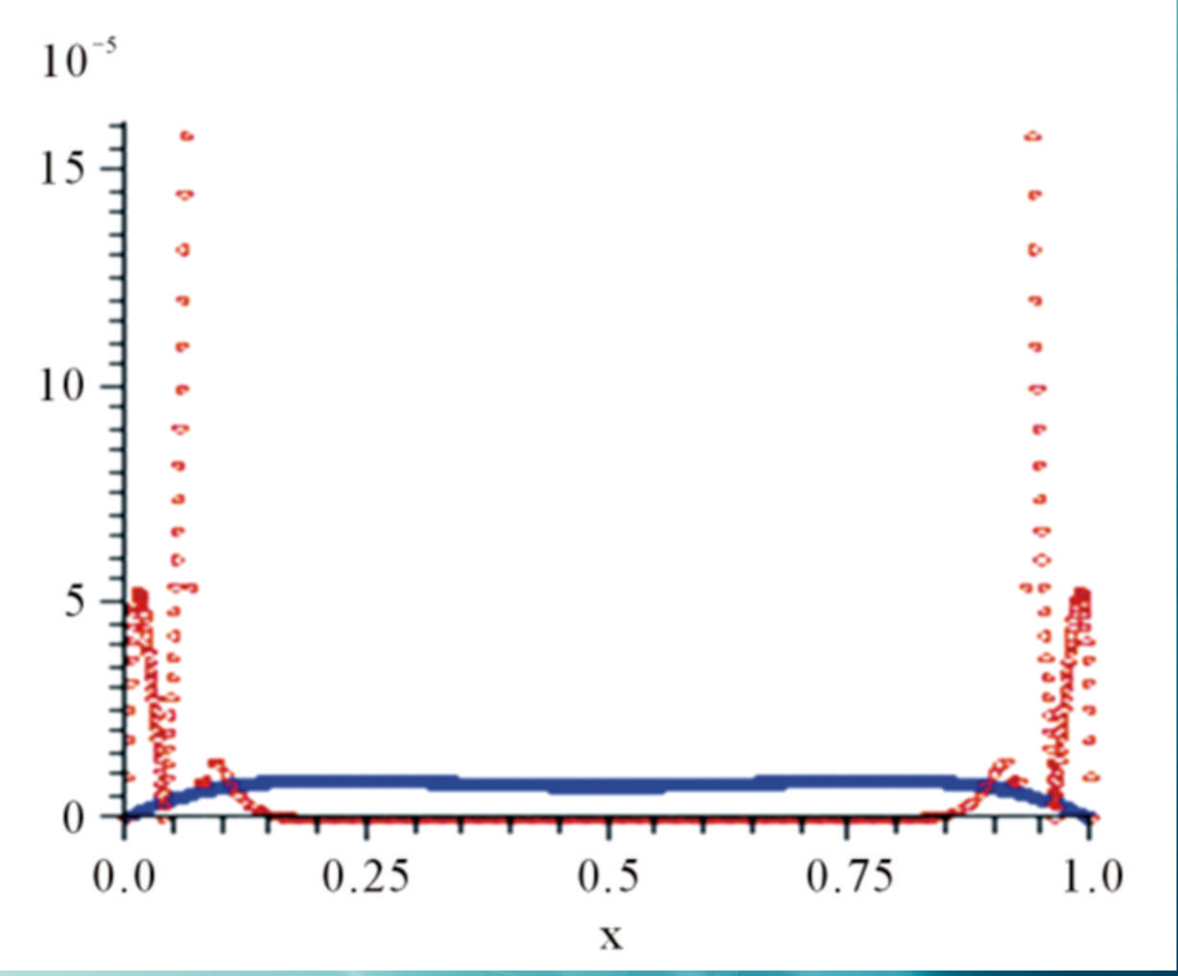

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On the Stringy Ghosts Which We Call the Missing Dark Energy of the Cosmos

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Abstract

Dark energy is explained using familiar notions and concepts used in quantum field theory, string theory and the exact mathematical theory of spacetime. The main result of the present work is first a new mathematical definition of pre-quantum spacetime (QST) as a multiset made of infinitely many empty Cantor sets connected to pre-quantum wave empty set (QW) and the pre-quantum particle (QP) zero set via the cobordism equation $\partial(QW) = (QP)U(QST)$. Second, and in turn, this new path of reasoning is used to validate the quantum splitting of Einstein's $E = mc^2$ into the sum of the ordinary energy $E = mc^2/22$ of the quantum particle and the dark energy $E = mc^2(21/22)$ of the quantum wave, used predominantly to explain the observed accelerated expansion of the universe.

Keywords

Quantum Spacetime, Quantum Ghost States, Dark Energy, Spacetime Cobordism, E-Infinity Theory, Fractal-Cantorian Spacetime, Noncommutative Geometry, 't Hooft-Susskind Holography

1. Introduction and Motivation

The theme of the present scientific essay is aptly captured by a remark ascribed to Wolfgang E. Pauli "*God made the bulk but the surface was invented by the devil*" [1]. It is evident that when the scale tends to an extremely small quantum and fractal scale [1]-[7], then the surface to volume ratio grows extremely large and the boundary effect dominates over the volume effect and consequently almost all of latent "space" energy is found located at the boundary [1] [2]. In anticipation of the main result of the present work, we can say that what is called dark energy [8]-[29] is closely related to this boundary energy but with a twist. Suppose now the universe is a single random version of the classical triadic Cantor set [3]-[6]. Given this theoretical toy model then at the limit when we have only Cantor set points separated by "nothing" as their "boundary", almost all the energy will be located in this "nothing" because this nothing is the surface of the vanishing Cantor "dust" [7]-[31]. What we just explained could be paraphrased in various equivalent languages and corresponding mental pictures. For instance we know that the unit interval is the simplest example of a cobordism [32]-[38]. Consequently by constructing a random triadic Cantor set on and in this interval we have de facto advanced a cobordism spacetime theory in one dimension harbouring two sets [8]-[17]. The first will be a substance like Cantor dust with a Hausdorff dimension equal to $\phi = \left(\sqrt{5}-1\right)/2$ and zero topological dimension [8]-[32]. This will be shown later to give rise to the pre-quantum particles and the associated ordinary energy. The second set is a spacetime-like non-substance fat Cantor set with a Hausdorff dimension $1-\phi = \phi^2$ and a topological dimension minus one (-1) indicating that it is essentially an empty set [7]-[9]. This will be shown later to give rise to the quantum wave which harbours what has been dubbed the missing dark energy of the cosmos [9]-[29]. From the preceding description even a reader with no advanced mathematical knowledge in topology and cobordism theory has probably got the right mental picture already, namely that the quantum wave empty set is the "border" or surface of the quantum particle zero set [7] [8]. One step is still missing to come to the quantitative result that confirms what is transpiring from the preceding toy model of our universe, namely that the "boundary" energy is nothing but the energy of the quantum wave and that is in fact what we call dark energy. This step is the following realization gained from the cobordism theory of spacetime [33] which simply put, means that Einstein's $D = 4$ spacetime is cobodent to Kaluza-Klein spacetime $D = 5$. Remembering that for a simple two-dimensional square area A is multiplicative $A = a^2$ while the length of the four border lines L is additive $L = 4a$ then generalizing to five dimensions we find that the quantum particle quasi volume must be ϕ^5 while the quantum wave quasi volume must be $5\phi^2$ [8]-[31].

In the present work we will give the details leading to a conclusion based on the above that the ordinary energy density of the cosmos is given by $E(O) = mc^2\left(\phi^5/2\right)$ while the dark energy density is simply the compliment of $E(O)$ and is given by $E(D) = mc^2\left(5\phi^2/2\right)$ [8]-[31]. Furthermore we show that this is actually a consequence of the measure concentration theorem of Dvoretzky and may be explained within conventional quantum field theory via string theory as being a consequence of *ad hoc* states referred to as zero norms and ghost states without realizing the pure mathematical origins of these states as explained here [28]-[32] which leads to the dissection of Einstein's $E = mc^2$ to $E = \left(mc^2/22\right) + mc^2\left(21/22\right)$ where $mc^2\left(21/22\right)$ is the 95.5% dark energy density of the cosmos. We were faced in the present work with the usual problem of wanting to make the work self contained while keeping the length of the paper as short as possible. That dilemma was solved as usual by including a large number of references [1]-[99].

2. An Ionescoian Scene from the Absurd Scientific Theatre of Reality

Waiter! There is a ghost in my soup. You see 95.5 percent of all the spoons I take to my mouth from this circular soup bowl you brought me turns out to be empty as soon as it reaches my mouth. No matter how hard I try, there is simply no soup in my spoon. It is both ridiculous and incredible. I counted accurately and only 4.5 percent of the spoons had soup in them which I could see, feels its temperature and taste. Yes of course there is the possibility that the soup bowl is empty, I mean almost empty because the capillary forces of the boundary of the bowl are attracting the soup there, leaving only 4.5% of the soup in the middle and that what my spoon is taking into my mouth… however this seems also ridiculous because there is no space to harbour at the boundary 95.5 percent of the soup… oh yes and on reflection there is again the possibility of a hyperbolic soup bowl so that the boundary is really far away at infinity and we have infinite hidden space where the 95.5 percent soup could be comfortably hiding at the finite infinity…

There is no doubt this absurd dialogue or more accurately internal monologue reminiscent of the writings of Eugene Ionesco or Samuel Beckett could not have taken place in an Italian or any other restaurant but in all earnest something similar has taken place and is still vividly discussed in cutting edge scientific research in quantum cosmology in connection with the question of the missing dark energy density of the cosmos [10]-[21].

The present short paper is intended to shed a relatively new light on the subject of the "mystery" of the "missing" dark energy. We will try to convey these advanced cutting edge scientific research results with a minimum of technical jargon and whenever this is unavoidable, we will attempt to explain these terms in a down to earth manner. Now that is nothing new and in fact it is the standard way when explaining a difficult subject to a wider scientifically educated but not specialized reader. However in the present context this issue of the language used is almost the crux of the matter. In a nutshell, the problem is that while Google translation with all its weaknesses has made a plethora of different languages accessible to almost all inhabitants of planet earth, something similar for science is still lacking. The scientific community is largely divided by private languages. To be more specific and to the point, the mystery of dark energy is in our opinion, no mystery if we accept as we do here that negative probabilities, phantoms, negative norms and ghosts as those familiar from quantum chromo dynamics as well as the empty set of the Menger-Urysohn dimensional theory means more or less the same thing and are what makes dark energy be called dark. This is our starting point and almost also our conclusion and end point. The rest of this article will be devoted to explaining in more detail what we have just said in considerable economy of thoughts.

3. The Fat Cantor Ghosts as Dark Energy

In describing a Cantor set, that is a zero length measure clopen interval, which is to say a simultaneously open and closed unit interval that possesses a positive value dimension (fractal-Hausdorff dimension), the word ghost crosses the mind and was in fact used by the author on numerous previous occasions [56]. How else could we describe something which has a length equal zero so that it is physically not in our "world" and yet has a substantial dimension, albeit Hausdorff dimension amounting to $\ell n2/\ell n3 \simeq 0.63$ for a deterministic Cantor set and $\phi = \left(\sqrt{5}-1\right)/2$ = 0.618033989 for a randomly constructed triadic Cantor set which means it is somehow connected to our world. At the time we encountered this mysterious and paradoxical nature of a Cantor set and utilized it for problems in quantum physics and relativity theory, we were not aware that this same term was in fact already in use for certain fundamental questions central to quantum field theory and in particular to string quantization and QCD [32]. The word ghost is used there to refer to norms which on its face value should not be part of "reality", ergo it should not be part of the "physical" Hilbert space. However these norms are there and not there for logical-mathematical reasons. This paradoxical situation was then helped by developing mathematical manoeuvres to have one's cake and get rid of it simultaneously [85]-[94]. In this respect the situation is rather reminiscent of our Cantor set and far more to the surface of a Cantor set which is a second even more puzzling Cantor set possessing a non-zero measure on the one side and a negative topological dimension, a so called Menger-Urysohn dimension on the other side. Such sets are well known in set theory and the theory of dimensions under the name of "empty" set. It took some time for the author to realize the obvious although, or maybe because it is very obvious to see at once. The language of the zero and empty set used by the author in his E-infinity theory is a pure mathematical tautology of the theory of zero and negative ghosts of Nambu-Gotto strings and its Gulpta-Bleuler quantization. By analogy to our previous result it becomes clear that ghosts are the origin of dark energy and that the rough density ratio rooted in the critical dimensions D_1 = 26 and D_2 = 25 may be found from the reduced dimensions [8]-[32]

$$\begin{aligned}\gamma(D) &= \frac{25-D^{(4)}}{26-D^{(4)}} \\ &= \frac{21}{22}\end{aligned} \tag{1}$$

or alternatively

$$\begin{aligned}\gamma(O) &= \frac{26-D^{(4)}}{|E8E8|-|SM|} \\ &= \frac{26-4}{496-12} \\ &= \frac{22}{484} \\ &= \frac{1}{22}\end{aligned} \tag{2}$$

so that we have again $\gamma(D) = 1-(1/22) = 21/22$. Assuming $E = mc^2$ to be not merely the maximal energy convergence possible but logically as well as intuitively the maximal average energy density of the universe we see that $\gamma_{\max} = \gamma = 1$ while $\gamma(O) = 1/22$ and $\gamma(D) = 21/22$ would imply that $E(O) = mc^2/22$ and $E(D) = mc^2(21/22)$ which is in hard to believe accuracy agreement with the overwhelming actual cosmic measurements, particularly COBE, WMAP and Type 1a supernova as well as Hubble and Plank projects [8]-[22].

4. Phantom in Hyperspace

Phantoms are supposed to be much stronger driving forces for cosmic expansion than the quintessence generalization of Einstein's cosmological constant. It is essentially negative energy and as such a radical proposal with no parallel in mainstream physics except other radical and related proposals such as negative gravity and the like. However if we pause for a second and ponder the quantum wave as viewed by E-infinity theory [8]-[29], then we realize that it is modelled by the empty set which is phantom enough in addition to being physically there via the Hausdorff dimension of the empty set (ϕ^2) and moreover it has its ties with the negative domain because the empty set, ergo the wave has the remarkable topological dimension minus one so that the combined von Neumann-Connes extended dimension reads $D(-1) \equiv (-1, \phi^2)$ in the noncommutative geometry theory notation or equivalently using the bijection formula $D_c^{-1} = (1/\phi)^{-2} = \phi^2$ in the fractal-Cantorian spacetime theory notation. Thus loosely speaking ghosts, negative norms as well as phantoms are mathematically fuzzy tautological equivalences of empty set, quantum wave and so on as explained in great detail in many publications by various authors where it was reasoned that in a *D*-5 Kaluza-Klein hyperspace of which Einstein's $D = 4$ is the cobordism [33]-[38], dark energy density is simply five times ϕ^2 divided by 2 leading to $E(D) = (5\phi^2/2)mc^2 \cong mc^2(21/22)$.

5. Almost Pitfall Free Approach

Now that we are coming to the end of the present work, a few words on the pitfalls and possible confusion are in order at this point. There are many different notions of ghosts in physics. There are for instance good ghosts and consequently bad ghosts. The archetypical good ghost is the Faddeev-Popov ghost. By contrast negative norms are considered bad ghosts. In addition there are subtle exceptions such as the ghosts associated with the Feynman-'t Hooft gauge theory and the mass of the Higgs. There are anti-commuting ghosts and ghost field Lagrangian [90]. All these more or less special cases and *ad hoc* techniques may easily be avoided by realizing the unifying nature of the present proposal of defining the quantum wave as the empty set and that the corresponding energy is a negative one which was long dubbed dark energy while not knowing that it is the same thing as the energy of the quantum wave. In view of the above the reader can appreciate the benefit of the clarity of using the sharp terms of the language of pure mathematics [8].

6. Renormalization, Fractal Spacetime and Dark Energy

Before ending our analysis, there is one more vital point left which is quite central to understanding the present approach. The essence of this point is the basic equivalence between the move from the energy level of electricity, magnetism and the weak force to that of the electroweak unification or more generally, all higher energy scales connected to electro chromodynamics and GUT and phase transition. Accepting this view we see that such energy scale transition corresponds to phase space transition expressed in a transition from smooth $D = 4$ classical and special relativistic spacetime to a first stage Cantorian-fractal spacetime dimensionality. This important dimensional transmutation is deeply linked to 't Hooft's dimensional regularization in its various forms used by G. 't Hooft [91] [98], M. Veltman and K. Wilson and characterized by $D = 4 - \in$ where it is shown by various authors that $\in$ may take the values discussed in the literature. However extensive analysis has shown that $\in = \phi^3(1-\phi^3)$ is twice Hardy's quantum entanglement probability ϕ^5 where $\phi = (\sqrt{5}-1)/2$. The reader may already have guessed where we are heading. The conjecture that dark energy is the energy of the ghost-like empty set leads directly to the exact result, namely the ratio of the first fractal dimension phase transition to the smooth dimension. That means [17]

$$\begin{aligned}\gamma(D) &= \frac{D-\in}{4} \\ &= \frac{4-k}{4} \\ &\cong 95.5\%\end{aligned}$$

in full agreement with cosmic measurements of COBE, WMAP and Type 1a supernova [7]-[12].

7. Discussion

Set theory in general and notions like the zero and empty set in particular [8]-[31] [39]-[55] is not part of the traditional mathematical training of physics. For such and other historical reasons when facing deep and foundational problems in quantum physics and quantum cosmology, research in these fields had naturally the needed ingenuity to reinvent the mathematical theory needed but this happened of course using different names and jargon [85]-[91]. As harmless as this may be, it did cause some confusion as documented in the present work where it became evident that dark energy did not come out of the blue. That way we discovered that we do not need to add any new notion to understand dark energy and instead we found that we are basically talking about the same things using different names and sometimes different languages ranging from that of pure mathematics and set theory to that of modern quantum field theory and the negative norms and ghosts of superstrings. The author is reminded in this context of what Ludwig Wittgenstein wrote in his Tractatus Logico-Philosphicus under paragraph No. 4.0031 "*All philosophy is a* '*critique of language*'." We just need here to change the words philosophy to natural philosophy and language to mathematics in the proposition of Wittgenstein to make it of universal validity in theoretical and mathematical physics [95].

8. Conclusions

In the present work we made some important insights into the nature of space, time and energy:

1) The zero set may be regarded as the quintessence of the quantum particle.

2) The empty set is the quintessence of the quantum wave.

3) Spacetime is a multi-set (multi-fractal) made of infinitely many empty sets with increasing degrees of emptiness.

4) The zero set quantum particle and the multi-set spacetime are cobordant via the empty set quantum wave. These pure mathematical objects correspond to negative norms and ghosts of string theory.

5) The relation between the quantum wave, the quantum particle and the spacetime manifolds could be roughly expressed symbolically in the following notation of the theory of cobordism [33]-[38]

$$\partial(QW) = (QP)U(QST)$$

where QW is the quantum wave, QP is the quantum particle and ST is spacetime [8] [32].

6) Einstein's $E = mc^2$ becomes far more illuminating when written in the dissected two quantum components $E = E(O) + E(D)$ for where $E(O) = mc^2/22$ is the ordinary energy density and $E = mc^2(21/22)$ is the dark energy density of the stringy ghosts [8] [32].

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The Physical Meaning of the Wave Function

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Abstract

Under some well-defined conditions, the mathematical formalism of quantum mechanics enables physicists, chemists and others to calculate and predict the outcome of a vast number of experiments. In fact, especially the Schrödinger equation which involves an imaginary quantity describes how a quantum state of a physical system changes with time and is one of the main pillars of modern quantum mechanics. The wave function itself is a determining part of the Schrödinger equation, but the physical meaning of the wave function is still not clear. Altogether, does the wave function represent a new kind of reality? This publication will solve the problem of the physical meaning of the wave function by investigating the relationship between the wave function and the theory of special relativity. It is shown that the wave function is determined by notion co-ordinate time of the special theory of relativity. Moreover, the result of this investigation suggests a new understanding of the wave function, according to which the wave function and co-ordinate time of the theory of special relativity are equivalent.

Keywords

Quantum Theory, Relativity Theory, Unified Field Theory, Causality

1. Introduction

The Schrödinger equation, published in 1926 by the Austrian physicist Erwin Rudolf Josef Alexander Schrödinger (1887-1961), is determined by Newton's (1642-1727) second law, in its original form known as "Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur" [1] and to some extent an analogue of Newton's second law in quantum mechanics. Leonhard Euler (1707-1783), a pioneering Swiss mathematician and physicist, formulated in 1752 Newton lexsecunda [2] in its mathematical form as

$$ {}_{O}\vec{F} = {}_{O}m \times {}_{O}\vec{a} \,. \tag{1}$$

The famous Schrödinger equation [3], a partial differential equation, describes how a quantum state of a system changes with time. The Schrödinger equation for any system, no matter whether relativistic or not, no matter how complicated, has the form

$$_{R}H \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = i\hbar \frac{\partial}{\partial t} {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \tag{2}$$

where i is the imaginary unit, $\hbar = \frac{h}{2 \times \pi}$ is Planck's constant divided by $2 \times \pi$, the symbol $\frac{\partial}{\partial t}$ indicates a partial derivative with respect to time $_{R}t$, $_{R}\Psi({}_{R}X, {}_{R}t)$ is the wave function of the quantum system, and $_{R}H$ is the Hamiltonian operator. In quantum mechanics, the Hamiltonian operator is a quantum mechanical operator which characterizes the total energy of a quantum mechanical system and is usually denoted by $_{R}H$. The Hamiltonian operator $_{R}H$ takes different forms depending upon situation.

The form of the Schrödinger equation itself depends on the physical situation and is determined by the wavefunction $_{R}\Psi({}_{R}X, {}_{R}t)$. The wave function itself is one of the most fundamental concepts of quantum mechanics. Schrödinger himself states that he has "*not attached a definite physical meaning to the wavefunction…*" [3].

The physical meaning of the wave function is in dispute in the alternative interpretations of quantum mechanics. The de Broglie-Bohm theory or the many-worlds interpretation has another view on the physical meaning of the wave function then the Copenhagen interpretation of the wave function. In view of this very unsatisfactory situation, it seems to be necessary to put some light on the problem of the physical meaning of the wave function from the standpoint of the theory of special relativity. In a similar way, hereafter, we shall restrict ourselves to a one-dimensional treatment in order to decrease the amount of notation needed, since in all cases, the generalization to four (*i.e.* quantum mechanics) or n-dimensions (*i.e.* quantum field theory) will be equally simple.

2. Material and Methods

2.1. Definitions

Definition: Proof by contradiction (Reductio ad Absurdum)

The logical background of a proof by contradiction is Aristotle's law of non-contradiction. A rigorous proof by contradiction of a theorem follows the standard method of contradiction used in science and mathematics and should be convincing as much as possible. For the first, we assume that a claim/a theorem/a proposition/a statement et cetera which has to be proved, is true. One then proceeds to demonstrate that a conclusion drawn from such a claim/a theorem/a proposition/a statement et cetera leads to a contradiction. Hence, the supposed claim/theorem/proposition/statement et cetera is deemed to be false.

Consequently, we are then led to conclude that it was wrong to assume the claim/the theorem/the proposition/the statement was true. Thus far, the claim/the theorem/the proposition/the statement is proved to be false.

Definition: Thought experiments

Properly constructed (real or) thought experiments (as devices of scientific investigation) can be used for diverse reasons in a variety of areas. Thought experiments can help us to investigate some basic properties of nature even under conditions when it is too difficult or too expensive to run a real experiment. Furthermore, a thought experiment can provide some evidence against or in favour of a theory. However, a thought experiment is not a substitute for a real experiment.

2.1.1. Newton's Axiomata sive leges motus

Newton's laws of motion, verified by experiment and observation over centuries, are valid in inertial reference frames. However, any reference frame moving in uniform motion with respect to an inertial frame is an inertial frame too. In quantum mechanics concepts such as force, momentum, and position are defined as linear operators which operate on a quantum state. Newton's three physical laws, the foundation of classical mechanics, have been expressed in several different ways and can be summarized as follows.

Definition: Newton's first law

"Lex. I. Corpus omne perseverare in statu suo quiescendi vel movendi uniformmiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare" [1].

Newton's first translated to English reads:

"Law. I. Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed."

Newton's first law first law states that if the net force (a vector sum of all forces acting on something) is zero, then the velocity, a vector (speed and direction of motion) quantity, of this something is constant. Newton's first law can be stated mathematically as

$$\sum_{t=0}^{n} {}_{t}\overrightarrow{F} = 0 . \tag{3}$$

Definition: Newton's second law

Newton defined his second law as:

"Lex. II. Mutationem motuspro portionalem esse vi motrici impressae, & fieri se-cundum lineam rectam qua vis illa imprimitur" [1].

Newton's second translated quite closely to English reads:

"Law. II. The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed."

In general, Newton's second law is stated mathematically as

$$\overrightarrow{{}_{R}F} = {}_{R}m \times \overrightarrow{{}_{R}a} . \tag{4}$$

Thus far, something at rest will stay at rest unless a net force acts upon it and equally something in motion will stay in motion (will not change its velocity) unless a net force acts upon it. In other words, if something is accelerating, then there must be a net force acting on it.

Definition: Newton's third law

Newton defined his own third law as:

"Lex. III. Actioni contrariam semper et aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi" [1].

Newton's third law (*actio* = *reactio*) translated to English reads:

"Law. III. To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

The two forces, equal but opposite with ${}_{O}F$ called the "action" and ${}_{R}F$ the "reaction" are part of every single interaction. It does not matter which force is called the reaction and which force is called the action. Newton's third law is stated mathematically as

$$\overrightarrow{{}_{O}F} = -\overleftarrow{{}_{R}F} . \tag{5}$$

2.1.2. Special Theory of Relativity

Definition: The relativistic energy ${}_{R}E$ (of a system)

In general, it is

$${}_{R}E = {}_{R}m \times c^{2} \tag{6}$$

where ${}_{R}E$ denotes the total ("relativistic") energy of a system, ${}_{R}m$ denotes the "relativistic" mass and c denotes the speed of the light in vacuum.

Scholium.

Einstein defined the matter/mass-energy equivalence as follows:

"Gibt ein Körper die Energie L in Form von Strahlung ab, so verkleinert sich seine Masse um L/V^2… Die Masse eines Körpers ist ein Maß für dessen Energieinhalt" [4].

In other words, due to Einstein, energy and mass are equivalent.

"Eines der wichtigsten Resultate der Relativitätstheorie ist die Erkenntnis, daß jegliche Energie E eine ihr proportionale Trägheit (E/c^2) besitzt" [5].

It is equally correct by Einstein to point out, matter/mass and energy are equivalent.

"Da Masse und Energie nach den Ergebnissen der speziellen Relativitätstheorie das Gleiche sind und die Energie formal durch den symmetrischen Energietensor ($T_{\mu\nu}$) beschrieben wird, so besagt dies, daß das G-Geld [gravitational field, *author*] durch den Energietensor der Materie bedingt und bestimmt ist" [6].

The term *relativistic mass* ${}_{R}m$ was coined by Gilbert and Tolman [7].

Definition: Einstein's mass-energy equivalence relation

The Einsteinian matter/mass-energy equivalence [4] lies at the core of today physics. In general, due to Einstein's special theory of relativity it is

$$ {}_{O}m = {}_{R}m \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{7} $$

or equally

$$ {}_{O}E \equiv {}_{O}m \times c^2 \equiv {}_{R}m \times c^2 \times \sqrt[2]{1-\frac{v^2}{c^2}} \equiv {}_{R}E \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{8} $$

or equally

$$ \frac{{}_{O}E}{{}_{R}E} = \frac{{}_{O}m \times c^2}{{}_{R}m \times c^2} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{9} $$

where ${}_{O}E$ denotes the "rest" energy, ${}_{O}$m denotes the "rest" mass, ${}_{R}E$ denotes the "relativistic" energy, ${}_{R}m$ denotes the "relativistic" mass, v denotes the relative velocity between the two observers and c denotes the speed of light in vacuum.

Scholium.

The constancy of the speed of the light c is something relative and nothing absolute. If we follow Einstein, the constancy of the speed of the light c is determined by a constant gravitational potential. "Dagegen bin ich der Ansicht, daß das Prinzip der Konstanz der Lichtgeschwindigkeit sich nur insoweit aufrecht erhalten läßt, als man sich auf raum-zeitliche Gebiete von konstantem Gravitationspotential beschränkt. Hier liegt nach meiner Meinung die Grenze der Gültigkeit… des Prinzips der Konstanz der Lichtgeschwindigkeit und damit unserer heutigen Relativitätstheorie" [5].

Definition: Normalized relativistic energy-momentum relation

The normalized relativistic energy momentum relation [8], a probability theory consistent formulation of Einstein's energy momentum relation, is determined as

$$ \frac{{}_{O}m^2}{{}_{R}m^2} + \frac{v^2}{c^2} = 1 \tag{10} $$

while the "particle-wave-dualism" [8] is determined as

$$ \frac{{}_{O}m^2}{{}_{R}m^2} + \frac{v^2}{c^2} \equiv \frac{{}_{O}m^2 \times c^2 \times c^2}{{}_{R}m^2 \times c^2 \times c^2} + \frac{v^2 \times {}_{R}m^2 \times c^2}{c^2 \times {}_{R}m^2 \times c^2} \equiv \frac{{}_{O}E^2}{{}_{R}E^2} + \frac{{}_{R}p^2 \times c^2}{{}_{R}E^2} \equiv \frac{{}_{O}E^2}{{}_{R}E^2} + \frac{{}_{W}E^2}{{}_{R}E^2} \equiv 1 \tag{11} $$

where ${}_{W}E = ({}_{R}p \times c)$ denotes the energy of an electro-magnetic wave and ${}_{R}p$ denotes the "relativistic" momentum while c is the speed of the light in vacuum.

Definition: The relativistic potential energy

Following Einstein in his path of thoughts, we *define* the relativistic potential energy ${}_{p}E$ [8] as

$$ {}_{P}E \equiv \frac{{}_{O}E \times {}_{O}E}{{}_{R}E} \equiv \frac{{}_{O}E}{{}_{R}E} \times {}_{O}E \equiv \sqrt[2]{1-\frac{v^2}{c^2}} \times {}_{O}E \tag{12} $$

Scholium.

The definition of the relativistic potential energy ${}_{p}E$ is supported by Einstein's publication in 1907. Einstein himself demands that there is something like a relativistic potential energy.

"Jeglicher Energie E kommt also im Gravitationsfelde eine Energie der Lage zu, die ebenso groß ist, wie die Energie der Lage einer 'ponderablen' Masse von der Größe E/c^2" [9].

Translated into English:

"Thus, to each energy E in the gravitational field there corresponds an energy of position that equals the potential energy of a 'ponderable' mass of magnitude E/c^2."

The relativistic potential energy ${}_{p}E$ can be viewed as the energy which is determined by an observer P which is at rest relative to the relativistic potential energy. The observer which is at rest relative to the relativistic potential energy will measure its own time, the relativistic potential time pt.

Definition: The relativistic kinetic energy (the "vis viva")

The relativistic kinetic energy $_{K}\boldsymbol{E}$ is defined [8] in general as

$$_{K}E \equiv \frac{_{W}E \times _{W}E}{_{R}E} \equiv \frac{_{R}m \times v \times c_{R} \times _{R}m \times v \times c}{_{R}m \times c^{2}} \equiv _{R}p \times v \equiv _{R}m \times v^{2} \tag{13}$$

where $_{R}m$ denotes the "relativistic mass" and v denotes the relative velocity. In general, it is

$$_{R}E \equiv _{R}H \equiv _{P}E + _{K}E \equiv _{P}H + _{K}H \tag{14}$$

where $_{P}E$ denotes the relativistic potential energy, $_{K}E$ denotes the relativistic kinetic energy, $_{P}H$ denotes the Hamiltonian of the relativistic potential energy, $_{K}H$ denotes the Hamiltonian of the relativistic kinetic energy. Multiplying this equation by the wave function $_{R}\Psi$, we obtain a relativity consistent form of Schrödinger's equation as

$$_{R}E \times _{R}\Psi \equiv _{R}H \times _{R}\Psi \equiv \left(_{P}E \times _{R}\Psi\right) + \left(_{K}E \times _{R}\Psi\right) \equiv \left(_{P}H \times _{R}\Psi\right) + \left(_{K}H \times _{R}\Psi\right). \tag{15}$$

Scholium.

The historical background of the relativistic kinetic energy $_{K}E$ is backgrounded by the long lasting and very famous dispute between Leibniz (1646-1716) and Newton (1642-1726). In fact, the core of this controversy was the dispute about the question, what is preserved through changes. Leibniz himself claimed, that "*vis viva*" [10], [11] or the relativistic kineticenergy $_{K}E = _{R}m \times v \times v$ was preserved through changes. In contrast to Leibniz, Newton was of the opinion that the momentum $_{R}p = _{R}m \times v$ was preserved through changes. The observer which is at rest relative to the relativistic kinetic energy will measure its own time, the relativistic kinetic time $_{k}t$.

Definition: Einstein's Relativistic Time Dilation Relation

An accurate clock in motion slows down with respect a stationary observer (observer at rest). The proper time $_{O}t$ of a clock moving at constant velocity v is related to a stationary observer's coordinate time $_{R}t$ by Einstein's relativistic time dilation [12] and defined as

$$_{O}t = _{R}t \times \sqrt[2]{1 - \frac{v^{2}}{c^{2}}} \tag{16}$$

where $_{O}t$ denotes the "proper" time, $_{R}t$ denotes the "relativistic" (*i.e.* stationary or coordinate) time, v denotes the relative velocity and c denotes the speed of light in vacuum.

Scholium.

Coordinate systems can be chosen freely, deepening upon circumstances. In many coordinate systems, an event can be specified by one time coordinate and three spatial coordinates. The time as specified by the time coordinate is denoted as coordinate time. Coordinate time is distinguished from proper time. The concept of proper time, introduced by Hermann Minkowski in 1908 and denoted as $_{O}t$, incorporates Einstein's time dilation effect. In principle, Einstein is defining time exclusively for every place where a watch, measuring this time, is located.

"… Definition… der… Zeit… für den Ort, an welchem sich die Uhr… befindet…" [12].

In general, a watch is treated as being at rest relative to the place, where the same watch is located.

"Es werde ferner mittels der *im ruhenden System* befindlichen *ruhenden Uhr*en die Zeit *t* [*i.e.* $_{R}t$, author] des ruhenden Systems … bestimmt, ebensowerde die Zeit τ [$_{O}t$, author] des *bewegten Systems*, in welchen sich relativ zu letzterem *ruhende Uhren* befinden, bestimmt…" [12].

Due to Einstein, it is necessary to distinguish between clocks as such which are qualified to mark the time $_{R}t$ when at rest relatively to the stationary system *R*, and the time $_{O}t$ when at rest relatively to the moving system O.

"Wir denken uns ferner eine der Uhren, welche relative zum ruhenden System ruhend die Zeit *t* [$_{R}t$, author], relative zum bewegten System ruhend die Zeit τ [$_{O}t$, author] anzugeben befähigt sind…" [12].

In other words, we have to take into account that both clocks *i.e.* observers have at least one point in common, the stationary observer *R* and the moving observer O are at rest, but at rest relative to what? The stationary observer *R* is at rest relative to a stationary co-ordinate system *R*, the moving observer *O* is at rest relative to a moving co-ordinate system *O*. Both co-ordinate systems can but must not be at rest relative to each other. The time $_{R}t$ of the stationary system R is determined by clocks which are at rest relatively to that stationary system *R*.

Similarly, the time ${}_{O}t$ of the moving system O is determined by clocks which are at rest relatively to that the moving system O. In last consequence, due to Einstein's theory of special relativity, a moving clock (${}_{O}t$) will measure a smaller elapsed time between two events than a non-moving (inertial) clock (${}_{R}t$) between the same two events.

Definition: The normalized relativistic time dilation relation

As defined above, due to Einstein's special relativity, it is

$$ {}_{O}t = {}_{R}t \times \sqrt[2]{1-\frac{v^2}{c^2}} \tag{17} $$

where ${}_{O}t$ denotes the "proper" time, ${}_{R}t$ denotes the "relativistic" (*i.e.* stationary or coordinate) time, v denotes the relative velocity and c denotes the speed of light in vacuum. Equally, it is

$$ \frac{{}_{O}t}{{}_{R}t} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{18} $$

or

$$ \frac{{}_{O}t}{c^2} \times \frac{c^2}{{}_{R}t} = \sqrt[2]{1-\frac{v^2}{c^2}} \tag{19} $$

or

$$ \frac{{}_{O}t^2}{{}_{R}t^2} = 1-\frac{v^2}{c^2} \tag{20} $$

The normalized relativistic time dilation is defined as

$$ \frac{{}_{O}t^2}{{}_{R}t^2} + \frac{v^2}{c^2} = 1 . \tag{21} $$

Definition: The relationship between ${}_{R}t$ and ${}_{R}g$

In general, we define the mathematical identity

$$ {}_{R}t \equiv c^2 \times {}_{R}g . \tag{22} $$

Scholium.

In our understanding, ${}_{R}g$ represents the gravitational field as determined by a stationary observer R, which has to be proofed.

Definition: The relationship between ${}_{O}t$ and ${}_{O}g$

We define another mathematical identity

$$ {}_{O}t \equiv c^2 \times {}_{O}g . \tag{23} $$

Scholium.

In our understanding, ${}_{O}g$ represents something similar to the gravitational field as determined by a co-moving observer O, which is at rest relative to the "rest" energy ${}_{O}E$.

Definition: The relationship between ${}_{W}t$ and ${}_{R}g$

We define the following mathematical identity.

$$ {}_{W}t \equiv v \times c \times {}_{R}g . \tag{24} $$

Scholium.

In our understanding, ${}_{W}t$ represents something similar to the time as associated with the electro-magnetic wave ${}_{W}E$.

Definition: The relationship between ${}_{K}t$ and ${}_{R}g$

We define another mathematical identity.

$$ {}_{K}t \equiv \frac{{}_{W}t \times {}_{W}t}{{}_{R}t} \equiv \frac{{}_{W}t}{{}_{R}t} \times {}_{W}t \equiv \frac{v^2 \times c^2 \times {}_{R}g^2}{c^2 \times {}_{R}g} \equiv v^2 \times {}_{R}g . \tag{25} $$

Scholium.

The time ${}_{K}t$ is the time as determined by an observer which is at rest relative to the relativistic kinetic energy ${}_{K}E$.

Definition: The relationship between ${}_{P}t$ and ${}_{O}t$

We define the following mathematical identity.

$$ {}_{P}t \equiv \frac{{}_{O}t \times {}_{O}t}{{}_{R}t} \equiv \frac{{}_{O}t}{{}_{R}t} \times {}_{O}t \equiv \sqrt[2]{1 - \frac{v^2}{c^2}} \times {}_{O}t . \tag{26} $$

Scholium.

The time ${}_{P}t$ is the time as determined by an observer which is at rest relative to the relativistic potential energy ${}_{P}E$.

Definition: The Relationship between ${}_{P}t$ and ${}_{K}t$ and ${}_{R}t$

We define the mathematical identity.

$$ {}_{R}t \equiv {}_{P}t + {}_{K}t . \tag{27} $$

Definition: The relationship between ${}_{P}g$ and ${}_{O}g$

We define another mathematical identity too.

$$ {}_{P}g \equiv \frac{{}_{P}t}{c^2} \equiv \frac{{}_{O}t \times {}_{O}t}{{}_{R}t \times c^2} \equiv \frac{{}_{O}t}{{}_{R}t \times c^2} \times {}_{O}t \equiv \sqrt[2]{1 - \frac{v^2}{c^2}} \times \frac{{}_{O}t}{c^2} = \sqrt[2]{1 - \frac{v^2}{c^2}} \times {}_{O}g . \tag{28} $$

Definition: The relationship between ${}_{R}g$ and ${}_{W}t$ and ${}_{R}t$

We define the mathematical identity.

$$ {}_{K}g \equiv \frac{{}_{K}t}{c^2} \equiv \frac{{}_{W}t \times {}_{W}t}{{}_{R}t \times c^2} \equiv \frac{{}_{W}t}{{}_{R}t \times c^2} \times {}_{W}t \equiv \frac{v^2 \times c^2 \times {}_{R}g^2}{c^2 \times {}_{R}g \times c^2} \equiv \frac{v^2 \times {}_{R}g}{c^2} . \tag{29} $$

Scholium.

In our understanding, ${}_{K}g$ represents something similar to the gravitational field as associated with the time as determined by an observer which is at rest relative to the relativistic kinetic energy ${}_{K}E$.

Definition: The relationship between ${}_{R}g$ and ${}_{Red}t$

We define another mathematical identity.

$$ {}_{Kred}t = v \times {}_{R}g . \tag{30} $$

Definition: The relationship between ${}_{R}g$ and ${}_{Red}t$ and ${}_{Red}g$

We define the mathematical identity.

$$ {}_{Kred}g \equiv \frac{{}_{Kred}t}{c^2} \equiv \frac{v \times {}_{R}g}{c^2} . \tag{31} $$

Definition: The Relationship between ${}_{W}g$ and ${}_{W}t$

We define another mathematical identity.

$$ {}_{W}g = \frac{{}_{W}t}{c^2} . \tag{32} $$

Equally, it is

$$ {}_{W}g^2 = \frac{{}_{W}t^2}{c^2 \times c^2} . \tag{33} $$

Scholium.

In our understanding, ${}_{W}g$ represents the gravitational waves.

Definition: The relationship between ${}_{W}g$ and ${}_{W}t$

We define the mathematical identity.

$$ {}_{W}g \equiv \frac{{}_{W}t}{c^2} \equiv \frac{v \times {}_{R}g \times c}{c^2} \equiv \frac{{}_{Kred}t \times c}{c^2} \equiv {}_{Kred}g \times c \equiv \frac{{}_{Kred}t}{c} = \frac{v}{c} \times {}_{R}g . \tag{34} $$

Definition: The distance ${}_{O}d$ and the distance ${}_{R}d$

In general it is *distance* = *speed* × *time*. The time as such depends on the frame of reference in which it is measured or in other words a moving clock will more slowly. In general, it follows that

$$ {}_{O}d \equiv c \times {}_{O}t \equiv c \times {}_{R}t \times \sqrt[2]{1 - \frac{v^2}{c^2}} \equiv {}_{R}d \times \sqrt[2]{1 - \frac{v^2}{c^2}} \tag{35} $$

where ${}_{O}d$ denotes the distance as measured by a moving observer, ${}_{O}t$ denotes time as measured by a moving observer, ${}_{R}d$ denotes the distance as measured by a stationary observer, ${}_{R}t$ denotes the time as measured by a stationary observer, v denotes the relative velocity between the moving O and the stationary R observer and c denotes the speed of light in vacuum.

2.1.3. Quantum Theory

Definition: The quantum mechanical energy operator ${}_{R}H$

In quantum mechanics, energy is defined in terms of an energy operator which itself is acting on the wavefunction ${}_{R}\Psi$ of the system. The Hamiltonian, named after Sir William Rowan Hamilton (1805-1865), an Irish mathematician, physicist and astronomer, is a quantum mechanical operator corresponding to the total energy of a quantum mechanical system and usually denoted by H or by ${}_{R}H$. By analogy with classical mechanics and special relativity, the Hamiltonian is the sum of operators *i.e.* is corresponding to the total energy (*i.e.* potential and kinetic energies) (of all the particles) associated with a quantum mechanical system and can take different forms depending on the situation. The total (relativistic or non-relativistic) energy of a system is transformed into the Hamiltonian which acts as *a source of the wavefunction* and upon the wavefunction to generate the evolution of the wavefunction in time and space. The Hamiltonian operator is Hermitian. According to the expansion postulate, the wavefunction can be expanded as *a series of its eigenfunctions* where *an eigenfunction belongs to an eigenvalue of the Hamiltonian operator*. Consequently, an eigenstate of the operator H is one in which the energy is perfectly defined. Thus far, an important property of Hermitian operators is that their eigenvalues are real. The total energy operator H is determined as

$$ {}_{R}H = H = \hat{H} = i\hbar \frac{\partial}{\partial t}. \tag{36} $$

For our purposes, the (non-relativistic or relativistic) Hamiltonian is corresponding to the total energy of a (quantum mechanical object or) system. Thus far, it is

$$ {}_{R}E \equiv {}_{R}H \equiv H \equiv \hat{H} = i\hbar \frac{\partial}{\partial t} \tag{37} $$

where ${}_{R}E$ is identical with the notion "relativistic" energy of a (quantum mechanical) system, i is the imaginary unit, $\hbar$ is the reduced Planck constant, and H is the Hamiltonian operator.

Definition: The quantum mechanical operator of matter ${}_{R}M$

In quantum mechanics, the Hamiltonian, named after the Irish mathematician Hamilton, is the total energy operator. Thus far we define the quantum mechanical operator of matter ${}_{R}M$ as

$$ {}_{R}M \equiv \hat{M} \equiv \frac{{}_{R}E}{c^2} \equiv \frac{H}{c^2} = \frac{\hat{H}}{c^2} = \frac{i\hbar}{c^2} \frac{\partial}{\partial t} \tag{38} $$

where ${}_{R}M$ is quantum mechanical operator of matter (and not only of mass), c is the speed of the light in vacuum and H is the Hamiltonian operator.

Scholium.

Matter and energy are equivalent but not identical and thus far not absolutely the same. Einstein himself defined matter not in relation to time but in relation to gravitational field. Due to Einstein's understanding of the relationship between matter and gravitational field, *all but the gravitational field is matter*. Consequently, matter as such includes matter in the ordinary sense and the electromagnetic field as well. In other words, there is no third between matter and gravitational field, *tertium non datur*. Einstein himself wrote:

"Wir unterscheiden im folgenden zwischen 'Gravitationsfeld' und 'Materie', in dem Sinne, daß alles außerdem Gravitationsfeld als 'Materie' bezeichnet wird, also nicht nur die 'Materie' im üblichen Sinne, sondern auch das elektro-magnetische Feld" [13].

Einstein's writing translated into English:

"We make a distinction hereafter between 'gravitational field' and 'matter' in this way, that we denote everything but the gravitational field as 'matter', the word matter therefore includes not only matter in the ordinary sense, but the electromagnetic field as well."

Definition: The eigenvalue $_{O}H$ and the anti eigenvalue $\underline{_{O}H}$ of the Hamiltonian operator $_{R}H$

Let us consider *i.e.* the Hamiltonian $_{R}H$, a physical quantity which characterizes the total energy of a quantum system. Strictly, we should speak in the following discussion not of one quantity, but of a complete set of them. For brevity and simplicity, the values which a given physical quantity like the Hamiltonian can take are called its *eigenvalues*. The set of these eigenvalues is referred to as the spectrum of eigenvalues of a given quantity. In cases where the eigenvalues of a given physical quantity occupy a continuous range, in such cases we speak of a continuous spectrum of eigenvalues. A given physical quantity can be determined by eigenvalues who form some discrete set. In such cases we speak of a discrete spectrum. We shall suppose for simplicity that the quantities considered here has a discrete spectrum. The case of a continuous spectrum can be discussed later.

The Hamiltonian operator, which plays a central role in quantum mechanics, corresponds to the total energy of a system. An energy eigenstate which does not change by time is called a stationary state. Energy eigenvalues of the Hamiltonian operator are related to the observed values in experimental measurements (*i.e.* after the "collapse" of the wavefunction). In a single experiment, a measured value $_{O}H$ of the Hamiltonian operator $_{R}H$ is an energy eigenvalue of the Hamiltonian operator. Thus far, let $_{O}H$ denote an *energy eigenvalue* of the Hamiltonian operator $_{R}H$, let $\underline{_{O}H}$ denote an *energy anti eigenvalue* of the Hamiltonian operator $_{R}H$, let $_{R}H$ denote the Hamiltonian operator. In general, we define

$$ {}_{0}H + \underline{{}_{0}H} \equiv {}_{R}H \, . \tag{39}$$

Scholium.

Eigenvalues $_{O}H$ of the Hamiltonian operator $_{R}H$ are related to observed values in experimental measurements and thus far determined by the necessities of special theory of relativity. In this context, it is possible that the energy eigenvalue $_{O}H$ is the part of the "relativistic" energy $_{R}E$ which is measured by a co-moving observer O. In other words, there are circumstances where $_{O}E = {}_{O}H$. Under these circumstances, it follows that $\underline{_{O}H}$ is equivalent with $\underline{_{O}H} = {}_{R}E - {}_{O}E = {}_{\Delta}E$ and quantum theory reduces to simple special theory of relativity. Following Bohm's terminology, $\underline{_{O}H}$ can be understood as a kind of a "local hidden" variable. If the energy eigenvalue is measured by a potential observer, it makes sense to denote the same as $_{P}H$. The *energy anti eigenvalue of a potential observer*, denoted by $\underline{_{P}H}$, is then $\underline{_{P}H} = {}_{R}H - {}_{P}H$.

Definition: Eigenfunctions of the Hamiltonian operator $_{R}H$

The Hamiltonian operator, equal to the total energy of a system, is Hermitean. An eigenstate of the operator $_{R}H$ is one in which the energy is perfectly defined and equal to the eigenvalue *i.e.* $_{i}H$. To each eigenvalue $_{i}H$ of the Hamiltonian operator $_{R}H$ is associated an eigenfunction. A state in which the energy is well defined is a state in which the probabilities remain constant with time and can therefore be called a *stationary state*.

Definition: The expectation value $E({}_{R}H)$ of the Hamiltonian operator $_{R}H$

Let $E({}_{R}H)$ denote the expectation value of the Hamiltonian operator $_{R}H$. In general, we define the expectation value of the Hamiltonian operator $_{R}H$ as

$$ E\left({}_{R}H\right) \equiv {}_{R}p\left({}_{R}H\right) \times {}_{R}H \equiv \left(1 - {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right)\right) \times {}_{R}H \, . \tag{40}$$

Definition: The complex number $_{R}Y$

Let $_{R}Y$ denote a (complex) number from the standpoint of a stationary observer R of preliminary unknown properties.

Definition: The wave function (of quantum mechanical system)

Let ${}_{R}\Psi({}_{R}X, {}_{R}t)$ denote the wavefunction of system from the standpoint of a stationary observer *R*. The wave function represents something like a probability amplitude. When only one spatial dimension ${}_{R}X$ is relevant we write

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \tag{41}$$

while the modulus squared $|{}_{R}\Psi({}_{R}X, {}_{R}t)|^2$ of the wavefunction gives the probability for finding a quantum mechanical entity at a given point ${}_{R}X$ in space at the given time ${}_{R}t$. The wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ itself is not an *eigenfunction* of an operator. However, every wavefunction (that is not an eigenfunction) can be expressed as a superposition of eigenfunctions of an operator.

Scholium.

The wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ corresponding to the system state can change (out of itself or by a third [*i.e.* measurement]) into an eigenfunction ${}_{O}\Psi({}_{O}X, {}_{O}t)$. The change of the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ into an eigenfunction ${}_{O}\Psi({}_{O}X, {}_{O}t)$ of an operator (corresponding to the measured quantity) is called wavefunction collapse. Corresponding to each *eigenvalue* is an "*eigenfunction*", corresponding to each *anti eigenvalue* is an "*anti eigenfunction*". Only certain eigenvalues with associated eigenfunctions are able to satisfy Schrödinger's equation. The eigenvalue ${}_{O}H$ as measured by a co-moving observer O is one of the eigenvalues of the quantum mechanical observable ${}_{R}H$. The eigenvalue (corresponding to some scalar) concept as such is not limited only to energy. Finding a specific function (*i.e.* eigenfunction) which describes an energy state (*i.e.* a solution to the Schrodinger equation) is very important. Under conditions where the eigenvalues are discrete, a physical variable is said to be "*quantized*" and an index *i* plays the role of a "quantum number" which is characterizing a specific state.

Definition: Born's rule

The probability of an existing quantum mechanical entity (photon, electron, X-ray, etc.) being somewhere in space is unity. Thus far, the wavefunction must fulfill some certain mathematical requirements, especially the normalization condition. In other words, the integration of the wavefunction over all space leads to a probability of 1. That is, the wavefunction is normalized. In general, the wavefunction as such represents the probability amplitude for finding a particle *ata given point* in space *at a given time*. Due to special theory of relativity, time as measured by a stationary observer can be different from time as determined in the same respect by a moving observer. Thus far, let us define the following.

Let the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ denote the single-valued probability amplitude at $({}_{R}X, {}_{R}t)$ where ${}_{R}X$ is position and ${}_{R}t$ is time *i.e.* from the standpoint of *a stationary observer*. Let ${}_{R}\Psi^{*}({}_{R}X, {}_{R}t)$ denote *the complex conjugate* of the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ *i.e.* from the standpoint of a stationary observer. Let ${}_{R}p({}_{R}\Psi({}_{R}X, {}_{R}t))$ denote the probability *i.e.* from the standpoint of a stationary observer R as associated with the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ that a "particle" will be found at $({}_{R}X, {}_{R}t)$.

Let the eigenfunction ${}_{O}\Psi({}_{O}X, {}_{O}t)$ denote the single-valued probability amplitude at $({}_{O}X, {}_{O}t)$ where ${}_{O}X$ is position and ${}_{O}t$ is time *i.e.* from the standpoint of *a co-moving observer*. Let ${}_{O}\Psi^{*}({}_{O}X, {}_{O}t)$ denote *the complex conjugate* of the eigenfunction ${}_{O}\Psi({}_{O}X, {}_{O}t)$ *i.e.* from the standpoint of *a co-moving observer*. Let ${}_{O}p({}_{O}\Psi({}_{O}X, {}_{O}t))$ denote the probability *i.e.* from the standpoint of *a co-moving observer O* as associated with an eigenfunction that a "particle" will be found at $({}_{O}X, {}_{O}t)$.

In general, due the Born's rule, named after Max Born [14], it is

$$ {}_{R}p\left({}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right) \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) \equiv \left|{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right|^{2} \tag{42}$$

and

$$ {}_{O}p\left({}_{O}\Psi\left({}_{O}X, {}_{O}t\right)\right) \equiv {}_{O}\Psi\left({}_{O}X, {}_{O}t\right) \times {}_{O}\Psi^{*}\left({}_{O}X, {}_{O}t\right) \equiv \left|{}_{O}\Psi\left({}_{O}X, {}_{O}t\right)\right|^{2} \tag{43}$$

where ${}_{R}X$ is the position as determined by the stationary observer, ${}_{R}t$ is the time as determined by the stationary observer, the asterix ${}_{R}\Psi({}_{R}X, {}_{R}t)^{*}$ indicates the complex conjugate of the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$, ${}_{R}p({}_{R}\Psi({}_{R}X, {}_{R}t))$ denotes the probability (mass function or the probability density function or the cumulative distribution function) as associated with the wavefunction. From these definitions follows that

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \equiv \frac{1}{{}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right)} \times {}_{R}p\left({}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right) \equiv \frac{\left|{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right|^{2}}{{}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right)} \tag{44}$$

and that

$$ {}_{O}\Psi({}_{O}X,{}_{O}t) \equiv \frac{1}{{}_{O}\Psi^{*}({}_{O}X,{}_{O}t)} \times {}_{O}p({}_{O}\Psi({}_{O}X,{}_{O}t)) \equiv \frac{\left|{}_{O}\Psi({}_{O}X,{}_{O}t)\right|^{2}}{{}_{O}\Psi^{*}({}_{O}X,{}_{O}t)}. \tag{45} $$

Scholium.

The wavefunction satisfies the Schrödinger equation, a first order differential equation in time. Other prefer to use the wavefunction as associated with a random variable instead of the probability as such. Due to the relationship above, this can be done in a contradiction free way.

Definition: The relationship between expansion coefficients and probability

In general, let ${}_{R}X$ denote a random variable, an operator, a quantum mechanical observable et cetera, let ${}_{i}X$ denote an exact numerical (eigen-) value of the random variable ${}_{R}X$. Let $p({}_{i}X)$ denote the probability that in a measurement a system will be found in a state in which the random variable ${}_{R}X$ has the exact numerical value ${}_{i}X$. Let $c({}_{i}X)$ denote an arbitrary (complex) coefficients. Let $c^{*}({}_{i}X)$ denote arbitrary (complex conjugate) coefficients. Thus, we define

$$ p({}_{i}X) \equiv c({}_{i}X) \times c^{*}({}_{i}X). \tag{46} $$

Definition: The expectation value of the wavefunction

Let $E({}_{R}\Psi({}_{R}X, {}_{R}t))$ denote the expectation value of the wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$. In general, we define the expectation value of the wave function as

$$ E({}_{R}\Psi({}_{R}X,{}_{R}t)) \equiv {}_{R}p({}_{R}\Psi({}_{R}X,{}_{R}t)) \times {}_{R}\Psi({}_{R}X,{}_{R}t) \equiv {}_{R}\Psi({}_{R}X,{}_{R}t)^{2} \times {}_{R}\Psi^{*}({}_{R}X,{}_{R}t). \tag{47} $$

Definition: The relationship between the complex number ${}_{R}Y$ and Born's rule

Let ${}_{R}Y$ denote an unknown complex number from the standpoint of a stationary observer R. Let the relationship between the unknown complex number ${}_{R}Y$ and Born's rule be given by

$$ {}_{R}p({}_{R}\Psi({}_{R}X,{}_{R}t)) \equiv {}_{R}\Psi({}_{R}X,{}_{R}t) \times {}_{R}\Psi^{*}({}_{R}X,{}_{R}t) \equiv {}_{R}\Psi({}_{R}X,{}_{R}t) \times {}_{R}Y . \tag{48} $$

Scholium.

Due to this definition it is

$$ {}_{R}Y \equiv {}_{R}\Psi^{*}({}_{R}X,{}_{R}t) \equiv \frac{{}_{R}p({}_{R}\Psi({}_{R}X,{}_{R}t))}{{}_{R}\Psi({}_{R}X,{}_{R}t)}. \tag{49} $$

Definition: The superposition principle

The wavefunction ${}_{R}\Psi({}_{R}X, {}_{R}t)$ itself is not an eigenfunction of an operator. However, every wavefunction can be expressed as a superposition of eigenfunctions of an operator. In general, a set of eigenfunctions of an operator can be used as a basis set. Thus far, any arbitrary wavefunction can be expanded in terms of a superposition of *eigenfunctions*. In general, assumed that only one spatial dimension is relevant, the wave function can be written briefier as

$$ {}_{R}\Psi({}_{R}X,{}_{R}t) \equiv \sum_{i=0}^{n} {}_{i}c \times {}_{i}\Psi({}_{i}X,{}_{i}t) \equiv {}_{0}c \times {}_{0}\Psi({}_{O}X,{}_{O}t) + \sum_{i=1}^{n} {}_{i}c \times {}_{i}\Psi({}_{i}X,{}_{i}t) \tag{50} $$

where ${}_{i}\Psi({}_{i}X, {}_{i}t)$ denotes the *eigenfunction* of an corresponding eigenvalue and ${}_{i}c$ denotes the associated arbitrary complex number. The equation before can be rewritten using Dirac's notation as

$$ \left|{}_{R}\Psi\right\rangle \equiv \sum_{i=0}^{n} {}_{i}c \times \left|{}_{i}\Psi\right\rangle. \tag{51} $$

The set of all eigenfunctions are called a complete set of states, or a basis. Multiplying with the complex conjugate of the wavefunction $\Psi({}_{R}X, {}_{R}t)^{*}$, we obtain

$$ {}_{R}p({}_{R}\Psi({}_{R}X,{}_{R}t)) \equiv {}_{R}\Psi({}_{R}X,{}_{R}t) \times {}_{R}\Psi^{*}({}_{R}X,{}_{R}t) \equiv \sum_{i=0}^{n} {}_{i}c \times {}_{i}\Psi({}_{i}X,{}_{i}t) \times {}_{R}\Psi^{*}({}_{R}X,{}_{R}t) \tag{52} $$

From this follows that

$$\sum_{i=0}^{n} {}_{i}c \times {}_{i}\Psi\left({}_{i}X, {}_{i}t\right) \times {}_{i}c^{*} \times {}_{i}\Psi^{*}\left({}_{i}X, {}_{i}t\right) \equiv \sum_{i=0}^{n} {}_{i}c \times {}_{i}c^{*} \int \Psi\Psi^{*} \mathrm{d}X \ . \tag{53}$$

Scholium.

Eigenfunctions play an important role in quantum physics. Corresponding to each eigenvalue ${}_{i}E$ is an eigenfunction ${}_{i}\Psi({}_{i}X, {}_{i}t)$.

Definition: The anti eigenfunction

Let us now clarify how eigenfunctions and anti eigenfunctions are related to what is observed in experiments. Thus far let us consider the physical quantity ${}_{R}H$, which characterizes the total energy of a quantum system. For brevity and simplicity we should speak in the following (from the standpoint of the stationary observer R) not of one quantity, but of a complete set at the same time. Generally speaking, the values which a given physical quantity like the Hamiltonian ${}_{R}H$ can take are called in quantum mechanics its eigenvalues. The set of these eigenvalues is referred to as the spectrum of eigenvalues of the given physical quantity. In classical mechanics, quantities run through a continuous series of values. Let us suppose for simplicity that the physical quantity ${}_{R}H$ considered here has a discrete spectrum. The eigenvalues of the physical quantity ${}_{R}H$ are denoted by ${}_{i}H$, where the suffix i takes the values $0, 1, 2, 3, \cdots$. The associated wave function of the system, in the state where the physical quantity ${}_{R}H$ has the value ${}_{i}H$ is denoted by ${}_{i}\Psi({}_{i}X, {}_{i}t)$. The wave functions ${}_{i}\Psi({}_{i}X, {}_{i}t)$ are called the eigenfunctions of the given physical quantity ${}_{R}H$. In accordance with the principle of superposition, the eigenfunctions form a complete set. Thus far, any wave function can be written as a linear combination of eigenfunctions. Let ${}_{O}\Psi({}_{O}X, {}_{O}t)$ denote the eigenfunction (*i.e.* as determined by the moving observer O). Let $\underline{{}_{O}\Psi}\left(\underline{{}_{O}X}, \underline{{}_{O}t}\right)$ denote the corresponding anti eigenfunction. In this respect, let $\Psi({}_{R}X, {}_{R}t)$ denote the wavefunction of a system from the standpoint of a stationary observer R. The wave function can be expanded in terms of the eigenfunctions and represented in the form of a series as

$${}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \equiv \sum_{i=O}^{n=\ldots} {}_{i}c \times {}_{i}\Psi\left({}_{i}X, {}_{i}t\right) \equiv {}_{O}c \times {}_{O}\Psi\left({}_{O}X, {}_{O}t\right) + \sum_{i=1}^{n=\ldots} {}_{i}c \times {}_{i}\Psi\left({}_{i}X, {}_{i}t\right) \tag{54}$$

where the summation extends over all n, and the ${}_{i}c$ are some arbitrary complex numbers. In general, we define an anti-eigenfunction $\underline{{}_{O}\Psi}\left(\underline{{}_{O}X}, \underline{{}_{O}t}\right)$ of the eigenfunction ${}_{O}\Psi({}_{O}X, {}_{O}t)$ as

$$\underline{{}_{O}c} \times \underline{{}_{O}\Psi}\left(\underline{{}_{O}X}, \underline{{}_{O}t}\right) = {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) - {}_{O}c \times {}_{O}\Psi\left({}_{O}X, {}_{O}t\right) \tag{55}$$

or as

$$\underline{{}_{O}c} \times \underline{{}_{O}\Psi}\left(\underline{{}_{O}X}, \underline{{}_{O}t}\right) \equiv \sum_{i=1}^{n} {}_{i}c \times {}_{i}\Psi\left({}_{i}X, {}_{i}t\right) \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) - {}_{O}c \times {}_{O}\Psi\left({}_{O}X, {}_{O}t\right). \tag{56}$$

The anti-eigenfunction of the observer k is defined as

$$\underline{{}_{k}c} \times \underline{{}_{k}\Psi}\left(\underline{{}_{k}X}, \underline{{}_{k}t}\right) \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) - {}_{k}c \times {}_{k}\Psi\left({}_{k}X, {}_{k}t\right). \tag{57}$$

Each coefficient ${}_{i}c$ in the expansion determines the probability of the corresponding physical quantity. The sum of the probabilities of all possible values must be equal to unity. In other words, at the end the relation

$$\sum_{i=0}^{n=\ldots} \left|{}_{i}c\right|^{2} \equiv 1 \tag{58}$$

must hold.

Scholium.

The above definition assures the compatibility of quantum theory with relativity theory and does not exclude the possibility that an eigenfunction can be determined by a potential observer P (*i.e.* from the standpoint of potential energy). In this case we would obtain

$$\underline{{}_{P}c} \times \underline{{}_{P}\Psi}\left(\underline{{}_{P}X}, \underline{{}_{P}t}\right) \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) - {}_{P}c \times {}_{P}\Psi\left({}_{P}X, {}_{P}t\right) \tag{59}$$

where ${}_{P}\Psi({}_{P}X, {}_{P}t)$ denotes the eigenfunction as determined by the observer P and $\underline{{}_{P}\Psi}\left(\underline{{}_{P}X}, \underline{{}_{P}t}\right)$ denotes the corresponding anti-eigenfunction.

Definition: The complex number $_RS$

Let $_RS$ denote something, *i.e.* a (complex) number from the standpoint of a stationary observer R. Let $_OS$ denote a (complex) number of the same system in the same respect from the standpoint of a co-moving observer. In general, we define

$$_RS \equiv {}_R\Psi\left({}_RX, {}_Rt\right) + {}_RH \equiv {}_R\Psi\left({}_RX, {}_Rt\right) + i\hbar\frac{\partial}{\partial {}_Rt} \tag{60}$$

where $_RH$ denotes the Hamiltonian from the standpoint of a stationary observer R, $_R\Psi(t)$ denote the wavefunction of the system from the standpoint of a stationary observer R, where i is the imaginary unit, $\hbar$ is the Planck constant divided by $2 \times \pi$, $\frac{\partial}{\partial t}$ indicates a partial derivative with respect to time t.

Perhaps there is no need for us to point out that there are other approaches to the notion energy. The topic of this definition is not—at least directly—energy as such; rather, it is the definition of the relationship between energy and time. In general, it is

$$_RE + {}_Rt = {}_RS\ . \tag{61}$$

Definition: The complex conjugate $_RS^*$ of the complex number $_RS$

The complex conjugate of $_RS^*$ of the complex number $_RS$ from the standpoint of a stationary observer R is defined as

$$_RS^* \equiv {}_R\Psi\left(t\right) - i\hbar\frac{\partial}{\partial t}\ . \tag{62}$$

Scholium.

In our understanding of the relationship between energy and time *everything but time is energy*. More precisely, there is *no third between energy and time*. Clearly, $_RS$ is a kind of a complex number. Thus far, the straight forward question is, are there circumstances where $_RS \times {}_RS^* = i^2 = 1$?

Definition: The relationship between the complex number $_RY$ and the complex number $_RS$

Let $_RY$ denote an unknown complex number from the standpoint of a stationary observer R. Let the relationship between the unknown complex number $_RY$ and the unknown complex number $_RS$ be given by

$$_RS \times {}_RY \equiv {}_R\Psi\left({}_RX, {}_Rt\right) \times {}_RY + {}_RH \times {}_RY \equiv {}_R\Psi\left({}_RX, {}_Rt\right) \times {}_RY + i\hbar\frac{\partial}{\partial {}_Rt} \times {}_RY \equiv 1\ . \tag{63}$$

Scholium.

Due to this definition it is

$$_RY \equiv \frac{1}{{}_RS}\ . \tag{64}$$

Definition: The complex number $_OS$ and $\underline{_OS}$

Let $_OS$ denote something, *i.e.* a (complex) number from the standpoint of an eigenvalue and eigenfunction (*i.e.* a co-moving observer O). Let $\underline{_OS}$ denote an anti (complex) number of the same system. In general, we define

$$_OS \equiv {}_OH + {}_Oc \times {}_O\Psi\left({}_OX, {}_Ot\right) \tag{65}$$

and

$$\underline{_OS} \equiv \underline{_OH} + \underline{{}_Oc \times {}_O\Psi\left({}_OX, {}_Ot\right)} \tag{66}$$

and

$$_RS \equiv {}_OS + \underline{_OS} = {}_R\Psi\left({}_RX, {}_Rt\right) + {}_RH\ . \tag{67}$$

Scholium.

The following picture may illustrate the definitions above.

Fig.		"Curvature"		
		yes	no	
"Momentum"	yes	${}_{O}H$	$\underline{{}_{O}H}$	${}_{R}H$
	no	${}_{O}c \times {}_{O}\Psi\left({}_{O}X, {}_{O}t\right)$	$\underline{{}_{O}c \times {}_{O}\Psi\left({}_{O}X, {}_{O}t\right)}$	${}_{R}\Psi\left({}_{R}X, {}_{R}t\right)$
		${}_{O}S$	$\underline{{}_{O}S}$	${}_{R}S$

The four notions (${}_{O}H$, $\underline{{}_{O}H}$, ${}_{O}\Psi$, $\underline{{}_{O}\Psi}$) can denote the four basic fields of nature too. From the above definitions follows that

$$ {}_{R}S - {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \equiv {}_{O}S + \underline{{}_{O}S} - {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = +{}_{R}H \ . \tag{68} $$

Let us assume, just for the sake of the argument, that there are conditions where

$$ \Lambda \equiv \underline{{}_{O}S} - {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \tag{69} $$

where Λ denotes Einstein's cosmological constant. Under these circumstances, we obtain

$$ {}_{O}S + \Lambda = +{}_{R}H \ . \tag{70} $$

Multiplying by Einstein's metric tensor $g_{\mu\nu}$ we obtain

$$ {}_{O}S \times g_{\mu\nu} + \Lambda \times g_{\mu\nu} = {}_{R}H \times g_{\mu\nu} \ . \tag{71} $$

Under conditions where $\underline{T}_{\mu\nu} = (8 \times \pi \times \gamma/c^4) \times T_{\mu\nu} - {}_{R}H \times g_{\mu\nu}$ it should be that is $\underline{G}_{\mu\nu} = G_{\mu\nu} - {}_{O}S \times g_{\mu\nu}$. "Historically the term containing the "cosmological constant" was introduced into the field equations in order to enable us to account theoretically for the existence of a finite mean density in a static universe. It now appears that in the dynamical case this end can be reached without the introduction of Λ" [15].

Definition: The complex number ${}_{R}U$

Let ${}_{R}U$ denote a complex number from the standpoint of a stationary observer. Let ${}_{O}U$ denote a complex number of the same system in the same respect from the standpoint of a co-moving observer. Let ${}_{R}S$ denote a complex number from the standpoint of a stationary observer. Let ${}_{O}S$ denote a complex number of the same system in the same respect from the standpoint of a co-moving observer. In general, we define

$$ {}_{R}U \equiv \frac{{}_{R}S}{c^2} \equiv \frac{{}_{R}\Psi(t)}{c^2} + \frac{{}_{R}H}{c^2} \equiv \frac{{}_{R}\Psi(t)}{c^2} + \frac{i\hbar}{c^2}\frac{\partial}{\partial t} . \tag{72} $$

The complex conjugate of ${}_{R}U^{*}$ of the complex number ${}_{R}U$ from the standpoint of a stationary observer R is defined as

$$ {}_{R}U^{*} \equiv \frac{{}_{R}S^{*}}{c^2} . \tag{73} $$

Definition: The quantum mechanical wave function of the gravitational field ${}_{R}G$

Let the wavefunction of the gravitational field (from the standpoint of a stationary observer R) describe the gravitational field completely. In general, let ${}_{R}G$ denote the wavefunction of the gravitational field from the standpoint of a stationary observer R, let ${}_{R}G^{*}$ denote the complex conjugate of the wavefunction of the gravitational field from the standpoint of a stationary observer R, let ${}_{O}G$ denote the wavefunction of the gravitational field from the standpoint of a co-moving observer O, let ${}_{O}G^{*}$ denote the complex conjugate of the wavefunction of the gravitational field from the standpoint of a co-moving observer O. Let ${}_{R}M$ denote matter or the quantum mechanical operator of matter (as defined above) from the standpoint of a stationary observer R. In general, we define

$$ {}_{R}G \equiv {}_{R}U - {}_{R}M \ . \tag{74} $$

Definition: The relationship between matter and gravitational field in general

In general, the relationship between matter and gravitational field is given by

$$ {}_{R}U \equiv \frac{{}_{R}S}{c^{2}} \equiv {}_{R}U + 0 \equiv {}_{R}U - {}_{R}M + {}_{R}M \equiv {}_{R}G + {}_{R}M \, . \tag{75} $$

Scholium.

There is no third between gravitational field and matter, a third is not given (*tertium non datur*). Aristotle's (384-322) B.C. *principium exclusitertii* seumedii inter duo contradictoria follows from this relationship. Still, there is a close relationship between matter *i.e.* energy and the gravitational field, both are related to each other in a certain manner. Especially, Einstein himself provided some very important views on the relationship between matter and gravitational field.

"Das G-Feld ist restlos durch die Massen der Körper bestimmt" [16].

Translated into English:

"The G-field [*gravitational field, author*] is completely determined by the masses of the bodies."

Einstein, while investigating the nature of the relationship between matter and the gravitational field, writes:

"Da *Masse und Energie* nach den Ergebnissen der speziellen Relativitätstheorie *das Gleiche* sind und die Energie formal durch den symmetrischen Energietensor ($T_{\mu\nu}$) beschrieben wird, so besagt dies, daß das G-Feld durch den Energietensor der Materie bedingt und bestimmt sei" [16].

Einstein's view translated into English:

"According to the results of the special theory of relativity *mass and energy are the same* and energy itself is described by the symmetric energy tensor ($T_{\mu\nu}$) so this indicates that the G-field [*gravitational field, author*] is conditioned and determined by the energy tensor of matter"

Gravitational field and matter are not only determined by each other. Both are related to each other in a very subtle way.

"Wir unterscheiden im folgenden zwischen 'Gravitationsfeld' und 'Materie', in dem Sinne, daß *alles außer dem Gravitationsfeld als 'Materie' bezeichnet wird*, also nicht nur die 'Materie' im üblichen Sinne, sondern auch das elektro-magnetische Feld" [13].

Einstein's position translated into English:

"We make a distinction hereafter between the 'gravitational field' and 'matter' in this way, that *we denote everything but the gravitational field as 'matter'*, the word matter therefore includes not only matter in the ordinary sense, but the electromagnetic field as well."

We see from the above that Einstein himself defined the relationship between matter (*i.e.* not only mass) and gravitational field *ex negativo*. More precisely, due to Einstein *everything but the gravitational field is matter*. Consequently, matter as such includes *matter in the ordinary sense* and the *electromagnetic field* as well. In other words, there is no third between matter and gravitational field. For the sake of illustration of the principles involved, let us consider **Table 1**.

This approach to the relationship between matter and gravitational field is backgrounded by *the de Broglie hypothesis* too. Louis de Broglie worked on Compton's momentum phenomena. In 1924, as part of his PhD thesis, de Broglie hypothesized that all matter (*i.e.* electrons) can exhibit wave-like behaviour. Louis de Broglie was awarded the Nobel Prize in Physics 1929 for the discovery of the wave nature of electrons (*i.e.* matter waves). These matter waves, a central part of today's theory of quantum mechanics, were first experimentally confirmed to occur in the Davisson-Germer experiment [17] for electrons.

Definition: The Schrödinger equation

The Schrödinger equation for any system, no matter whether relativistic or not, no matter how complicated, has the form

$$ {}_{R}H \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = i\hbar \frac{\partial}{\partial t} {}_{R}\Psi\left({}_{R}X, {}_{R}t\right), \tag{76} $$

where i is the imaginary unit, $\hbar = \frac{h}{2 \times \pi}$ is Planck's constant divided by $2 \times \pi$, the symbol $\frac{\partial}{\partial t}$ indicates a partial derivative with respect to time ${}_{R}t$, ${}_{R}\Psi({}_{R}X, {}_{R}t)$ is the wave function of the quantum system, and ${}_{R}H$ is the Hamiltonian operator.

Table 1. The relationship between matter and gravitational field..

	Matter $_{R}M$	Gravitational field $_{R}G$

Matter and gravitational field.

2.2. Axioms

Axiom I. (Lex identitatis).

The following theory is based on the following axiom:

$$+1 = +1. \tag{77}$$

Scholium.

Why does it make sense to define such an axiom and to take all transformations and proofs from the same? An axiom should be so evident or well-established, that it is or can be accepted without any controversy or question by the scientific community. Thus far, such an axiom can be used as a premise or as a starting point for further theorems, arguments and reasoning. As we will see, a lot of proofs in this paper are repeated to demonstrate the strategic potential of axiom I. All Newtonian axioms can be derived from the axiom I, Einstein's field equation can be derived from the axiom I. Consequently, axiom I has the potential to serve as the common ground for the unification of quantum and relativity theory into a unique mathematical framework.

3. Results

3.1. Theorem: Newton's First Axiom Can Be Derived from Axiom I

Claim. (Theorem. Proposition. Statement.)

In general, Newton first law can be derived from axiom I as

$$\sum_{t=0}^{n} \overrightarrow{{}_{t}F} = 0. \tag{78}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1. \tag{79}$$

Multiplying this equation by 0, we obtain

$$0 \times 1 = 0 \times 1. \tag{80}$$

In general it is

$$0 = 0. \tag{81}$$

As stated before, Newton's first law first law demands that the net force (a vector sum of all forces acting on something) is zero. We obtain

$$\sum_{t=0}^{n} \overrightarrow{{}_{t}F} = 0. \tag{82}$$

Quod erat demonstrandum.

3.2. Theorem: Newton's Second Axiom Can Be Derived from Axiom I

Claim. (Theorem. Proposition. Statement.)

In general, Newton second law can be derived from axiom I as

$$\overrightarrow{{}_{O}F} = {}_{O}m \times \overrightarrow{{}_{O}a}. \tag{83}$$

Direct proof.
In general, axiom I is determined as

$$+1 = +1\,. \tag{84}$$

Multiplying this equation by ${}_{O}F$, we obtain

$$\overrightarrow{{}_{O}F} \times 1 = \overrightarrow{{}_{O}F} \times 1\,. \tag{85}$$

In general it is ${}_{O}F = {}_{O}m \times {}_{O}a$ and we obtain

$$\overrightarrow{{}_{O}F} = {}_{O}m \times \overrightarrow{{}_{O}a}\,. \tag{86}$$

Quod erat demonstrandum.

3.3. Theorem: Newton's Third Axiom Can Be Derived from Axiom I

Claim. (Theorem. Proposition. Statement.)
In general, Newton third law is defined as

$$\overrightarrow{{}_{O}F} = -\overleftarrow{{}_{R}F}\,. \tag{87}$$

Direct proof.
In general, axiom I is determined as

$$+1 = +1\,. \tag{88}$$

Multiplying this equation by ${}_{O}F$, we obtain

$$\overrightarrow{{}_{O}F} = \overrightarrow{{}_{O}F}\,. \tag{89}$$

Adding ${}_{R}F$, we obtain

$$\overrightarrow{{}_{O}F} + \overleftarrow{{}_{R}F} = \overrightarrow{{}_{O}F} + \overleftarrow{{}_{R}F} = \overleftrightarrow{{}_{R}C}\,. \tag{90}$$

Newton's laws of motion are valid in inertial reference frames where the net force is equal to zero. We obtain

$$\overrightarrow{{}_{O}F} + \overleftarrow{{}_{R}F} = \overrightarrow{{}_{O}F} + \overleftarrow{{}_{R}F} = \overleftrightarrow{{}_{R}C} = 0\,. \tag{91}$$

At the end, Newton's third law follows as

$$\overrightarrow{{}_{O}F} = -\overleftarrow{{}_{R}F}\,. \tag{92}$$

Quod erat demonstrandum.
Scholium.
In particular, an axiom serves as a starting point from which other theorems, laws, equations, statements et cetera are logically derived and should be something which is so well-established or evident that it can be accepted without question or controversy or demonstration. However, an axiom in one theory or system may be only one theorem in another theory or system, and vice versa. This theory is based on the axiom +1 = +1. From this axiom, Newton's axioms can be derived without any contradiction. *Consequently, in this theory, Newton's axioms are only theorems*. The axiom +1 = +1 is very general and of use outside of physics too. Axiom I can be the foundation for an axiomatic formulations of quantum mechanics.

3.4. Theorem: Einstein's Field Equations

Einstein's field equations can be derived from axiom I.
Claim. (Theorem. Proposition. Statement.)
In general, Einstein's field equations are derived as

$$G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right). \tag{93}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1. \tag{94}$$

Multiplying this equation by the stress-energy tensor of general relativity $((4 \times 2 \times \pi \times \gamma)/(c^4)) \times T_{\mu\nu}$, it is

$$+1 \times \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) = +1 \times \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right) \tag{95}$$

where γ is Newton's gravitational "constant" [18], c is the speed of light in vacuum and π, sometimes referred to as "Archimedes' constant", is the ratio of a circle's circumference to its diameter. Due to Einstein's general relativity, Equation (95) is equivalent with

$$R_{\mu\nu} - \left(\frac{R}{2} \times g_{\mu\nu}\right) + \left(\Lambda \times g_{\mu\nu}\right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right). \tag{96}$$

$R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature, $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant and $T_{\mu\nu}$ is the stress-energy tensor. By defining the Einstein tensor as $G_{\mu\nu} = R_{\mu\nu} - (R/2) \times g_{\mu\nu}$, it is possible to write the Einstein field equations in a more compact as

$$G_{\mu\nu} + \left(\Lambda \times g_{\mu\nu}\right) = \left(\frac{4 \times 2 \times \pi \times \gamma}{c^4} \times T_{\mu\nu}\right). \tag{97}$$

Quod erat demonstrandum.

3.5. Theorem: The Normalized Relativistic Energy-Momentum Relation

Claim. (Theorem. Proposition. Statement.)

In general, it is

$$\frac{{}_{O}E^2}{{}_{R}E^2} + \frac{v^2}{c^2} = 1. \tag{98}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1. \tag{99}$$

Multiplying this equation by "rest" mass ${}_{O}m$, it is

$${}_{O}m \times 1 = {}_{O}m \times 1. \tag{100}$$

Due to special relativity, we rearrange this equation to

$${}_{O}m = {}_{R}m \times \sqrt[2]{1 - \frac{v^2}{c^2}}. \tag{101}$$

Multiplying by the speed of the light c square, we obtain

$${}_{O}m \times c^2 = {}_{R}m \times c^2 \times \sqrt[2]{\left(1 - \frac{v^2}{c^2}\right)}. \tag{102}$$

This equation is equivalent with

$${}_{O}E = {}_{R}E \times \sqrt[2]{\left(1 - \frac{v^2}{c^2}\right)}. \tag{103}$$

Rearranging this equation, we obtain

$$\frac{{}_{O}E}{{}_{R}E} = \sqrt[2]{\left(1 - \frac{v^2}{c^2}\right)}. \tag{104}$$

A further mathematical manipulation of the equation before leads to

$$\frac{{}_{O}E^2}{{}_{R}E^2} = \left(1 - \frac{v^2}{c^2}\right). \tag{105}$$

At the end, we obtain the relationship

$$\frac{{}_{O}E^2}{{}_{R}E^2} + \frac{v^2}{c^2} = 1. \tag{106}$$

Quod erat demonstrandum.

3.6. Theorem: The Normalized Relativistic Time Dilation Relation

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$\frac{{}_{O}t^2}{{}_{R}t^2} + \frac{v^2}{c^2} = 1. \tag{107}$$

Direct proof.
In general, axiom I is determined as

$$+1 = +1. \tag{108}$$

Multiplying this equation by time ${}_{O}t$ as measured by a co-moving observer, it is

$${}_{O}t \times 1 = {}_{O}t \times 1. \tag{109}$$

Due to special relativity, we rearrange this equation as

$${}_{O}t = {}_{R}t \times \sqrt[2]{1 - \frac{v^2}{c^2}}. \tag{110}$$

This equation is equivalent with

$$\frac{{}_{O}t}{{}_{R}t} = \sqrt[2]{\left(1 - \frac{v^2}{c^2}\right)}. \tag{111}$$

A further mathematical manipulation of the equation before leads to

$$\frac{{}_{O}t^2}{{}_{R}t^2} = \left(1 - \frac{v^2}{c^2}\right). \tag{112}$$

At the end, we obtain the relationship

$$\frac{{}_{O}t^2}{{}_{R}t^2} + \frac{v^2}{c^2} = 1. \tag{113}$$

Quod erat demonstrandum.

3.7. Theorem: The First Basic Law of Special Relativity

Special relativity implies basic physical law of far reaching consequences.
Claim. (Theorem. Proposition. Statement.)

Under conditions of special relativity, it is

$$ {}_{O}E \times {}_{R}t = {}_{O}t \times {}_{R}E . \tag{114} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 . \tag{115} $$

Due to the theorem before, it is (${}_{O}E/{}_{R}E + v^2/c^2 = 1$). We obtain

$$ \frac{{}_{O}E}{{}_{R}E} + \frac{v^2}{c^2} = 1 . \tag{116} $$

Due to another theorem before, it is (${}_{O}t/{}_{R}t + v^2/c^2 = 1$). We obtain

$$ \frac{{}_{O}E}{{}_{R}E} + \frac{v^2}{c^2} = \frac{{}_{O}t}{{}_{R}t} + \frac{v^2}{c^2} . \tag{117} $$

After subtraction of (v^2/c^2), it follows that

$$ \frac{{}_{O}E}{{}_{R}E} = \frac{{}_{O}t}{{}_{R}t} . \tag{118} $$

Multiplying this equation by the term (${}_{R}E \times {}_{R}t$), it is

$$ \frac{{}_{O}E \times {}_{R}E \times {}_{R}t}{{}_{R}E} = \frac{{}_{O}t \times {}_{R}E \times {}_{R}t}{{}_{R}t} . \tag{119} $$

Simplifying this equation, we obtain

$$ {}_{O}E \times {}_{R}t = {}_{O}t \times {}_{R}E . \tag{120} $$

Quod erat demonstrandum.

3.8. Theorem: The Normalization of the Relationship between ${}_{O}E$ and ${}_{\Delta}E$

Claim. (Theorem. Proposition. Statement.)
In general, we must accept that

$$ \frac{{}_{O}E}{{}_{R}E} + \frac{{}_{\Delta}E}{{}_{R}E} = 1 . \tag{121} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 . \tag{122} $$

Multiplying this equation by ${}_{R}E$, we obtain

$$ {}_{R}E = {}_{R}E . \tag{123} $$

Due to our definition ${}_{R}E = {}_{O}E + \underline{{}_{O}E}$ or ${}_{R}E = {}_{O}E + {}_{\Delta}E$ it is $\underline{{}_{O}E} = {}_{\Delta}E$ and we obtain

$$ {}_{O}E + {}_{\Delta}E = {}_{R}E . \tag{124} $$

Dividing this equation by ${}_{R}E$ it is

$$ \frac{{}_{O}E}{{}_{R}E} + \frac{{}_{\Delta}E}{{}_{R}E} = 1 . \tag{125} $$

Quod erat demonstrandum.

3.9. Theorem: The Normalization of the Relationship between $_{O}t$ and $_{\Delta}t$

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$\frac{_{O}t}{_{R}t}+\frac{_{\Delta}t}{_{R}t}=1. \tag{126}$$

Direct proof.
In general, axiom I is determined as

$$+1=+1. \tag{127}$$

Multiplying this equation by $_{R}t$, we obtain

$$_{R}t={}_{R}t. \tag{128}$$

Our definition was $_{R}t={}_{O}t+{}_{\Delta}t$. The equation before changes to

$$_{O}t+{}_{\Delta}t={}_{R}t. \tag{129}$$

Dividing this equation by $_{R}t$ it is

$$\frac{_{O}t}{_{R}t}+\frac{_{\Delta}t}{_{R}t}=1. \tag{130}$$

Quod erat demonstrandum.

3.10. Theorem: The Second Basic Law of Special Relativity

Special relativity implies another basic physical law of no less reaching consequences.
Claim. (Theorem. Proposition. Statement.)
Under conditions of special relativity, it is

$$_{\Delta}E\times{}_{R}t={}_{\Delta}t\times{}_{R}E. \tag{131}$$

Direct proof.
In general, axiom I is determined as

$$+1=+1. \tag{132}$$

Due to the theorem before, it is ($_{O}E/_{R}E+{}_{\Delta}E/_{R}E=1$). We obtain

$$\frac{_{O}E}{_{R}E}+\frac{_{\Delta}E}{_{R}E}=1. \tag{133}$$

Due to another theorem before, it is ($_{O}t/_{R}t+{}_{\Delta}t/_{R}t=1$). We obtain

$$\frac{_{O}E}{_{R}E}+\frac{_{\Delta}E}{_{R}E}=\frac{_{O}t}{_{R}t}+\frac{_{\Delta}t}{_{R}t}. \tag{134}$$

After multiplication with ($_{R}E\times{}_{R}t$), it follows that

$$\frac{_{O}E\times{}_{R}E\times{}_{R}t}{_{R}E}+\frac{_{\Delta}E\times{}_{R}E\times{}_{R}t}{_{R}E}=\frac{_{O}t\times{}_{R}E\times{}_{R}t}{_{R}t}+\frac{_{\Delta}t\times{}_{R}E\times{}_{R}t}{_{R}t}. \tag{135}$$

Rearranging equation, it is

$$_{O}E\times{}_{R}t+{}_{\Delta}E\times{}_{R}t={}_{O}t\times{}_{R}E+{}_{\Delta}t\times{}_{R}E. \tag{136}$$

According to the first basic law of special relativity, it is $_{O}E\times{}_{R}t={}_{R}E\times{}_{O}t$. Based on this insight, we rearrange the equation before and do obtain

$$ {}_{O}E \times {}_{R}t + {}_{\Delta}E \times {}_{R}t = {}_{O}E \times {}_{R}t + {}_{\Delta}t \times {}_{R}E \tag{137} $$

or at the end similar *the law of the lever* as provided and proven by Archimedes (~287 BC - ~212 BC)

$$ {}_{\Delta}E \times {}_{R}t = {}_{\Delta}t \times {}_{R}E \,. \tag{138} $$

Quod erat demonstrandum.

3.11. Theorem: The Relationship between the (Complex) Number ${}_{R}S$ and the Unknown Complex Number ${}_{R}Y$

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$ {}_{R}Y \equiv \frac{1}{{}_{R}S} \,. \tag{139} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 \,. \tag{140} $$

Multiplying this equation by ${}_{R}S \times {}_{R}Y$, we obtain

$$ {}_{R}S \times {}_{R}Y \times 1 \equiv {}_{R}S \times {}_{R}Y \times 1 \tag{141} $$

Due to our above definition, it is ${}_{R}S \times {}_{R}Y = 1$. Consequently, Equation (141) changes too

$$ {}_{R}S \times {}_{R}Y \equiv 1 \,. \tag{142} $$

We obtain

$$ {}_{R}Y \equiv \frac{1}{{}_{R}S} \,. \tag{143} $$

Quod erat demonstrandum.

3.12. Theorem: The Relationship between the Complex Conjugate ${}_{R}\Psi^{*}({}_{R}X, {}_{R}t)$ of the Wave Function ${}_{R}\Psi({}_{R}X, {}_{R}t)$ and the Unknown (Complex) Number ${}_{R}Y$

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$ {}_{R}Y \equiv {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) . \tag{144} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 \,. \tag{145} $$

Multiplying this equation by ${}_{R}\Psi({}_{R}X, {}_{R}t) \times {}_{R}\Psi^{*}({}_{R}X, {}_{R}t)$, we obtain

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) \times 1 \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) \times 1 \,. \tag{146} $$

Due to our above definition, it is ${}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}Y = {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right)$. Consequently, Equation (146) changes too

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}Y \equiv {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) . \tag{147} $$

At the end, we obtain

$$ {}_{R}Y \equiv {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) . \tag{148} $$

Quod erat demonstrandum.

3.13. Theorem: The Relationship between the (Complex) Number ${}_RS$ and the Complex Conjugate ${}_R\Psi^*({}_RX, {}_Rt)$ of the Wavefunction ${}_R\Psi({}_RX, {}_Rt)$

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$ {}_R\Psi^*\left({}_RX, {}_Rt\right) = \frac{1}{{}_RS}. \tag{149} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1. \tag{150} $$

Multiplying this equation by ${}_RY$, we obtain

$$ {}_RY \times 1 = {}_RY \times 1. \tag{151} $$

Due to Equation (148), it is ${}_RY = {}_R\Psi^*({}_RX, {}_Rt)$. Consequently, substituting this equation into Equation (151) we obtain

$$ {}_R\Psi^*\left({}_RX, {}_Rt\right) = {}_RY. \tag{152} $$

Due to Equation (143), it is ${}_RY = 1/{}_RS$. Consequently, substituting this equation into Equation (152) we obtain

$$ {}_R\Psi^*\left({}_RX, {}_Rt\right) = \frac{1}{{}_RS}. \tag{153} $$

Quod erat demonstrandum.

3.14. Theorem: The Probability 1-${}_Rp({}_R\Psi({}_RX, {}_Rt))$ as Associated with the Hamiltonian ${}_RH$

Claim. (Theorem. Proposition. Statement.)
The probability 1-${}_Rp({}_R\Psi({}_RX, {}_Rt))$ as associated with the Hamiltonian ${}_RH$ is determined as

$$ 1 - {}_Rp\left({}_R\Psi\left({}_RX, {}_Rt\right)\right) = {}_RH \times {}_R\Psi^*\left({}_RX, {}_Rt\right). \tag{154} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1. \tag{155} $$

Multiplying Equation (155) by ${}_RS$, we obtain

$$ {}_RS \times 1 = {}_RS \times 1. \tag{156} $$

Due to our definition above, we obtain

$$ {}_RH + {}_R\Psi\left({}_RX, {}_Rt\right) = {}_RS. \tag{157} $$

Multiplying Equation (157) by the complex conjugate of the wave function ${}_R\Psi^*({}_RX, {}_Rt)$, it is

$$ {}_RH \times {}_R\Psi^*\left({}_RX, {}_Rt\right) + {}_R\Psi\left({}_RX, {}_Rt\right) \times {}_R\Psi^*\left({}_RX, {}_Rt\right) = {}_RS \times {}_R\Psi^*\left({}_RX, {}_Rt\right). \tag{158} $$

Due to Equation (153) it is ${}_RS \times {}_R\Psi^*({}_RX, {}_Rt) = 1$. Thus far, Equation (158) changes to

$$ {}_RH \times {}_R\Psi^*\left({}_RX, {}_Rt\right) + {}_R\Psi\left({}_RX, {}_Rt\right) \times {}_R\Psi^*\left({}_RX, {}_Rt\right) = 1. \tag{159} $$

Following Born's rule, it is ${}_Rp\left({}_R\Psi\left({}_RX, {}_Rt\right)\right) = {}_R\Psi\left({}_RX, {}_Rt\right) \times {}_R\Psi^*\left({}_RX, {}_Rt\right)$. We obtain

$$ {}_{R}H \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right) + {}_{R}p\left({}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right) = 1. \tag{160} $$

At the end, it is

$$ 1 - {}_{R}p\left({}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right) = {}_{R}H \times {}_{R}\Psi^{*}\left({}_{R}X, {}_{R}t\right). \tag{161} $$

Quod erat demonstrandum.

3.15. Theorem: The Normalization of the Relationship between Energy and Time

Claim. (Theorem. Proposition. Statement.)
The relationship between Energy ${}_{R}E$ and time ${}_{R}t$ can be normalized as

$$ \frac{{}_{R}E}{{}_{R}S} + \frac{{}_{R}t}{{}_{R}S} = +1. \tag{162} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1. \tag{163} $$

Multiplying Equation (163) by ${}_{R}S$, we obtain

$$ {}_{R}S = {}_{R}S. \tag{164} $$

Due to our definition, we obtain

$$ {}_{R}E + {}_{R}t = {}_{R}S. \tag{165} $$

We divide the Equation (165) by ${}_{R}S$. The relationship between energy and time is normalized to 1 as

$$ \frac{{}_{R}E}{{}_{R}S} + \frac{{}_{R}t}{{}_{R}S} = +1. \tag{166} $$

Quod erat demonstrandum.

3.16. Theorem: The Normalization of the Relationship between Matter and Gravitational Field

In general, the modification of our understanding of space and time undergone through Einstein's relativity theory is indeed a profound one. But even Einstein's relativity theory does not [19] give satisfactory answers to a lot of questions. Einstein's successful geometrization of the gravitational field in his general theory of relativity does not include a geometrized theory of the electromagnetic field too. The theoretical physicists working in the field of the general theory of relativity were not able to succeed in finding a convincing geometrical formulation of the gravitational and electromagnetic field. Still, electromagnetic fields are not described by Riemannian metrics. More serious from the conceptual point of view, in order to achieve unification, with the development of quantum theory any conceptual unification of the gravitational and electromagnetic field should introduce a possibility that the fields can be quantized. In our striving toward *unification of the foundations of physics a relativistic field theory* we are looking for should therefore be an extension of the general theory of relativity and equally and of no less importance a generalization of the theory of the gravitational field. Evidently, following up these train of thoughts and in view of all these difficulties, the following theorem is based on a (gravitational) field of more complex nature. Still, in our attempt to obtain a deeper knowledge of the foundations of physics the new and basic concepts are in accordance with general relativity theory from the beginning but with philosophy too. The purpose of this theorem is to provide some new and basic fundamental insights of the relationship between matter and the gravitational field.

Claim. (Theorem. Proposition. Statement.)
The relationship between the quantum mechanical operator of matter and the wavefunction of the gravitational field can be normalized as

$$\frac{{}_{R}G}{{}_{R}U}+\frac{{}_{R}M}{{}_{R}U}=+1. \tag{167}$$

Direct proof.

In general, axiom I is determined as

$$+1=+1. \tag{168}$$

Multiplying Equation (168) by ${}_{R}U$, we obtain

$${}_{R}U={}_{R}U \tag{169}$$

which is equivalent to

$${}_{R}U+0={}_{R}U. \tag{170}$$

In general it is ${}_{R}M-{}_{R}M=0$. In our understanding ${}_{R}M$ is a determining part of ${}_{R}U$. Thus far, we obtain

$${}_{R}U-{}_{R}M+{}_{R}M={}_{R}U. \tag{171}$$

Following, Einstein all but matter is gravitational field. We obtain ${}_{R}G={}_{R}U-{}_{R}M$. Thus far, it follows that

$${}_{R}G+{}_{R}M={}_{R}U. \tag{172}$$

We divide the Equation (172) by ${}_{R}U$. The relationship between the quantum mechanical operator of matter ${}_{R}M$ and the wavefunction of the gravitational field ${}_{R}G$ is normalized to 1 as

$$\frac{{}_{R}G}{{}_{R}U}+\frac{{}_{R}M}{{}_{R}U}=+1. \tag{173}$$

Quod erat demonstrandum.

3.17. Theorem: The Equivalence of Gravitational Field and Time

Time has always featured prominently discussion in science and is especially associated with the logic of change. A good deal of the scientific work with respect to time has been especially important since the beginning of science as such. Under which conditions are we allowed assuming a period of time during which nothing changes? Further coverage, ahistorical overview and a general presentation of these and other topics related to time as such are available or can be found in literature. A proper discussion of the various views of the nature of the relationship between time and gravitational field and the different issues related to time and gravitational field as such would take us far beyond the scope of this simple theorem. For our purposes, time and gravitational field are related somehow. In any case, especially due to Einstein's relativity theory, there is a very close relationship between time the gravitational field and vice versa. According to this line of thought, we will analyze the relationship between time and gravitational field from the standpoint of Einstein's theory of special relativity. Questions about the nature of time appear to be closely connected to the issue of gravitational field itself. Under conditions of special theory of relativity, clocks which are moving with respect to an inertial system of observation are found to be running more slowly. A clock which is closer to the gravitational mass (*i.e.* deeper in its "gravity well") appears to go more slowly than a clock which is more distant from the same mass (energy). For example, clocks on the former Space Shuttle where found to ran slightly slower than reference clocks on the Earth. Thus far, the stronger the gravitational potential (*i.e.* the closer a clock is to the source of the gravitational field, the slower time passes). The gravitational time dilation has been repeatedly confirmed experimentally by several tests of general relativity. Altogether, the behavior of time as such is linked to the behavior of gravitational field and vice versa even if we still don't know how. The aim of this theorem is to work out the interior logic between gravitational field and time. As we will see, the relationship between time and gravitational field is similar to the relationship between mass and energy or the gravitational field is equivalent to time and vice versa, both are equivalent [20] or identical. Finally, we must accept the gravitational field-time equivalence.

Claim. (Theorem. Proposition. Statement.)

The gravitational field and time are equivalent. In general, it is

$$ {}_R G \equiv \frac{{}_R t}{c^2} \,. \tag{174} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 \,. \tag{175} $$

Due to Equation (166), it is

$$ +1 = \frac{{}_R E}{{}_R S} + \frac{{}_R t}{{}_R S} \,. \tag{176} $$

Due to Equation (173), Equation (176) changes to

$$ \frac{{}_R G}{{}_R U} + \frac{{}_R M}{{}_R U} = \frac{{}_R E}{{}_R S} + \frac{{}_R t}{{}_R S} \,. \tag{177} $$

Multiplying this equation by ${}_R U$, it is

$$ {}_R G + {}_R M = \frac{{}_R U}{{}_R S} \times {}_R E + \frac{{}_R U}{{}_R S} \times {}_R t \,. \tag{178} $$

According to our definition, it is ${}_R S = c^2 \times {}_R U$. Thus far, it is $1/c^2 = {}_R U / {}_R S$. Equation (178) changes to

$$ {}_R G + {}_R M = \frac{{}_R E}{c^2} + \frac{{}_R t}{c^2} \,. \tag{179} $$

Due to our definition of matter as ${}_R M = {}_R E / c^2$. Equation (179) changes to

$$ {}_R G + {}_R M = {}_R M + \frac{{}_R t}{c^2} \,. \tag{180} $$

The matter term ${}_R M$ drops out, and what is left is the wavefunction of the gravitational field ${}_R G$ as

$$ {}_R G = \frac{{}_R t}{c^2} \,. \tag{181} $$

Quod erat demonstrandum.

3.18. Theorem: The Normalization of the Relationship between the Hamitonian and the Wavefunction

Claim. (Theorem. Proposition. Statement.)
The relationship between the Hamiltonian operator and the wavefunction can be normalized as

$$ \frac{{}_R H}{{}_R S} + \frac{{}_R \Psi\left({}_R X, {}_R t\right)}{{}_R S} = +1 \,. \tag{182} $$

Direct proof.
In general, axiom I is determined as

$$ +1 = +1 \,. \tag{183} $$

Multiplying Equation (183) by ${}_R S$, we obtain

$$ {}_R S = {}_R S \,. \tag{184} $$

Due to our definition, we obtain

$$ {}_R H + {}_R \Psi\left({}_R X, {}_R t\right) = {}_R S \,. \tag{185} $$

Divide Equation (185) by ${}_{R}S$. The normalization of the relationship between the Hamiltonian and the wave-function follows as

$$\frac{{}_{R}H}{{}_{R}S} + \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{{}_{R}S} = +1. \tag{186}$$

Quod erat demonstrandum.

3.19. Theorem: The Wavefunction of the Gravitational Field ${}_{R}G$

The hottest and one of the major unresolved problems of today's quantum mechanics is the physical meaning of the wave function. The debate about the physical meaning of the wave function raises broader issues as well. In brief, the difficulties stemmed from an apparent conflict about the existence of an objective reality existing independent of the human mind and consciousness. The purpose of this theorem is to investigate the meaning of the wave function by analyzing the relationship between the wave function and the wave function of the gravitational field itself. As we will see, the wavefunction of a system and the wave function of the gravitational of the same system are equivalent.

Claim. (Theorem. Proposition. Statement.)

In general, the wavefunction of the gravitational field ${}_{R}G$ is determined as

$${}_{R}G \equiv \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2}. \tag{187}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1. \tag{188}$$

Due to Equation (186), it is

$$+1 = \frac{{}_{R}H}{{}_{R}S} + \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{{}_{R}S}. \tag{189}$$

Due Equation (173), Equation (189) changes to

$$\frac{{}_{R}G}{{}_{R}U} + \frac{{}_{R}M}{{}_{R}U} = \frac{{}_{R}H}{{}_{R}S} + \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{{}_{R}S}. \tag{190}$$

Multiplying this equation by ${}_{R}U$, it is

$${}_{R}G + {}_{R}M = \frac{{}_{R}U}{{}_{R}S} \times {}_{R}H + \frac{{}_{R}U}{{}_{R}S} \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right). \tag{191}$$

According to our definition, it is ${}_{R}S = c^2 \times {}_{R}U$. Thus far, it is $1/c^2 = {}_{R}U/{}_{R}S$. Equation (191) changes to

$${}_{R}G + {}_{R}M = \frac{{}_{R}H}{c^2} + \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2}. \tag{192}$$

Due to our definition of matter as ${}_{R}M = {}_{R}H/c^2$, Equation (192) changes to

$${}_{R}G + {}_{R}M = {}_{R}M + \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2}. \tag{193}$$

Subtracting ${}_{R}M$ on both sides of the equation before, the wavefunction of the gravitational field ${}_{R}G$ follows that

$${}_{R}G = \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2}. \tag{194}$$

Quod erat demonstrandum.

3.20. Theorem: The Equivalence of Time and Wavefunction

The (quantum mechanical) system, and with it the wavefunction varies with time. Theoretically, in this sense the wave function can be regarded as a function of time also. In general, the actual dependence of the wave function on time is determined by equations thus that the wave function ${}_R\Psi({}_RX, {}_Rt)$ and the time ${}_Rt$ as measured by a stationary observer R are equivalent.

Claim. (Theorem. Proposition. Statement.)

Under conditions of the special theory of relativity from the standpoint of a stationary observer R it is

$$ {}_Rt = {}_R\Psi\left({}_RX, {}_Rt\right). \tag{195}$$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1. \tag{196}$$

Multiplying this equation by ${}_RG$, the wave function of the gravitational field, we obtain

$$ {}_RG \times 1 = {}_RG \times 1. \tag{197}$$

Due to Equation (181), it is ${}_RG = {}_Rt/c^2$. We obtain

$$ \frac{{}_Rt}{c^2} = {}_RG. \tag{198}$$

According to Equation (194), t is ${}_RG = {}_R\Psi({}_RX, {}_Rt)/c^2$. Rearranging equation, we obtain

$$ \frac{{}_Rt}{c^2} = \frac{{}_R\Psi\left({}_RX, {}_Rt\right)}{c^2}. \tag{199}$$

Rearranging Equation (199), we obtain

$$ {}_Rt = {}_R\Psi\left({}_RX, {}_Rt\right). \tag{200}$$

Quod erat demonstrandum.

Scholium.

Any state of a system at a given moment can be described by a definite (in general complex) wave function (sometimes also *the probability amplitude*). The square of the modulus of this wave function determines the probability of the various results of any other measurement of a system. The sum of the probabilities of all possible values of a system must, by definition, must be equal to unity. It is therefore necessary that the result of integrating $|\Psi|^2$ should be equal to unity too (*the normalisation condition for wave function*). However, it is possible to use wave functions which are not normalized. In any physical system with waves, the waveform at a given time is a function of the source. In quantum mechanics, a wave can be described by a wave function. Moreover, the chief positive principle of quantum mechanics called *the principle of superposition* demands that a wave function is determined by a superposition (called "quantum superposition") of (possibly infinitely many) other wave functions of a certain type (called "eigenfunction"). The probability for a specified configuration is given by the square of the absolute value of a complex coefficient.

3.21. Theorem: The Relationship between Energy ${}_RE$, Time ${}_Rt$ and Space ${}_RS$

Claim. (Theorem. Proposition. Statement.)

In general, it is

$$ {}_RE + {}_Rt = {}_RS. \tag{201}$$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1. \tag{202}$$

Multiplying this equation by $({}_{R}H + {}_{R}\Psi({}_{R}X, {}_{R}t))$ we obtain

$$\left({}_{R}H + {}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right)\times 1 = \left({}_{R}H + {}_{R}\Psi\left({}_{R}X, {}_{R}t\right)\right)\times 1. \tag{203}$$

Due to our definition of ${}_{R}S = ({}_{R}H + {}_{R}\Psi({}_{R}X, {}_{R}t))$, Equation (203) changes to

$${}_{R}H + {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = {}_{R}S. \tag{204}$$

In our understanding, it is ${}_{R}H = {}_{R}E$. Substituting this relationship into Equation (204) it follows that

$${}_{R}E + {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = {}_{R}S. \tag{205}$$

Due to Equation (200) it is ${}_{R}t = {}_{R}\Psi({}_{R}X, {}_{R}t)$. Substituting this relationship into Equation (205) it follows in general that

$${}_{R}E + {}_{R}t = {}_{R}S. \tag{206}$$

Quod erat demonstrandum.

3.22. Theorem: The Normalization of the Relationship between the Gravitational Field ${}_{O}g$ and the Gravitational Wave ${}_{W}g$

Claim. (Theorem. Proposition. Statement.)
In general, it is

$$\frac{{}_{O}g^{2}}{{}_{R}g^{2}} + \frac{{}_{W}g^{2}}{{}_{R}g^{2}} = 1. \tag{206a}$$

Direct proof.
In general, axiom I is determined as

$$+1 = +1. \tag{207}$$

Multiplying Equation (207) by $(({}_{O}t^2/{}_{R}t^2) + (v^2/c^2))$, we obtain

$$\left(\frac{{}_{O}t^{2}}{{}_{R}t^{2}} + \frac{v^{2}}{c^{2}}\right)\times 1 \equiv \left(\frac{{}_{O}t^{2}}{{}_{R}t^{2}} + \frac{v^{2}}{c^{2}}\right)\times 1. \tag{208}$$

According to the relationship $(({}_{O}t^2/{}_{R}t^2) + (v^2/c^2)) = 1$, Equation (208) simplifies as

$$\left(\frac{{}_{O}t^{2}}{{}_{R}t^{2}} + \frac{v^{2}}{c^{2}}\right) = 1. \tag{209}$$

Equation (209) can be rearranged as

$$\frac{{}_{O}t^{2}\times c^{2}\times c^{2}}{{}_{R}t^{2}\times c^{2}\times c^{2}} + \frac{v^{2}}{c^{2}} = 1. \tag{210}$$

and equally as

$$\frac{{}_{O}t^{2}\times c^{2}\times c^{2}}{c^{2}\times c^{2}\times {}_{R}t^{2}} + \frac{v^{2}\times c^{2}\times {}_{R}g^{2}}{c^{2}\times c^{2}\times {}_{R}g^{2}} = 1. \tag{211}$$

Due to our definition it is ${}_{O}g = {}_{O}t/c^2$ and equally ${}_{R}g = {}_{R}t/c^2$, Equation (211) simplifies as to

$$\frac{{}_{O}g^{2}}{{}_{R}g^{2}} + \frac{v^{2}\times c^{2}\times {}_{R}g^{2}}{c^{2}\times c^{2}\times {}_{R}g^{2}} = 1. \tag{212}$$

According to our definition it is ${}_{W}t = v \times c \times {}_{R}g$ and thus far ${}_{W}t^2 = (v \times c \times {}_{R}g)^2$. Equation (212) changes to

$$\frac{{}_O g^2}{{}_R g^2} + \frac{{}_W t^2}{c^2 \times c^2 \times {}_R g^2} = 1. \tag{213}$$

The definition of the gravitational waves was ${}_W g = {}_W t/c^2$. Thus far, it is ${}_W g^2 = ({}_W t/c^2)^2$. The normalization of the relationship between the gravitational field ${}_O g$ and the gravitational waves ${}_W g$ follows as

$$\frac{{}_O g^2}{{}_R g^2} + \frac{{}_W g^2}{{}_R g^2} = 1. \tag{214}$$

Quod erat demonstrandum.

Scholium.

This theorem leads to some consequences. The relationship

$$\frac{{}_O g^2}{{}_R g^2} + \frac{{}_W g^2}{{}_R g^2} = 1 \tag{215}$$

is equivalent with the relationship

$$\frac{{}_O t^2}{{}_R t^2} + \frac{{}_W g^2 \times c^2 \times c^2}{{}_R t^2} = 1 \tag{216}$$

and thus far with the relation

$$\frac{{}_W g^2 \times c^2 \times c^2}{{}_R t^2} = \left(1 - \frac{{}_O t^2}{{}_R t^2}\right) \tag{217}$$

or with

$${}_W g^2 = \frac{{}_R t^2}{c^2 \times c^2} \times \left(1 - \frac{{}_O t^2}{{}_R t^2}\right). \tag{218}$$

Due to special relativity it is $\left(\left({}_O t^2 / {}_R t^2\right) + \left(v^2/c^2\right)\right) = 1$ and thus far $\left(v^2/c^2\right) = 1 - \left({}_O t^2 / {}_R t^2\right)$. We obtain

$${}_W g^2 = \frac{{}_R t^2}{c^2 \times c^2} \times \frac{v^2}{c^2}. \tag{219}$$

As we will see later, the quantum-mechanical wave function ${}_R\Psi({}_R X, {}_R t)$ and the time ${}_R t$ are identical. Consequently, the quantum-mechanical wave function ${}_R\Psi({}_R X, {}_R t)$ can be used to describe the gravitational waves ${}_W g$ as

$${}_W g^2 = \frac{v^2}{c^2 \times c^2 \times c^2} \times {}_R\Psi\left({}_R X, {}_R t\right) \times {}_R\Psi\left({}_R X, {}_R t\right). \tag{220}$$

3.23. Theorem: The Quantization of the Gravitational Field

The current understanding of gravitation and the gravitational field is based on Albert Einstein's general theory of relativity. In contrast to Einstein's general relativity, all non-gravitational forces are described within the framework of quantum mechanics. Clearly there is something unsatisfactory about the framework of quantum mechanics, a quantum mechanical description of gravitation and the gravitational field consistent with Einstein's theory of relativity is still not in sight. More and more, one often gets the impression that we are far away from the verge of a *Theory of Everything* and that combining gravity with all the others fundamental forces of nature is much more than just a matter of technical details.

From a technical point of view, the conceptual and technical problems associated with the approaches to quantize the gravitational field, known in the literature as "covariant" (particle physics) and "canonical" quantum gravity, disabled until today to construct a theory in which the gravitational field is treated quantum-mechanically. As a result, many theorists have taken up more radical approaches to the problem of the quantization of the gravitational field, quantum gravity, the most popular approaches being loop quantum gravity and string theory. This publication makes no pretence to be either complete or up-to-date of today's most popular approaches of the quantization of the gravitational field. A review of the contemporary methods of quantization of the gravitational field will not be given. The goal of this theorem is to follow a completely new approach and to solve the problem of the quantization of the gravitational field. To do this and to reconcile general relativity with the principles of quantum mechanics, we followed strictly Einstein's understanding of the relationship between matter and gravitational field.

Claim. (Theorem. Proposition. Statement.)

In general, the quantization of the gravitational field is determined by the equation

$$ {}_{R}M \times {}_{R}G = \frac{i\hbar}{c^2} \frac{\partial}{\partial t} \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} . \tag{221}$$

Direct proof.

In general, axiom I is determined as

$$ +1 = +1 . \tag{222}$$

Multiplying the Schrödinger equation, we obtain

$$ {}_{R}H \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times 1 = {}_{R}H \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \times 1 . \tag{223}$$

or equally

$$ {}_{R}H \times {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = i\hbar \frac{\partial}{\partial t} {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) . \tag{224}$$

Dividing by the speed of the light squared, we obtain

$$ \frac{{}_{R}H}{c^2} \times \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} = \frac{i\hbar}{c^2} \frac{\partial}{\partial t} \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} . \tag{225}$$

Due to our definition of matter it is ${}_{R}M = {}_{R}H/c^2$. The equation before changes to

$$ {}_{R}M \times \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} = \frac{i\hbar}{c^2} \frac{\partial}{\partial t} \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} \tag{226}$$

Due to Equation (194) it is ${}_{R}G = {}_{R}\Psi({}_{R}X, {}_{R}t)/c^2$. The quantization of the gravitational field follows as

$$ {}_{R}M \times {}_{R}G = \frac{i\hbar}{c^2} \frac{\partial}{\partial t} \frac{{}_{R}\Psi\left({}_{R}X, {}_{R}t\right)}{c^2} . \tag{227}$$

Quod erat demonstrandum.

Scholium.

Thus far, quantum mechanics strictly predicts that all matter is quantum while general relativity describes the gravitational effects of classical matter. Then again, one might argue that we cannot reconcile general relativity with the principles of quantum mechanics at all. In view of many possible approaches to quantize the gravitational field already developed, we described the dynamics of the gravitational field by the principles of quantum mechanics while following strictly the rules of Einstein's theory of relativity.

3.24. Theorem: The Physical Meaning of the Wavefunction

Conceptual difficulties are associated with quantum mechanics since its inception, despite of quantum mechanics extraordinary predictive successes. One "defining" part of quantum mechanics is the Schrödinger's equation. Quite naturally, Schrödinger's equation is to quantum mechanics what Newton's second law of motion is to classical mechanics. A determining part of Schrödinger's equation is the wave function. The nice fact is that quantum mechanics with its revolutionary implications meets still with serious difficulties in telling us *what is the physical meaning of the wave function*. As already stated, reconsidering basic concepts of one of the most successful theories in the history of human science, a very natural question which all scientists who are concerned about the physical meaning of the wave function is, what could be the correct starting point to solve this problem. This theorem will overcome the difficulties about the physical meaning of the wave function posed by quantum mechanics by strictly following Einstein in his proposal of to accept a physical reality as a kind of objective reality, a physical reality which is independent of the human being, his consciousness and his mind. The purpose of the present theorem is not to put forward a new and complete self-consistent interpretation of quantum mechanics which is able fully to highlight the explicative power of this revolutionary theory. But to fully appreciate the novel aspects contained in this theorem, it is more than necessary and highly useful to recall, clarify and elaborate about some basic notions needed to solve the problem of the physical meaning of the wave function. To complete this short overview, it is straightforward to convince oneself that the Hamiltonian, usually denoted by H or as ${}_RH$, is a quantum mechanical operator corresponding to the total energy of a (quantum mechanical) system. In most of the cases, *the spectrum of the Hamiltonian H* is the set of the possible outcomes when one measures the total energy of a system. In special relativity, the total energy of the system is denoted by ${}_RE$ and is equally the set of possible outcomes at least ${}_OE + {}_\Delta E = {}_OE + {}_O\underline{E} = {}_RE$.

Claim. (Theorem. Proposition. Statement.)

Under conditions of the special theory of relativity from the standpoint of a stationary observer R it is

$${}_Rt = {}_R\Psi\left({}_RX, {}_Rt\right). \tag{228}$$

Direct proof.

In general, axiom I is determined as

$$+1 = +1. \tag{229}$$

Multiplying this equation by ${}_RE \times {}_Rt$, we obtain

$${}_RE \times {}_Rt = {}_RE \times {}_Rt. \tag{230}$$

The system analyzed by us is completely described by this equation. The same system, no matter how complicated, is equally completely described by Schrödinger equation too. We equate both equations.

$${}_RE \times {}_Rt = {}_RH \times {}_R\Psi\left({}_RX, {}_Rt\right). \tag{231}$$

The Hamiltonian is a quantum mechanical equivalent of the total energy of a system, the set of all possible outcomes. In special relativity, the total energy of the system is denoted by ${}_RE$. Therefore, we equate both notions as ${}_RE = {}_RH$. Rearranging the equation above, it follows that

$${}_RE \times {}_Rt = {}_RE \times {}_R\Psi\left({}_RX, {}_Rt\right). \tag{232}$$

Finally, our proof is complete. In general, it is

$${}_Rt = {}_R\Psi\left({}_RX, {}_Rt\right). \tag{233}$$

Quod erat demonstrandum.

Scholium.

The general requirements of consistency and unity of a theorem assumes of course the absence of contradictions with experiment. The core of the Copenhagen Interpretation of quantum mechanics is the reduction of the wave function which is linked to the measurement processes. Due to Copenhagen Interpretation of quantum mechanics, during the measurement process, the quantum-mechanical wave function ${}_R\Psi({}_RX, {}_Rt)$ reduces to an eigenfunction of the system. Consequently, due to the Copenhagen Interpretation of quantum mechanics *the wave function does not possess any real physical meaning*. However, questions of this kind deal with only one aspect of quantum theory. To explain the main lines of thought, on this point opinions seem to be divided. In opposite to the canonical Copenhagen Interpretation of quantum mechanics, there are already interpretations of quantum mechanics which attributed a real physical meaning to the wave function. In contrast to the more or less hidden agenda of scientists blocking progress toward a unification of the incompatible descriptions of reality provided by relativity and quantum theory a realistic description of reality at its most fundamental level is possible in principle and necessary.

Especially David Bohm (1917-1992), a theoretical U. S. physicist, rediscovered an interpretation of the quantum theory made by de Broglie [21] in 1926 (de Broglie's pilot-wave theory), but later given up by him, and presented a principal alternative to the very dogmatic worldview as pushed by the Copenhagen interpretation of quantum mechanics. Bohm's nonrelativistic interpretation of quantum mechanics [22], known as "hidden variable" theory, is still held at a distance by most physicists. In the Bohmian mechanical version of nonrelativistic quantum theory, the wave function, obeying Schrödinger's equation, does not provide a complete description of a quantum system. Still, in Bohm's nonrelativistic interpretation of quantum mechanics the wave function gets a real physical meaning. "*In contrast to the usual interpretation, this alternative interpretation permits us to conceive of each individual system as being in a precisely definable state, whose changes with time are determined by definite laws, analogous to (but not identical with) the classical equations of motion*." [23]. Not long after second world war (1939-1945) Bohm was charged before the Committee on Un-American Activities in May and June 1949, but refused to cooperate with McCarthy's committee. As a consequence, Princeton University fired him, Bohm academic career was derailed. In 1600, the Roman Catholic Church burned *Giordano Bruno* (1548-1600) alive. In the 1950s the U. S. Administration blacklisted *David Bohm* by U.S. universities and effectively exiled him to Brazil, Israel and finally to England. On the first sight, it is possible to claim that ${}_{R}t$ has nothing in common with ${}_{R}\Psi({}_{R}X, {}_{R}t)$. This relationship can be rearranged something as

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) = {}_{R}t \times \frac{1}{1} = \frac{{}_{R}t}{\mathrm{e}^{-\frac{E\times t}{\hbar}}} \times \mathrm{e}^{-\frac{E\times t}{\hbar}} \equiv A \times \mathrm{e}^{-\frac{E\times t}{\hbar}} . \tag{234} $$

In the above equation we defined $A = {}_{R}t/(\mathrm{e}^{-(\dots)})$. At the end we obtain something like

$$ {}_{R}\Psi\left({}_{R}X, {}_{R}t\right) \equiv A \times \mathrm{e}^{-\frac{E\times t}{\hbar}} . \tag{235} $$

Thus far, all the changes above have no influence on the fact that ${}_{R}t = {}_{R}\Psi({}_{R}X, {}_{R}t)$.

4. Discussion

The hegemonic standard a causal Copenhagen Interpretation of quantum mechanics established by a small elite of physicists in the 1920s centers around the uncertainty principle (which is meanwhile refuted [24]-[25]) and opened a very wide door to mysticism, logical fallacies and a wishful thinking especially in quantum physics. The political attitude and the ideology of this small elite of physicists (Niels Bohr, Werner Heisenberg, Max Born and view other) played a major role in the construction of the Copenhagen Interpretation of quantum mechanics in the 1920s. Regardless of observation, the associating ideas, the political attitude or the ideology of physicists following strictly the *acausal*, *indeterministic Copenhagen Interpretation of quantum mechanics* objective reality (or the properties of matter) is the way the same is. We still find, years after quantum mechanics inception that quantum mechanics is fundamentally only about observation or results of measurement but not about reality as such. On the basis of more or less the same or similar considerations as those above it is claimed that a deterministic completion or reinterpretation of quantum theory is not possible. Strictly speaking, the wave function is still one of the pillars of quantum mechanics. Especially, due to the Copenhagen Interpretation of quantum mechanics *the wave function does not possess any physical meaning*. Therefore, the theory quantum mechanics, perhaps the most revolutionary theory in the history of science, still raises innumerable questions to physicists, chemists and philosophers of science.

The guideline for Einstein in his physical research was an objective reality existing independently of human mind and consciousness. In particular Einstein was one of the most famous critics of the Copenhagen Interpretation of quantum mechanics. Einstein has always regarded quantum theory as incomplete [26].

Due to the excellent agreement with an extremely wide range of experiments, the most physicists regarded the objections as raised by Einstein and other as not relevant.

The wave function is existing independently and outside of human mind and consciousness and something objective and real. The wave function is the quantum mechanical equivalent of the time ${}_{R}t$ of the special theory of relativity as measured by a stationary observer *R*. Thus far, there is nothing mysterious with the wave function. In contrast to the representatives of the Copenhagen interpretation, a realistic interpretation of quantum theory will expel any kind of mysticism from physics.

In particular, there is another crucial aspect of quantum mechanics called *the reduction of the state vector* (*i.e.* collapse of the wavefunction). The collapse of the wave function and the correct understanding of the collapse of the wave function addresses several distinct, important and far reaching issues of the foundations of today's physics and science as such. Originally, the concept of wavefunction collapse was introduced by Werner Heisenberg in his 1927 paper "Über den anschaulichen Inhalt der quantentheoretischen Kinematic und Mechanik" and later incorporated into the mathematical formalism of quantum mechanics by John von Neumann in his 1932 publication "Mathematische Grundlagen der Quantenmechanik". In his 1927 paper, Heisenberg writes

"*durch die experimentelle Feststellung*: '*Zustand m*' *wählen wir aus der Fülle der verschiedenen Möglichkeiten* (c_{nm}) *eine bestimmte: m aus, zerstören aber gleichzeitig, wie nachher erläutert wird, alles, was an Phasenbeziehungen noch in den Größen* c_{nm} *enthalten war*" [27].

It is easy understand the core of this problem. A (relativistic) system evolves in time by the continuous evolution via the Schrödinger equation or some relativistic equivalent. Under appropriate circumstances, the wave function, initially in a superposition of several eigenstates, collapses or reduces to a single eigenstate, that what is measured by a moving observer O. However, after the collapse of the wave function, a physical system is determined or described again by a wave function.

Thus far, the continuous evolution via the Schrödinger equation and the collapse of the wave function are the two basic processes by which quantum systems evolve in time. However, let us focus on the appropriate notion of the collapse of the wave function. Is the collapse of the wavefunction a fundamental and objective physical phenomenon of its own, rather than a non-real theoretical mathematical construct? Does the collapse of the wave function takes any time, the collapse time?

The Copenhagen Interpretation of quantum mechanics is grounded on a self-closing chain of circular hypotheses, which are in principle unverifiable. The hypotheses even if found to be inadequate can be made to agree with experiment by some yet unknown changes in the mathematical formulation of the theory without requiring any fundamental changes in the physical interpretation. Thus far, as long as we accept the usual Copenhagen Interpretation of quantum mechanics, it is impossible to give up this interpretation, even if the same interpretation is proofed to be wrong by some (thought) experiments.

5. Conclusion

The wave function is one of the most important concepts of quantum mechanics. Schrödinger himself originally attempted a largely classical interpretation of the wave function as a description of real physical wave. Born's probability interpretation of the wave function operating with a mysterious collapse of the wave function during measurement replaced Schrödinger's view and became the standard interpretation of the wave function today. However, the standard interpretation of the wave function is based on measurement and still unsatisfying in principle. Thus far, it is neither possible nor necessary to interpret the wave function as something objective and physically real. Especially, due to standard Copenhagen interpretation, the wave function does not possess any physical meaning; the wave function is not referring to something physically real. In view of these problems, alternative and realistic interpretations [22]-[23], [28]-[30] of the wave function have been proposed. According to the de Broglie-Bohm theory, the wave function is generally taken as an objective physical field. In this paper, we have demonstrated that a correct local realistic interpretation of the wave function is possible without any contradiction. Especially, the problem of the physical meaning of the wave function is solved. The wave function is the quantum mechanical analogue of the notion time ${}_{R}t$ as determined by a stationary observer R of the theory of special relativity.

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A New Conjugate Gradient Projection Method for Solving Stochastic Generalized Linear Complementarity Problems

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Abstract

In this paper, a class of the stochastic generalized linear complementarity problems with finitely many elements is proposed for the first time. Based on the Fischer-Burmeister function, a new conjugate gradient projection method is given for solving the stochastic generalized linear complementarity problems. The global convergence of the conjugate gradient projection method is proved and the related numerical results are also reported.

Keywords

Stochastic Generalized Linear Complementarity Problems, Fischer-Burmeister Function, Conjugate Gradient Projection Method, Global Convergence

1. Introduction

Suppose (Ω_1, F, G, P) is a probability space with $\Omega_1 \subseteq \Re^n$; P is a known probability distribution. The stochastic generalized linear complementarity problems (denoted by SGLCP) is to find $x \in \Re^n$, such that

$$F(x,\omega) := M_1(\omega)x + q_1(\omega) \geq 0, G(x,\omega) := M_2(\omega)x + q_2(\omega) \geq 0, F^{\mathrm{T}}(x,\omega)G(x,\omega) = 0, \tag{1}$$

where $M_1(\omega), M_2(\omega) \in \Re^{n\times n}$ and $q_1(\omega), q_2(\omega) \in \Re^n$ for $\omega \in \Omega_1$, are random matrices and vectors. When $G(x,\omega) = x$, stochastic generalized linear complementarity problems reduce to the classic Stochastic Linear Complementarity Problems (SLCP), which has been studied in [1]-[7]. Generally, they usually apply the Expected Value (EV) method and Expected Residual Minimization (ERM) method to solve this kind of problem.

If Ω_1 only contains a single realization, then (1) reduces to the following standard Generalized Linear Complementarity Problem (GLCP), which is to find a vector $x \in \Re^n$ such that

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$$F(x) := M_1 x + q_1 \geq 0, G(x) := M_2 x + q_2 \geq 0, F^{\mathrm{T}}(x) G(x) = 0,$$

where $M_1, M_2 \in \Re^{n \times n}$ and $q_1, q_2 \in \Re^n$.

In this paper, we consider the following generalized stochastic linear complementarity problems. Denote $\Omega_1 = \{\omega_1, \omega_2, \cdots, \omega_m\}$, to find an $x \in \Re^n$ such that

$$\begin{aligned} &F(x, \omega_i) := M_1(\omega_i) x + q_1(\omega_i) \geq 0, \\ &G(x, \omega_i) := M_2(\omega_i) x + q_2(\omega_i) \geq 0, \quad i = 1, \cdots, m, \ m > 1. \\ &F^{\mathrm{T}}(x, \omega_i) \cdot G(x, \omega_i) = 0. \end{aligned} \tag{2}$$

Let $\bar{M}_j = \sum_{i=1}^{m} p_i M_j(\omega_i), \bar{q}_j = \sum_{i=1}^{m} p_i q_j(\omega_i)$, where $p_i = P(\omega_i \in \Omega_1) > 0$, $i = 1, \cdots, m$, $j = 1, 2$. Then (2) is equivalent to (3) and (4)

$$\bar{M}_1 x + \bar{q}_1 \geq 0, \bar{M}_2 x + \bar{q}_2 \geq 0, \left(\bar{M}_1 x + \bar{q}_1\right)^{\mathrm{T}} \cdot \left(\bar{M}_2 x + \bar{q}_2\right) = 0, \tag{3}$$

$$\begin{aligned} &M_1(\omega_i) x + q_1(\omega_i) \geq 0, \\ &M_2(\omega_i) x + q_2(\omega_i) \geq 0, \quad i = 1, \cdots, m. \end{aligned} \tag{4}$$

In the following of this paper, we consider to give a new conjugate gradient projection method for solving (2). The method is based on a suitable reformulation. Base on the Fischer-Burmeister function, x is a solution of (3) $\Leftrightarrow \phi(x) = 0$, where

$$\phi(x) = \begin{pmatrix} \phi\left(\left(\bar{M}_1 x + \bar{q}_1\right)_1, \left(\bar{M}_2 x + \bar{q}_2\right)_1\right) \\ \vdots \\ \phi\left(\left(\bar{M}_1 x + \bar{q}_1\right)_n, \left(\bar{M}_2 x + \bar{q}_2\right)_n\right) \end{pmatrix}.$$

Define

$$\Psi(x) = \frac{1}{2} \|\phi(x)\|^2.$$

Then solving (3) is equivalent to find a global solution of the minimization problem

$$\min_{x \in \Re^n} \Psi(x).$$

So, (3) and (4) can be rewritten as

$$H(x, y) = 0, \quad y \geq 0, \tag{5}$$

where

$$H(x, y) = \begin{pmatrix} \phi(x) \\ M_1(\omega_1) x + q_1(\omega_1) - y_1 \\ \vdots \\ M_1(\omega_m) x + q_1(\omega_m) - y_m \\ M_2(\omega_1) x + q_2(\omega_1) - y_{m+1} \\ \vdots \\ M_2(\omega_m) x + q_2(\omega_m) - y_{2m} \end{pmatrix},$$

$y = \left[y_1^{\mathrm{T}}, y_2^{\mathrm{T}}, \cdots, y_{2m}^{\mathrm{T}}\right]^{\mathrm{T}} \in \Re^{2m \times n}$ is slack variable with $y_i \in \Re^n$, $i = 1, \cdots, 2m$.

Let $x = x' - x''$, where $x', x'' \in \Re^n$ and $x', x'' \geq 0$. Then we know that $H(x', x'', y) = 0$ has $(2m+2)n$ equations with $(2m+2)n$ variables.

Let $t = (x', x'', y) \in \Re_+^{(2m+2)n}$ and define a merit function of (5) by

$$\theta(t)=\frac{1}{2}\|H(t)\|^2.$$

If (2) has a solution, then solving (5) is equivalent to find a global solution of the following minimization problem

$$\min \theta(t) \qquad (6)$$
$$s.t. \quad t\in\Omega$$

where $\Omega=\left\{t \middle| t\in\Re_+^{(2m+2)n}\right\}$.

2. Preliminaries

In this section, we give some Lemmas, which are taken from [8]-[10].

Lemma 1. Let P be the projection onto Ω, let $t(s)=P(t+s)$ for given $t\in\Omega$ and $s\in\Re^{(2m+2)n}$, then
1) $\langle t(s)-(t+s), y-t(s)\rangle\ge 0$, for all $y\in\Omega$.
2) P is a non-expansive operator, that is, $\|P(y)-P(x)\|\le\|y-x\|$ for all $x,y\in\Re^{(2m+2)n}$.
3) $\langle -s, t-t(s)\rangle\ge\|t(s)-t\|^2$.

Lemma 2. Let $\nabla_\Omega\theta(t)$ be the projected gradient of θ at $t\in\Omega$.
1) $\min\{\nabla\theta(t), v: v\in T(t), \|v\|\le 1\}=-\|\nabla_\Omega\theta(t)\|$.
2) The mapping $\|\nabla_\Omega\theta(\cdot)\|$ is lower semicontinuous on Ω, that is, if $\lim\limits_{k\to\infty} t_k\to t$, then

$$\|\nabla_\Omega\theta(t)\|\le\liminf_{k\to\infty}\|\nabla_\Omega\theta(t_k)\|.$$

3) The point $t^*\in\Omega$ is a stationary point of problem (6) ⇔ $\nabla_\Omega\theta(t^*)=0$.

3. The Conjugate Gradient Projection Method and Its Convergence Analysis

In this section, we give a new conjugate gradient projection method and give some discussions about this method.

Given an iterate $t_k\in\Omega=\left\{t \middle| t\in\Re_+^{(2m+2)n}\right\}$, we let $\overline{t_k}(s_k)=P\left[t_k-\nabla\theta(t_k)\right]$,

$$t_{k+1}=t_k(s_k)=P\left[t_k+s_k\right], \qquad (7)$$

where $s_k=\begin{cases}-\nabla\theta(t_k) & k=1\\ -\nabla\theta(t_k)+\beta_k d_{k-1} & k>1\end{cases}$. Inspired by the literature [8]-[11], we take

$$|\beta_k|=\frac{\left\|\overline{t_k}(s_k)-t_k\right\|^2}{(1+\lambda)\|\nabla\theta(t_k)\|\|d_{k-1}\|}, \qquad (8)$$

with $\lambda>0$.

And d_k is defined by

$$d_k=t_k(s_k)-t_k. \qquad (9)$$

Method 1. Conjugate Gradient Projection Method (CGPM)
Step 0: Let $t_1\in\Omega$, $0\le\varepsilon\le 1$, $\sigma_1,\sigma_2\in(0,1)$, $\beta_1=0$, $d_0=0$, set $k=1$.
Step 1: Compute α_k, such that

$$\theta(t_k+\alpha_k d_k)\le\theta(t_k)+\sigma_1\alpha_k\nabla\theta(t_k)^{\mathrm{T}}d_k,$$

$$\nabla\theta(t_k+\alpha_k d_k)^{\mathrm{T}}d_k\ge\sigma_2\nabla\theta(t_k)^{\mathrm{T}}d_k.$$

Set $t_{k+1}=t_k+\alpha_k d_k$.
Step 2: If $\|t_k-t_k(s_k)\|\le\varepsilon$, stop, $t^*=t_k(s_k)$.
Step 3: Let $k:=k+1$, and go to Step 1.
In order to prove the global convergence of the Method 1, we give the following assumptions.

Assumptions 1

1) $\theta(t)$ has a lower bound on the level set $L_0 = \left\{ t_1 \in \Re^{(2m+2)n} \middle| \theta(t) \le \theta(t_1) \right\}$, where t_1 is initial point.

2) $\theta(t)$ is continuously differentiable on the L_0, and its gradient is Lipschitz continuous, that is, there exists a positive constant L such that

$$\|g(u) - g(v)\| \le L\|u - v\| \quad \forall u, v \in L_0 .$$

Lemma 3. If t_k is not the stability point of (6), $t_k \ne t_k(s_k)$, then search direction d_k generated by (9) descent direction, which is $\langle \nabla\theta(t_k), d_k \rangle \le -\frac{\lambda}{1+\lambda}\|\nabla\theta(t_k)\|^2 < 0$.

Proof. From (7), Lemma 1, and (8), we have

$$\begin{aligned}
&\langle \nabla\theta(t_k), d_k \rangle = \langle \nabla\theta(t_k), t_k(s_k) - t_k \rangle \\
&= \left[\langle \nabla\theta(t_k), t_k(s_k) - \overline{t_k}(s_k) \rangle + \langle \nabla\theta(t_k), \overline{t_k}(s_k) - t_k \rangle \right] \\
&\le \|\nabla\theta(t_k)\| \|t_k(s_k) - \overline{t_k}(s_k)\| - \langle \nabla\theta(t_k), t_k - \overline{t_k}(s_k) \rangle \\
&\le |\beta_k| \|\nabla\theta(t_k)\| \|d_{k-1}\| - \|\overline{t_k}(s_k) - t_k\|^2 \\
&\le \left(\frac{1}{1+\lambda} - 1 \right) \|\overline{t_k}(s_k) - t_k\|^2 \\
&\le \frac{-\lambda}{1+\lambda} \|\nabla\theta(t_k)\|^2 < 0.
\end{aligned}$$

Lemma 4. [11] Suppose that Assumptions 1 holds. Let $\theta(t)$ continuously differentiable and lower bound on the Ω, $\nabla\theta(t)$ is uniformly continuous on the Ω and $\{\nabla\theta(t_k)\}$ is bounded, then $\{t_k\}$ generated by Method 1 are satisfied

$$\lim_{k\to\infty} \|t_k - t_k(s_k)\| = 0, \quad \lim_{k\to\infty} \|t_k - \overline{t_k}(s_k)\| = 0 .$$

Theorem 1. Let $\theta(t)$ continuously differentiable and lower bound on the Ω, $\nabla\theta(t)$ is uniformly continuous on the Ω, $\{t_k\}$ is a sequence generated by Method 1, then $\lim_{k\to\infty} \|\nabla_\Omega \theta(t_k)\| = 0$, and any accumulation point of $\{t_k\}$ is a stationary point of (6).

Proof. By Lemma 2, we have $\forall \varepsilon > 0$, $\exists v_k \in T_\Omega(t_k)$, $\|v_k\| \le 1$, satisfy

$$\|\nabla_\Omega \theta(t_k)\| \le \langle -\nabla\theta(t_k), v_k \rangle + \varepsilon , \tag{10}$$

for $\forall z \in \Omega$, by Lemma 1, we know that $\langle t_k(s_k) - (t_k + s_k), z - t_k(s_k) \rangle \ge 0$, and we have $\langle s_k, z - t_k(s_k) \rangle \le \langle t_k(s_k) - t_k, z - t_k(s_k) \rangle \le \|t_k(s_k) - t_k\| \|z - t_k(s_k)\|$, so,

$$\langle s_k, z - t_k(s_k) \rangle \le \|t_k(s_k) - t_k\| \|z - t_k(s_k)\| . \tag{11}$$

Let $v_{k+1} = z - t_k(s_k) \in T_\Omega(t_{k+1})$, $\|v_{k+1}\| \le 1$, from (11), we have

$$\langle s_k, v_{k+1} \rangle = \langle -\nabla\theta(t_k) + \beta_k d_{k-1}, v_{k+1} \rangle \le \|t_k(s_k) - t_k\| .$$

By the above formula, (8) and Lemma 1, we get

$$\begin{aligned}
\langle -\nabla\theta(t_k), v_{k+1} \rangle &\le \|t_k(s_k) - t_k\| + |\beta_k| \|d_{k-1}\| \\
&\le \|t_k(s_k) - t_k\| + \frac{1}{(1+\lambda)\|\nabla\theta(t_k)\|} \|\overline{t_k}(s_k) - t_k\|^2 \\
&\le \|t_k(s_k) - t_k\| + \frac{1}{1+\lambda} \|\overline{t_k}(s_k) - t_k\|.
\end{aligned}$$

Taking limit on both sides and by Lemma 4, we know that

$$\lim_{k\to\infty}\sup\left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle = 0. \tag{12}$$

Because

$$\begin{aligned}\left\langle -\nabla\theta(t_k(s_k)), v_{k+1}\right\rangle &= \left\langle \nabla\theta(t_k)-\nabla\theta(t_k(s_k)), v_{k+1}\right\rangle + \left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle \\ &\le \left\|\nabla\theta(t_k)-\nabla\theta(t_k(s_k))\right\| + \left\langle -\nabla\theta(t_k), v_{k+1}\right\rangle\end{aligned} \tag{13}$$

and Lemma 4, we have

$$\lim_{k\to\infty}\left\|t_k - t_k(s_k)\right\| = 0. \tag{14}$$

By (12), (13), (14) and $\nabla\theta(t)$ is uniformly continuous on the Ω, we get

$$\lim_{k\to\infty}\sup\left\langle -\nabla\theta(t_k(s_k)), v_{k+1}\right\rangle = 0.$$

By (10), we know that

$$\lim_{k\to\infty}\left\|\nabla_\Omega\theta(t_k)\right\| = 0. \tag{15}$$

Let $\lim_{k\in N_0, k\to\infty} t_k = t$, where $N_0 \subseteq N$, by Lemma 2 and (15), we have

$$\left\|\nabla_\Omega\theta(t)\right\| \le \lim_{k\in N_0, k\to\infty}\inf\left\|\nabla_\Omega\theta(t_k)\right\| = 0.$$

From Lemma 2 3), we get any accumulation point of $\{t_k\}$ is a stationary point of (6).

4. Numerical Results

In this section, we give the numerical results of the conjugate gradient projection method for the following given test problems, which are all given for the first time. We present different initial point t_0, which indicates that Method 1 is global convergence.

Throughout the computational experiments, according to Method 1 for determining the parameters, we set the parameters as

$$\sigma_1 = 0.49,\ \sigma_2 = 0.5,\ \lambda = 1.067.$$

The stopping criterion for the method is $\|g_k\| \le 10^{-6}$ or $k_{\max} = 100000$.

In the table of the test results, t_0 denotes initial point, x^* denotes the solution, val denotes the final value of $\theta(t) = \frac{1}{2}\|H(t)\|^2$, Itr denotes the number of iteration.

Example 1. Considering SGLCP with

$$M_1(\omega) = \begin{pmatrix} \frac{3}{2}+\omega & -1 & 0 \\ -1 & \frac{3}{2}+\omega & -1 \\ 0 & -1 & \frac{3}{2}+\omega \end{pmatrix},\quad q_1(\omega) = \begin{pmatrix} \frac{1}{2}+\omega \\ \frac{1}{2}+\omega \\ \frac{1}{2}+\omega \end{pmatrix},$$

$$M_2(\omega) = \begin{pmatrix} \frac{5}{2}+\omega & -1 & 0 \\ -1 & \frac{5}{2}+\omega & -1 \\ 0 & -1 & \frac{5}{2}+\omega \end{pmatrix},\quad q_2(\omega) = \begin{pmatrix} 1+\omega \\ 1+\omega \\ 1+\omega \end{pmatrix},$$

$$\Omega_1 = \{\omega_1, \omega_2\} = \{0,1\} \text{ and } p_i = P(\omega_i \in \Omega_1) = 0.5,\ i = 1,2.$$

The test results are listed in "**Table 1**" using different initial points.

Table 1. Results of the numerical Example 1-2 using method 1.

Problem	t_0	x^*	val	Itr
Example 1	$0.5\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	1465
	$(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	1701
	$5\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	2670
	$10\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	3261
	$20\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	3847
	$50\times(1,1,\cdots,1)$	$(-0.8385,-1.0548,-0.8385)$	3.3×10^{-3}	4704
Example 2	$0.5\times(1,1,\cdots,1)$	$(-0.3747,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	62788
	$(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	65528
	$5\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	66962
	$10\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000
	$20\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000
	$50\times(1,1,\cdots,1)$	$(-0.3746,0.1516,-0.0276,-0.0770,0.2306,-0.9539,1.4488)$	0.7299	100,000

Example 2. Considering SGLCP with

$$M_1(\omega)=\begin{pmatrix} \frac{1}{2}+\omega & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & \frac{1}{2}+\omega & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & \frac{1}{2}+\omega & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & \frac{1}{2}+\omega & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & \frac{1}{2}+\omega & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+\omega & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}+\omega \end{pmatrix},\quad q_1(\omega)=\begin{pmatrix} -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \\ -\frac{3}{2}+\omega \end{pmatrix},$$

$$M_1(\omega)=\begin{pmatrix} \frac{3}{2}+\omega & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & \frac{3}{2}+\omega & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & \frac{3}{2}+\omega & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & \frac{3}{2}+\omega & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & \frac{3}{2}+\omega & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2}+\omega & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{2}+\omega \end{pmatrix},\quad q_1(\omega)=\begin{pmatrix} -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \\ -1+\omega \end{pmatrix},$$

$$\Omega_1=\{\omega_1,\omega_2\}=\{0,1\} \quad \text{and} \quad p_i=P(\omega_i\in\Omega_1)=0.5, \quad i=1,2.$$

The test results are listed in "**Table 1**" using different initial points.

5. Conclusion

In this paper, we present a new conjugate gradient projection method for solving stochastic generalized linear complementarity problems. The global convergence of the method is analyzed and numerical results show that Method 1 is effective. In future work, large-scale stochastic generalized linear complementarity problems need to be studied and developed.

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Spatio-Temporal Pulsating Dissipative Solitons through Collective Variable Methods

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Abstract

A semi-analytical approach for the pulsating solutions of the 3D complex Cubic-quintic Ginzburg-Landau Equation (CGLE) is presented in this article. A collective variable approach is used to obtain a system of variational equations which give the evolution of the light pulses parameters as a function of the propagation distance. The collective coordinate approach is incomparably faster than the direct numerical simulation of the propagation equation. This allows us to obtain, efficiently, a global mapping of the 3D pulsating soliton. In addition it allows describing the influence of the parameters of the equation on the various physical parameters of the pulse and their dynamics.

Keywords

Dissipative Soliton, Pulsating Light Pulse, Spatiotemporal Pulses, Collective Variable Approach, Complex Cubic-Quintic Ginzburg-Landau Equation, Bifurcation

1. Introduction

Dissipative systems in nonlinear optics have a dynamic involving many different phenomena such as nonlinear gain, the saturable losses, the dispersion and others effects. The interaction between these different physical manifestations leads to a rich variety of structures [1]. Dissipative solitons are stable elementary localized structures that appear in dissipative nonlinear medium and depend on the balance between nonlinearity and dispersion/diffraction, and the balance between gain and loss. They have several properties such as their internal

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energy exchange mechanism which is similar to that of biological structures or the possibility to reveal pulsating dynamics [2]. These properties make them attractive for research. The concept of dissipative solitons has been widely studied in several fields of nonlinear science [3] such as nonlinear dissipative optics. Among the important applications there are passively mode-locked laser systems and optical transmission lines.

For dissipative systems in nonlinear optics, stable solitons can arise in one, two, and three dimensions [4]. For example, in case of the one-dimensional (1D), new types of localized waves [5]-[7] have been experimentally [8] observed in lasers cavities.

With a good choice of the parameters of the system, stable solitons over very large distances of propagation are perfectly obtained. However, a variation of the parameters of the system changes the types of solutions via bifurcations. Dissipative soliton stability depends crucially on the energy balance and exists as long as there is a continuous energy supply to the system. Its shape, amplitude, and velocity are all fixed and defined by the parameters of the system [9] rather than by the initial condition.

Properties and conditions of the existence of solitons have been extensively studied in (1D) one-dimension and (2D) two-dimension. The (3D) three-dimensional case is still largely in its infancy and requires an extremely lengthy and costly procedure. Indeed solving numerically a (3D) equation for a given set of parameters and a given initial condition can take up to several days in a standard PC. In this context, it is important to develop theoretical tools that can perceive soliton solutions more efficiently and envisage their domains of existence.

It has been recently demonstrated in previous works that the collective variable approach is a useful tool which can reduce significantly the computation time. This approach is useful for predicting approximately the domains of existence of stable light bullets in the parameter space of the (3D) complex cubic-quintic Ginzburg-Landau equation [10].

Here, previous studies are focusing on this type of (3D) optical pulse in order to obtain the properties and conditions of their existence. According to the specific choice of the trial function, different internal dynamics of the pulsating dissipative soliton are shown.

2. Complex Cubic-Quintic Ginzburg-Landau Equation (CGLE)

In this study, the propagation of pulse in a system described by an extended complex Cubic-quintic Ginzburg-Landau equation model is considered. This equation (CGLE) is one of the universal equations used to describe dissipative systems. Many nonequilibrium phenomena, such as the generation of spatio-temporal dissipative structure in lasers [11], soliton propagation in optical fiber systems with linear and nonlinear gain and spectral filtering (such as communication links with lumped fast saturable absorbers or fiber lasers with additive-pulse mode-locking or nonlinear polarization rotation) may all be described by the CGLE [12]. The quintic dissipative term in CGLE is essential to provide the stability of the optical pulse [13]. Moreover this equation could be applied to the modeling of a wide-aperture laser cavity in the short pulse regime of operation. The model includes the effects of two-dimensional transverse diffraction of the beam, longitudinal dispersion of the pulse, and its evolution along the cavity. Dissipative terms describe the gain and loss of the pulse in the cavity. The propagation equation reads:

$$\psi_z - i\frac{D}{2}\psi_{tt} - i\frac{1}{2}\psi_{xx} - i\frac{1}{2}\psi_{yy} - i\gamma|\psi|^2\psi - i\nu|\psi|^4\psi = \delta\psi + \varepsilon|\psi|^2\psi + \beta\psi_{tt} + \mu|\psi|^4\psi \tag{1}$$

Equation (1) is written in normalized form. The optical envelope ψ is a complex function of four real variables $\psi = \psi(x, y, t, z)$, where t is the retarded time in the frame moving with the pulse, z is the propagation distance, and x and y are the two transverse coordinates.

The left-hand-side contains the conservative terms, namely $D = +1(-1)$ which is for the anomalous (normal) dispersion propagation regime and ν which represents, if negative, the saturation coefficient of the Kerr nonlinearity. In the following, the dispersion is anomalous, and ν is kept relatively small. The right-hand-side of Equation (1) includes all dissipative terms: δ, ε, β and μ are the coefficients for linear loss (if negative), nonlinear gain (if positive), spectral filtering (if positive) and saturation of the nonlinear gain (if negative), respectively. This distributed equation finds application in modeling for instance a wide-aperture active optical cavity in the regime of short pulse operation. The model includes the effects of two-dimensional transverse diffraction of the beam, longitudinal dispersion of the pulse and its evolution along the cavity. Dissipative terms describe gain and loss of the pulse in the cavity. Higher-order dissipative terms are responsible for the nonlinear transmission characteristics of the cavity which allows, for example, passive mode-locking. This equation is a natural extension of the one-dimensional CGLE.

Now, to best of our Knowledge, there is no analytical solution for the (3D) complex Cubic-quintic Ginzburg-Landau equation. Indeed, research conducted to date use namely direct numerical simulations of the CGLE. This procedure is extremely and costly [14].

The lack of general analytical solutions for the complex Cubic-quintic Ginzburg-Landau equation CGLE leads us to the necessity of using simple approaches to explore the existence of certain class of solutions. A few approximate, semi-analytical methods based on various physical backgrounds were developed and applied to study nonlinear pulse propagation. Several of them make use of a trial function and its associated finite-dimensional dynamical system.

Our study is therefore to develop theoretical tools that can perceive soliton solutions more efficiently and envisage their domains of existence. Using collective variable approach allow to rich variety of solutions that include stationary and pulsating dissipative solitons.

3. Collective Variable Approach

The dynamics of ultra-short light pulses in a fiber system can be described by the Equation (1). But the field $\psi=\psi\left(x,y,t,z\right)$ describes not only the pulse as a collective entity (localized in time and space) but also all other localized or non-localized excitations, such as noise or radiation, which are always more or less present in the real system. The pulse may not only be able to translate as a whole entity but it may also execute more or less complex internal vibrations depending on the type of the perturbations in the system. In this context it is useful, then, to somehow simplify the characterization of the pulse by use of a low dimensional equivalent mechanical system based on a finite number of degrees of freedom [15]. Each degree of freedom can then be described by means of a coordinate called the *collective variable*. The mean idea in the collective variable approach is to associate collective variables with the pulse's parameters of interest for which equations of motion may be derived. One may introduce N collective variables, z dependent, say X_i with $i=1,2,\cdots,N$, in a way such that each of them can correctly describe a fundamental parameter of the pulse (amplitude, width, chirp, …) [16]. To this end, one can decompose the field $\psi\left(x,y,t,z\right)$ in the following way:

$$\psi\left(x,y,t,z\right)=f\left(X_1,X_i,\cdots,X_N\right)+q\left(z,t\right) \tag{2}$$

where f, the ansatz function or trial function is a function of the collective variables. And $q\left(z,t\right)$ is a residual field that represents all other excitations in the system (noise, radiation, dressing field, etc.) [16]. The precise form of the ansatz function that introduces the collective variables in the theory is rather crucial, especially when approximations are made.

The choice of the trial function is important for the success of the technique. It is impossible to get any idea of the exact profile without solving numerically the CGLE (1). Hence, the choice of a particular trial function has a certain degree of arbitrariness, stressing furthermore the approximate nature of the collective variable approach. After choosing the ansatz function one can pursue the process of characterization of the pulse by neglecting the residual field. This approximation is called *the bare approximation* [16]. In this way one can consider the fact that the pulse propagation can be completely characterized that the ansatz function $\psi=\psi\left(x,y,t,z\right)$. This approximation depends very strongly on the choice of the trial function. Consequently, when approximations are made, such as the use of the variational approach, the precise shape of the trial function that introduces the collective variables in the analysis of the dynamics of soliton becomes rather crucial.

4. Variational Equations

In this study the propagation of solitons in a system described by an extended (3D) complex Cubic-quintic Ginzburg-Landau equation model is considered. This model includes cubic and quintic nonlinearities of dispersive and dissipative types. Transverse operators are added to take account of the spatial diffraction in the wave paraxial approximation.

To obtain a better understanding of dynamic processes which influence the behavior of pulse during its propagation, it is considered that the real field $\psi\left(x,y,t,z\right)$ represents the ansatz function which is endowed with the usual physical meaning of the pulse parameters.

Thus, it is possible to reduce the equation of impulsive field to an ordinary differential equation (ODE) describing the evolution of the parameters of the soliton during propagation. The advantage is that, the ordinary differential equation can be solved numerically with relative ease.

However, the precise shape of the trial function is crucial to have solutions with the desired properties. For the complex systems, this choice is preceded by careful analysis and based on a comparison of some analytical approximate solutions before an exact numerical solving of the CGLE can be made for comparison.

In previous work [10], it has been demonstrated with different trial functions that the collective variable approach is a useful tool for predicting approximately the domains of existence of stable light bullets in the parameter space of the complex cubic-quintic. These predictions were confirmed by the direct numerical solutions of the CGLE, qualitatively as well as quantitatively, that give us confidence in our collective variable approach.

On the basis of preliminary results of previous works [10] and to describe a richer variety of dynamical behaviors, a Gaussian trial function which admits asymmetric pulse shapes in the transverse plane (x, y) is used and is given by

$$f = A\mathrm{e}^{\left(-\frac{t^2}{w_t^2}-\frac{x^2}{w_x^2}-\frac{y^2}{w_y^2}+\frac{i}{2}c_t t^2+\frac{i}{2}c_x x^2+\frac{i}{2}c_y y^2+ip\right)} \tag{3}$$

A, w_t, w_x, w_y, c_t, c_x, c_y, and p represent the collective variables. With t, x and y, the temporal and transverses variables along x and y axis respectively. A stands for soliton amplitude, w_t the temporal width along t, w_x the transverse width along x axis and w_y the transverse width along y axis. c_t represents the temporal chirp parameter, c_x the transverse chirp along x, c_y the transverse chirp along y and p is the global phase that evolves along with propagation. When a stationary regime is reached, the phase becomes a linear function of the propagation distance z.

The choice of the trial function is done according to the master Equation (1) and the type of solutions pursued. Consequently, the precise shape of the trial function that introduces the collective variable in the analysis of the dynamics of soliton becomes rather crucial. After choosing the trial function one can pursue the process of characterization of the pulse in two completely different ways depending on the level of accuracy desired. First, one can make use of the exact pulse field to obtain the pulse parameters. The second approach of characterization would be to carry out a variational analysis neglecting the residual field. The approximation of neglecting the residual field is called *the bare approximation* [16], just like those in previous works [10]. This approach leads generally to results which are less accurate than the first one, but its application to the characterization of the propagation of soliton would be too times consuming.

After choosing the trial function one can carry out variational analysis neglecting the residual field (*the bare approximation*). Applying the bare approximation to the 3D CGLE, that is, substituting the field ψ by the given trial function $f(\psi = f)$ and projecting the resulting equations in the following direction

$$\frac{\mathrm{d}f^*}{\mathrm{d}X}\left(X = A, w_t, w_x, w_y, c_t, c_x, c_y, p\right)$$

The collective variables evolve according to the following set of eight coupled ordinary differential equation are easily obtained:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= A\delta + \frac{7}{16}\sqrt{2}A^3\varepsilon - \frac{2\beta}{w_t^2}A + \frac{2}{9}\mu\sqrt{3}A^5 - \frac{1}{2}ADc_t - \frac{1}{2}Ac_x - \frac{1}{2}Ac_y,\\
\frac{\mathrm{d}w_t}{\mathrm{d}t} &= w_t c_t D - \frac{1}{8}\sqrt{2}w_t A^2\varepsilon - \frac{2}{27}\mu\sqrt{3}A^4 w_t + \left(4 - w_t^4 c_t^2\right)\frac{\beta}{2w_t},\\
\frac{\mathrm{d}w_x}{\mathrm{d}z} &= w_x c_x - \frac{1}{8}\sqrt{2}w_x A^2\varepsilon - \frac{2}{27}\mu\sqrt{3}A^4 w_x,\\
\frac{\mathrm{d}w_y}{\mathrm{d}z} &= w_y c_y - \frac{1}{8}\sqrt{2}w_y A^2\varepsilon - \frac{2}{27}\mu\sqrt{3}A^4 w_y,\\
\frac{\mathrm{d}c_t}{\mathrm{d}z} &= -Dc_t^2 - \frac{1}{2w_t^2}\sqrt{2}A^2 + \frac{4D}{w_t^4} - \frac{8\sqrt{3}}{27w_t^2}\nu A^4 - \frac{8\beta}{w_t^2}c_t,\\
\frac{\mathrm{d}c_x}{\mathrm{d}z} &= -c_x^2 - \frac{1}{2w_x^2}\sqrt{2}A^2 + \frac{4}{w_x^4} - \frac{8\sqrt{3}}{27w_x^2}\nu A^4,\\
\frac{\mathrm{d}c_y}{\mathrm{d}z} &= -c_y^2 - \frac{1}{2w_y^2}\sqrt{2}A^2 + \frac{4}{w_y^4} - \frac{8\sqrt{3}}{27w_y^2}\nu A^4,\\
\frac{\mathrm{d}p}{\mathrm{d}z} &= \beta c_t + \frac{7}{16}\sqrt{2}A^2 - \frac{D}{w_t^2} - \frac{1}{w_x^2} - \frac{1}{w_y^2} + \frac{2\sqrt{3}}{9}\nu A^4.
\end{aligned} \tag{4}$$

Obviously the ordinary differential equations in Equation (4) depend on the choice the Equation (3).

These equations give no explicit information with regard to the different solutions of the Equation (1) and their stability. They give us the first idea on the dynamic of the light pulse. The variational equations allow seeing the influence of each Equation (1) parameters on the various physical parameters of the soliton. They are usually functions of time that evolve subject to the constraints of the system and finally converge to fixed point or a limit cycle.

A meticulous analysis of the variational equations show that the evolution of the amplitude (A) is dominated by the linear loss (δ), the nonlinear gain (ε) and its saturation (μ), as well as that the terms of spectral filtering (β) and dispersion term (D). This confirms quite well that the perfect balance between losses and gains is required to maintain the shape and stability of the soliton. The temporal (w_t) and spatial widths (w_x, w_y) also depend on the nonlinear gain (ε) and its saturation (μ). As expected, the terms of spectral filtering (β) and dispersion term (D) affect the temporal width. As well the spatial parameters c_x, c_y and temporal parameters c_t are influenced in the same way by the Kerr term saturation of the optical nonlinearity (ν), but the temporal term is also affected by the terms of spectral filtering (β) and dispersion term (D). Finally, not any parameters of the soliton are influenced by (p), the global phase.

One advantage of the collective variable approach is that the total energy can also expressed as function of the trial function parameters. Here it is interesting to gain insight from this simple and useful quantity, which is defined as

$$Q = \frac{\pi\sqrt{2\pi}}{4} A^4 w_t w_x w_y \tag{5}$$

This above equation point out that the total energy is strongly ruled by the amplitude (A) the temporal (w_t) and spatial widths (w_x, w_y).

Hence, the significance of our collective variable approach helps to analyze the variational equations and the influence of various parameters. Following this in-depth analysis, the first major step in our study is to provide a mapping of the regions of existence of stable solutions in the parameter space of the (3D) CGLE.

5. (3D) Pulsating Dissipative Soliton

From the analytical results of the variational Equation (4), the fixed points are carefully analyzed and their stability studied. The fixed points (FPs) of the system are found by imposing the left-hand side of Equation (4) to be zero. The threshold of existence of FPs can be estimated by the relation $\varepsilon_s \approx 2\sqrt{\delta\mu}$ [17]. If $\varepsilon > \varepsilon_s$, both stable and unstable fixed points appear. The stable fixed points correspond to stationary solutions. The unstable fixed points correspond to the pulsating and non-stationary solutions.

A major goal of our study is to provide a quick approximate mapping of the regions of existence of stable and unstable solutions in the parameter space of the Equation (1). Stationary and pulsating solutions exist in regions defined by the space of the parameters of the equation, but it is extremely difficult to map these regions by varying all these parameters. To remedy this, it is convenient to set all parameters of the equation except two chosen as variable parameters. Thus the different solutions obtained are defined in this plan. Chosen variable parameters are the dispersion D and the cubic nonlinear gain ε. So for each set of parameters, the existence of the fixed point and its stability is studied.

By investigating the parameter regions situated in the neighborhood of the parameters $\mu = -0.1$, $\delta = -0.4$, $\beta = 0.1$, $\nu = -0.08$ and $\gamma = 1$ and according to our previous study [17] [18], a rich variety of solutions of Equation (1) including stationary and pulsating dissipative solitons is observed in the plane (D, ε).

For a given set of CGLE parameters (μ, δ, β, ν, et γ), with an initial pulse

$$\psi(x,y,t,z) = 4\mathrm{e}^{\left(-\frac{t^2}{1.3}-\frac{x^2}{1.4}-\frac{y^2}{0.9}\right)}$$

For each value pair (D, ε), the fourth-order Runge-Kutta algorithm provides the fixed point whose stability is analyzed.

The mapping **Figure 1** brings to light this rigorous analysis. The dispersion coefficient D and the non-linear gain coefficient ε are the parameters that can be modified easily in most experiments. It is the case of experiments using lasers passively locked mode.

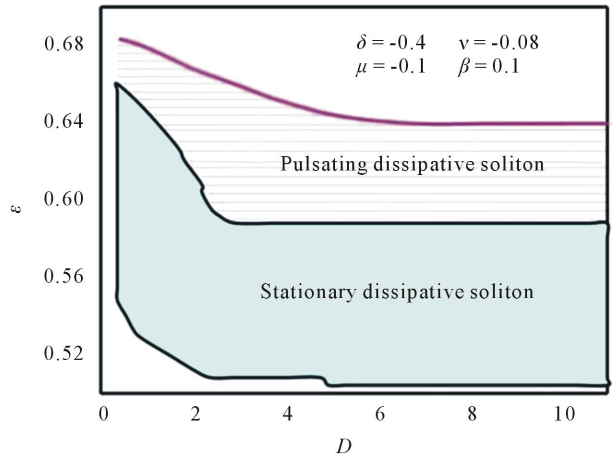

Figure 1. Cartography of stationary (color domain) and pulsating (dotted line domain) dissipative soliton found from collective variable approach in the (D, ε) plane using the Gaussian trial function. Other CGLE parameters appear inside the figure.

Regions of existence of stable light bullets following the same technique that we used for dissipative solitons in the (3 + 1)D dimensional case [18] have been determined.

Inside this cartography, two different regions according to the values of the dispersion and the nonlinear gain are observed. The coloured area contains the stable fixed points, which are the basin of shows. And near this points, all the others points converge. The stable fixed points regions represents the domain of stationary solitons of the Equation (1) found from collective variable approach. In this domain, all the solitons parameters (amplitude, width, chirp…) stay stationary throughout propagation. Above the stationary domain, instable fixed points which can be dived in two categories: the limit-cycle attractor and the instable solutions could be observed. Our main interest is to study the dynamic of the pulse in the limit-cycle attractor area (dotted line domain). Indeed close to boundaries of the existence domains of stationary solitons solutions, there is more often an intermediate region in which pulsating solutions can be found. The existence of pulsating solutions is indeed a general feature of most nonlinear dissipative systems. This behavior of pulsating soliton can be attributed of a limit-cycle attractor; it then possesses inherent stability the same way as stationary stable solutions do. Above the pulsating domain, instable solitons which are little physical interest in this current study are noticed. The cartography (**Figure 1**) is quite like those achieved in others parameters space (v, ε), (μ, ε), (β, ε) and (μ, v) [17] [18]. Thus according the value of the cubic nonlinear gain ε, the stability diagram shows types of dynamic. In the two-dimensional plane of parameters, the regions of existence of stable stationary solitons presented above are contained between an upper and a lower values of ε. Below the lower limit, the energy pumped into the system is not enough to support the solitons. In this case the localized solution dissipates and eventually vanishes. On the contrary, when is above the upper value, the energy supply inflates the soliton to the extent that it grows indefinitely in size. However, this process does not occur in one single step. There is a small intermediate region of ε values where stationary solutions are transformed into pulsating ones before the continuous inflating begins at even higher ε. On can notice that in the pulsating domain, for each a paired value of (D, ε), there is one and only one type of pulsating soliton with its own characteristics (amplitude, widths, energy, …). From **Figure 2** and **Figure 3**, pulsating solutions in the form of stable limit cycles are observed. These solutions can be considered as attractors. These figures are obtained thanks to our collective variable approach for $\mu = -0.1$, $\delta = -0.4$, $\varepsilon = 0.59$, $\beta = 0.1$, $D = 6$, $v = -0.08$, and $\gamma = 1$.

Figure 2 (below the enlarged views) shows the evolution on the total energy of the soliton for a given set of parameters corresponding in the pulsating domain (dotted line domain in **Figure 1**). Dynamics begins with a short transitional phase, which is followed by a permanent dynamics with an increase of the energy are noticed. The energy remains in this stable state when the permanent dynamic is reached. The total energy gives us the main information about the soliton dynamics. It's not conserved but evolves in accordance with the so-called balance equation. When a stationary solution is reached, the total energy converges to a constant value. However,

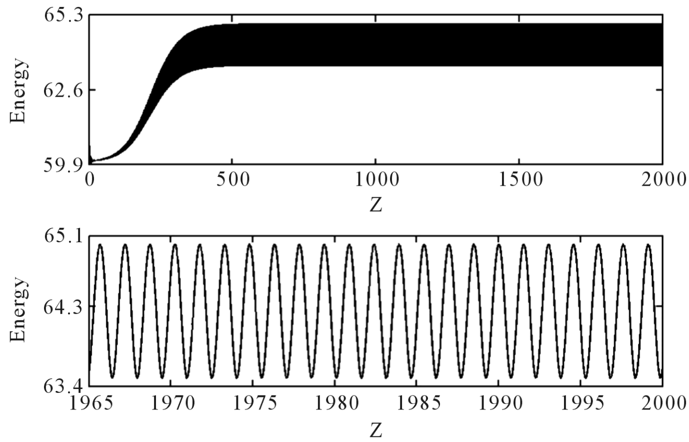

Figure 2. Evolution of the total pulse energy, enlarged view of this pulsation is plotted bottom with $\mu = -0.1$, $\varepsilon = 0.59$, $D = 6$, $\delta = -0.4$, $\beta = 0.1$, and $\gamma = 1$.

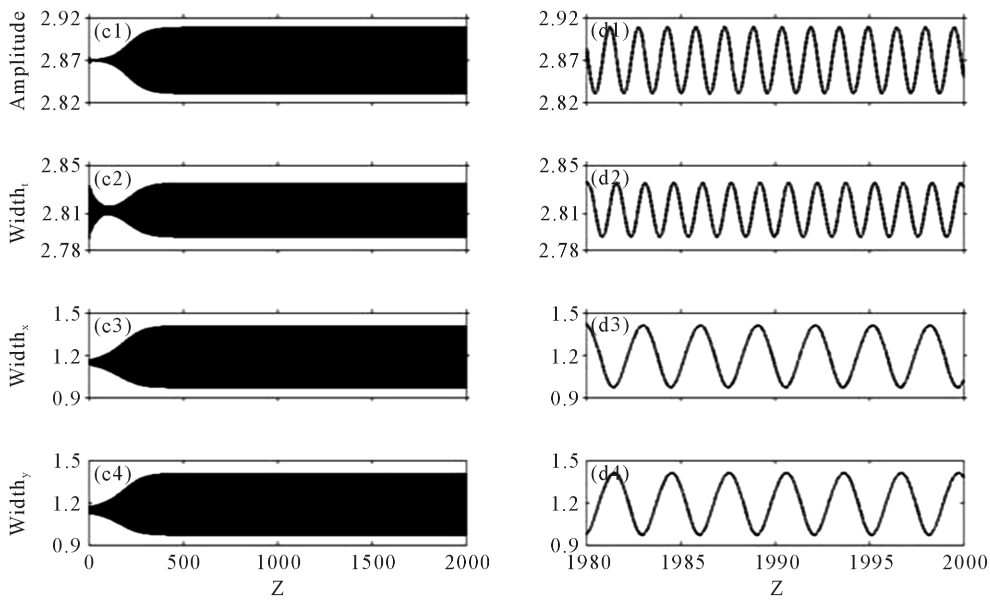

Figure 3. Radially asymmetric pulsations obtained with the use of the collective variable approach and the trial function. The evolution of the following collective variables with respect to z is shown: (c1) the amplitude, (c2) the temporal width, (c3) the width along the x axis, and (c4) the width along the y axis. The onset of stable harmonic pulsations appears clearly. Enlarged views of those pulsations are plotted to the right in (d1)-(d4). The pulsations along the x and y axes are out of phase and share the same amplitude $\mu = -0.1$, $\varepsilon = 0.59$, $D = 6$, $\delta = -0.4$, $\beta = 0.1$, and $\gamma = 1$.

the soliton, is a pulsating one, the total energy is an oscillating function of z. In **Figure 3**, all the others parameters (amplitude, widths, …) of the soliton follow the same dynamics. The dynamics are illustrated in this picture [d(i) are the enlarged views of the c(i)]. It shows the evolution of the soliton amplitude (**Figure 3(c1)**), temporal (**Figure 3(c2)**) width and the transverse width along x (**Figure 3(c3)**), and y (**Figure 3(c4)**), axis. After a brief transitional period characterized by small oscillations due to the adjustment of the initial condition, and after the onset of oscillations, the pulsating dynamics becomes steady. It is characterized by large oscillations between the two constant limits. The pulse breathing in the transverse and in the temporal domain can be seen from the evolution plots **Figure 3(d3)** and **Figure 3(d4)**. The amplitude oscillates between 2.82 and 2.92, the temporal width between 2.78 and 2.85 and the transverse widths between 0.9 and 1.5. It could be clearly seen that the difference oscillation between the temporal and the transverse width. As well, on the same propagation distance z, the amplitude of the soliton evolves much faster compared with its spatial widths. The same holds true for the temporal width. The close-up view of oscillations (**Figure 3(d3)** and **Figure 3(d4)**) shows a nearly harmonic evolution of the collective variables. The x and y oscillations are out of phase and have the same magnitude. This shows periodic out of phase consecutive contractions of the soliton in the x and y directions. At the same time, the total energy is not a constant and the amplitude and temporal width oscillate at a doubled frequency.

5.1. Bifurcation Diagram

In **Figure 1**, two different kind of stable 3D solutions in the specific domain appear. When a pulsating solution can be attributed to the existence of a limit-cycle attractor, it then possesses inherent stability the same way as stationary stable solutions do. For a given set of the parameters and varying the nonlinear gain parameter, the soliton move from a stationary solution to pulsating one. So the total energy leads to a pulsation mode whose spectrum contains two (three) main frequencies. To show pulsations with two oscillation periods, the system has to undergo two bifurcations from the stationary solution. And for the three oscillation periods, the system has to undergo four bifurcations from the two bifurcations. The picture **Figure 4** shows a bifurcation diagram obtained when varying the nonlinear gain parameter ε from 0.575 to 0.595, while keeping the rest of parameters (written in **Figure 4**) fixed. The curve represents a local maximum or minimum of the total energy. For the nonlinear gain value inferior to 0.583 the energy has a single value (maximum equal to minimum). This means that stationary solution is reach. From the up value to 0.583, a bifurcation occurs, which is related to the onset of pulsations with single frequency. In the case discussed above, energy pulsations remain very close to harmonic pulsation.

5.2. The Effect of the Saturation of the Kerr Nonlinearity

The influence of the saturation coefficient of the Kerr nonlinearity ν on the oscillations is investigated. To this end, parameters a $\mu = -0.1$, $\varepsilon = 0.6$, $D = 6$, $\delta = -0.4$, $\beta = 0.1$, $\gamma = 1$ are set with different values of ν (-0.07, -0.08, -0.09). The dynamics of the energy with respect to the propagation distance z are followed. **Figure 5** stands for the result of this detailed analysis [$b(i)$ are the enlarged views of the $a(i)$]. Obviously, the value of saturation of kerr nonlinearity for which the soliton oscillates is chosen, but this dynamics changes when ν is modified. So for different values of the saturation nonlinear parameter ν, the other parameters (written below the picture **Figure 5**) are fixed. When ν decrease (from −0.07 to −0.09), the amplitude of the pulsations increase, e.g. the energy augment. In **Figure 5(b1)** and **Figure 5(b2)** the same dynamics with stable oscillation is obtained, but in **Figure 5(b3)** the motion changes. More precisely, pulsations that consist of two oscillation periods instead of one are found. With precision, the asymptotic evolution of the total energy leads to a pulsation whose spectrum contains two main frequencies, associated to intensities (**Figure 5(b3)**) of the same magnitude. It could also be noticed that the internal dynamic change including many oscillations in the same period. It emerges from this study that according to the value of saturation nonlinearity and to the choice of the trial function, it is possible to find the other types of pulsating dissipative soliton and the dynamic within the same period differs, from harmonic to non-harmonic pulsations.

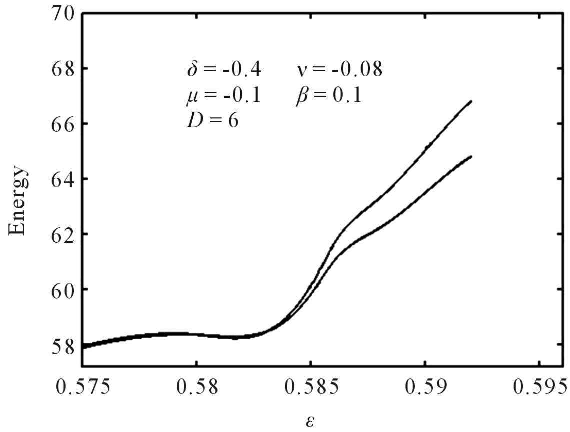

Figure 4. Bifurcation diagram of the dissipative pulsating soliton.

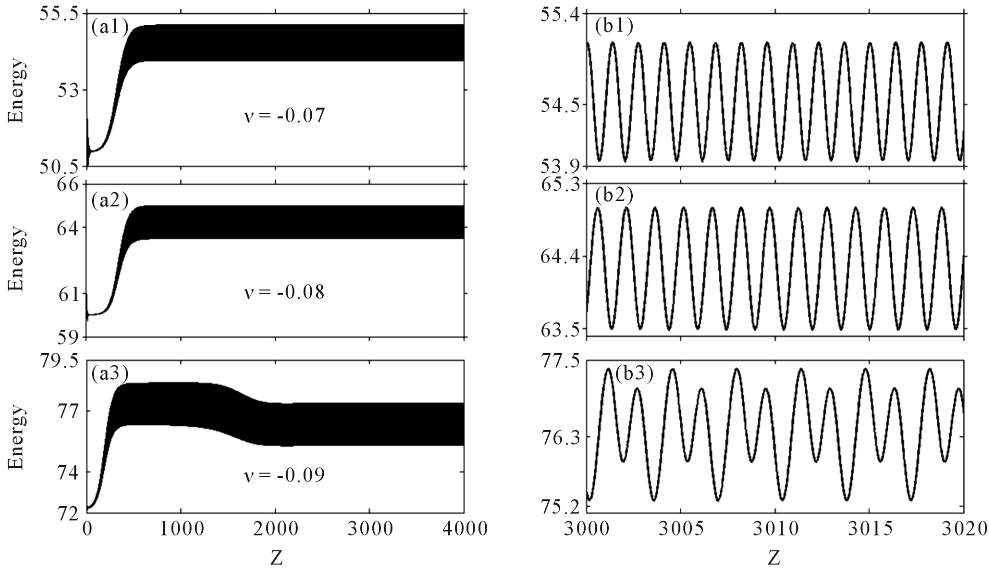

Figure 5. Evolution of the total pulse energy for different values of the Kerr nonlinearity saturation, with $\mu = -0.1$, $\varepsilon = 0.6$, $D = 6$, $\delta = -0.4$, $\beta = 0.1$ and $\gamma = 1$. Enlarged views of those pulsations are plotted to the right in (b1)-(b3). The pulsation (b3) contains two main frequencies.

6. Conclusion

In this work, based on collective variable approach, we have expanded the studies of 3D dissipative pulsating solitons in the complex Ginzburg-Landau equation with the cubic-quintic nonlinearity. In particular, the regions of coexistence of stationary and pulsating dissipative soliton are obtained. The collective variable approach is very efficient for approximating stable pulsating solutions when a suitable trial function is chosen. A nearly harmonic evolution of the widths along x and y axis is shown and the x and y oscillations are out of phase with the same magnitude. The dynamics of soliton can be controlled by the choice of the system parameters. So according to the values of the nonlinear gain, the system has to undergo a bifurcation from the stationary solution, to obtain pulsations with two oscillation periods. A pulsating soliton whose spectrum contains two main frequencies (two oscillation periods instead of one), associated to intensities of the same magnitude could also be predicted. The latter effect may be used, in principle, to grow photonic channels and multichannel arrays in bulk optical media. The collective variable technique is incomparably quicker than direct numerical computations. Of course, it should be used at the final stage of studies to confirm, complement, or invalidate the collective variable approach predictions.

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Epstein Barr Virus—The Cause of Multiple Sclerosis

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Abstract

Although many studies have found a kind of a relationship between an Epstein-Barr Virus (EBV) and the development of Multiple Sclerosis (MS), a fundamental aspect of this relationship remains uncertain. What is the cause of Multiple Sclerosis (MS)? In this study, we re-analysed the data as published by Wandinger *et al.* and were able to establish a new insight: without an Epstein-Barr Virus (EBV) infection no development of Multiple Sclerosis (MS). Furthermore, we determined a highly significant causal relationship between Epstein-Barr Virus (EBV) and multiple sclerosis. Altogether, Epstein-Barr Virus (EBV) is the cause of multiple sclerosis (*p*-value 0.0004251570).

Keywords

Epstein Barr Virus, Multiple Sclerosis

1. Introduction

Multiple Sclerosis (MS) is an unpredictable disease of the central nervous system which disrupts the communication between the brain and other parts of the body. Multiple Sclerosis (MS) can range from relatively benign to somewhat disabling and devastating symptoms. Some of today approved drugs to treat multiple sclerosis include Novantrone (mitoxantrone), teriflunomide, dimethyl fumarate, copolymer I (Copaxone) and forms of beta interferon. Steroids are used to reduce the duration and severity of attacks in some patients suffering from multiple sclerosis. Exercise and physical therapy can help to preserve remaining function. Various aids such as foot braces, canes, and walkers are of use to help patients to remain independent and mobile. Thus far, there is as yet no cure for multiple sclerosis while millions of people are suffering from this many times deadly disease.

Epstein-Barr Virus (EBV), a herpes virus, is a primary cause of Infectious Mononucleosis (IM) and associated with several malignancies including such as Hodgkin lymphoma, non-Hodgkin lymphoma, Burkitt lymphoma

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and other. Epidemiological, molecular virology and other [1]-[6] studies have been able to establish EBV as a risk factor for the development of Multiple Sclerosis (MS) and provided some evidence that the pathogenesis of multiple sclerosis might involve a response to an EBV infection. Still, the cause of Multiple sclerosis is not identified.

2. Material and Methods

2.1. Definitions

Definition. Bernoulli random variable

Let $t = +1, \cdots, +N$ denote an individual Bernoulli trial each with *constant* success probability *p*. Let *N* denote the number of independent Bernoulli trials (the size of a random sample or of the population).

Definition. The 2 × 2 table

Let A_t denote a Bernoulli/Binomial distributed random variable. Let $p(A_t)$ denote the probability of A_t. Let B_t denote a Bernoulli/Binomial distributed random variable. Let $p(B_t)$ denote the probability of B_t. Let $p(a_t) = p(A_t \cap B_t)$ denote joint distribution of A_t and B_t. Let $p(b_t) = p(A_t \cap \underline{B}_t)$ denote joint distribution of A_t and $\underline{B}_t$. Let $p(c_t) = p(\underline{A}_t \cap B_t)$ denote joint distribution of $\underline{A}_t$ and B_t. Let $p(d_t) = p(\underline{A}_t \cap \underline{B}_t)$ denote joint distribution of $\underline{A}_t$ and $\underline{B}_t$. In general, it is $p(a_t) + p(b_t) + p(c_t) + p(d_t) = 1$. Thus far, the relationships before are expressed in the 2 × 2 table (**Table 1**).

Thus far, let $A = N \times p(A_t)$ denote the expectation value. Let $\underline{A} = N - A = N \times (1 - p(A_t))$ denote the expectation value. Let $B = N \times p(B_t)$ denote the expectation value. Let $\underline{B} = N - B = N \times (1 - p(B_t))$ denote the expectation value. Let $a = N \times p(a_t) = N \times p(A_t \cap B_t)$ denote the expectation value. Let $b = N \times p(b_t) = N \times p(A_t \cap B_t)$ denote the expectation value. Let $c = N \times p(c_t) = N \times p(\underline{A}_t \cap B_t)$ denote the expectation value. Let $d = N \times p(d_t) = N \times p(\underline{A}_t \cap \underline{B}_t)$ denote the expectation value. Let $N = a + b + c + d = N \times (p(a_t) + p(b_t) + p(c_t) + p(d_t))$ denote the size of the sample or the size of the population. Let $A = a + b$ denote the expectation value of the condition (*i.e.* a risk factor, the *verum* population, the exposed group). Let $\underline{A} = c + d$ denote the expectation value of the non-condition (*i.e.* the non-exposed group, the *control* population). Let $B = a + c$ denote the expectation value of the conditioned. Let $\underline{B} = b + d$ denote the expectation value of the not conditioned. Thus far, the relationships before are expressed in the 2 × 2 table (**Table 2**).

Definition. Risk ratio or relative risk

Various quantitative techniques are used in Biostatistics to the describe and evaluate relationships among biologic and medical phenomena. Relative risk, defined by Fischer [7] as ψ, is an important [8] [9] statistical method used in epidemiologic studies and clinical trials. Let *RR*(*A*, *B*) denote the relative risk. Based on the 2 by 2 table above, the relative risk *RR*(*A*, *B*) is defined as

$$RR(A, B) \equiv \frac{a/(a+b)}{c/(c+d)} \tag{1}$$

In epidemiology and statistics, Relative Risk (*RR*) is the ratio of *the probability of an event a occurring under conditions of being exposed to* (*a* + *b*), the non-exposed to *the probability of c occurring under conditions of being exposed to* (*c* + *d*), the non-exposed group. The Relative Risk (*RR*) is a widely used *measure of association* in epidemiology. A risk ratio *RR*(*A*, *B*) < 1 suggest that an exposure can be considered as being associated with a reduction in risk. A risk ratio *RR*(*A*, *B*) > 1 suggest that an exposure can be considered as being associated with an increase in risk.

Conditions

The following relationships are taken with friendly permission by Ilija Barukčić [10].

Definition. Conditio sine qua non relationship

Let $p(A_t \leftarrow B_t)$ denote [10] the extent to which a condition *A* is a *conditio sine qua non* of the conditioned *B*. The conditio sine qua non relationship is calculated as

$$p(A_t \leftarrow B_t) \equiv p(a_t) + p(b_t) + p(d_t) = \frac{N}{N} \times (p(a_t) + p(b_t) + p(d_t)) = \frac{a+b+d}{N} \equiv \frac{A+d}{N} \equiv \frac{a+\underline{B}}{N} \tag{2}$$

The relationship before is expressed in the following 2 × 2 (**Table 3**).

Table 1. The 2 × 2 table. Probabilities.

		Conditioned B_t		
		Yes	No	
Condition A_t	Yes	$p(a_t)$	$p(b_t)$	$p(a_t)+p(b_t)=p(A_t)$
	No	$p(c_t)$	$p(d_t)$	$p(c_t)+p(d_t)=p(\underline{A}_t)$
		$p(a_t)+p(c_t)=p(B_t)$	$p(b_t)+p(d_t)=p(\underline{B}_t)$	1

Table 2. The 2 × 2 table. Expectation values.

		Conditioned B		
		Yes	No	
Condition A	Yes	a	b	$a+b=A$
	No	c	d	$c+d=\underline{A}$
		$a+c=B$	$b+d=\underline{B}$	N

Table 3. Conditio sine qua non.

		Conditioned B		
		Yes	No	
Condition A	Yes	a	b	$a+b=A$
	No	$c=0$	d	$c+d=\underline{A}$
		$a+c=B$	$b+d=\underline{B}$	N

Definition. Anti conditio sine qua non relationship

Let $p(A_t -< B_t)$ denote [10] the extent to which a condition A is not a *conditio sine qua non* of the conditioned B. The *anti conditio sine qua non* relationship is calculated as

$$p(A_t -< B_t) \equiv p(c_t) = \frac{N}{N}(p(c_t)) \equiv \frac{N-a-b-d}{N} \equiv \frac{c}{N} \equiv 1-p(A_t \leftarrow B_t) \tag{3}$$

The relationship before is expressed in the following 2 × 2 table (**Table 4**).

Definition. Conditio per quam relationship

Let $p(A_t \rightarrow B_t)$ denote [10] the extent to which a condition A is a *conditio per quam* of the conditioned B. The *conditio per quam* is calculated as

$$p(A_t \rightarrow B_t) \equiv p(a_t)+p(c_t)+p(d_t) = \frac{N}{N}\times(p(a_t)+p(c_t)+p(d_t)) = \frac{a+c+d}{N} \equiv \frac{B+d}{N} \equiv \frac{a+\underline{A}}{N} \tag{4}$$

The relationship before is expressed in the following 2 × 2 table (**Table 5**).

Definition. Anti conditio per quam relationship

Let $p(A_t > -B_t)$ denote [10] the extent to which a condition A is not a *condition per quam* of the conditioned B. The anti *conditio per quam* relationship is calculated as

$$p(A_t > -B_t) \equiv p(b_t) = \frac{N}{N}\times(p(b_t)) \equiv \frac{b}{N} \equiv \frac{N-a-c-d}{N} \equiv 1-p(A_t \rightarrow B_t) \tag{5}$$

The relationship before is expressed in the following 2 × 2 table (**Table 6**).

Definition. Conjunction. A and B relationship

Let $p(A_t \cap B_t)$ denote [10] the extent to which a condition A is *conjugated* with the conditioned B. The conjunction is calculated as conjunction of the two events

Table 4. Anti conditio sine qua non.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	$a = 0$	$b = 0$	$a + b = A$
	No	c	$d = 0$	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

Table 5. Conditio per quam.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	a	$b = 0$	$a + b = A$
	No	c	d	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

Table 6. Anti conditio per quam.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	$a = 0$	b	$a + b = A$
	No	$c = 0$	$d = 0$	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

$$p(A_t \cap B_t) \equiv p(a_t) = \frac{N}{N} \times (p(a_t)) = \frac{a}{N} \equiv \frac{N - b - c - d}{N} \equiv 1 - p(A_t \underline{\cap} B_t) \tag{6}$$

The relationship before is expressed in the following 2 × 2 table (**Table 7**).

Definition. Exclusion relationship

Let $p(A_t \underline{\cap} B_t)$ denote [10] the extent to which a condition *A excludes* the conditioned *B and vice versa*. The exculsion relationship (the Sheffer stroke) named after Henry M. Sheffer is written as a vertical bar or an upwards arrow and calculated as

$$p(A_t \underline{\cap} B_t) \equiv p(A_t | B_t) \equiv p(A_t \uparrow B_t) \equiv p(b_t) + p(c_t) + p(d_t) = \frac{N}{N} \times (p(b_t) + p(c_t) + p(d_t)) \tag{7}$$

or

$$p(A_t \underline{\cap} B_t) \equiv p(A_t | B_t) \equiv p(A_t \uparrow B_t) \equiv \frac{b + c + d}{N} \equiv \frac{N - a}{N} \equiv 1 - p(A_t \cap B_t) \tag{8}$$

The relationship before is expressed in the following 2 × 2 table (**Table 8**).

Definition. Disjunction. *A* or *B* relationship

Let $p(A_t \cup B_t)$ denote [10] the extent to which the condition *A* or the conditioned *B* are given. The inclusive disjunction also known as *alternation* is calculated as

$$p(A_t \cup B_t) \equiv p(a_t) + p(b_t) + p(c_t) = \frac{N}{N} \times (p(a_t) + p(b_t) + p(c_t)) \equiv 1 - p(A_t \underline{\cup} B_t) \tag{9}$$

or

$$p(A_t \cup B_t) \equiv p(A_t) + p(B_t) - p(a_t) = \frac{a + b + c}{N} \equiv \frac{A + B - a}{N} \equiv 1 - p(A_t \underline{\cup} B_t) \tag{10}$$

The relationship before is expressed in the following 2 × 2 table (**Table 9**).

Definition. Neither *A* nor *B* relationship

Let $p\left(A_t \underline{\cup} B_t\right)$ denote [10] the extent to which neither a condition *A* nor the conditioned *B* is given. The *neither A nor B* relationship was introduced by Charles Sanders Peirce and is known also as Peirce's arrow too and can be calculated as

$$p\left(A_t \underline{\cup} B_t\right) \equiv p\left(A_t \downarrow B_t\right) \equiv p\left(d_t\right) = \frac{N}{N} \times \left(p\left(d_t\right)\right) = \frac{d}{N} \equiv \frac{N-a-b-c}{N} \equiv 1 - p\left(A_t \cup B_t\right) \tag{11}$$

The relationship before is expressed in the following 2 × 2 table (**Table 10**).

Definition. Equivalence of *A* and *B* relationship

Let $p\left(A_t <=> B_t\right)$ denote [10] the extent to which a condition *A* and the conditioned *B* are equivalent. The equivalence of *A* and *B* is calculated as

$$p\left(A_t <=> B_t\right) \equiv p\left(a_t\right) + p\left(d_t\right) = \frac{N}{N} \times \left(p\left(a_t\right) + p\left(d_t\right)\right) = \frac{a+d}{N} \equiv 1 - p\left(A_t >=< B_t\right) \tag{12}$$

The relationship before is expressed in the following 2 × 2 table (**Table 11**).

Definition. Either *A* or *B* relationship

Let $p\left(A_t >=< B_t\right)$ denote [10] the extent to which *either* the condition *A* *or* the conditioned *B* is given. The *either A or B* relationship can be calculated as

$$p\left(A_t >=< B_t\right) \equiv p\left(b_t\right) + p\left(c_t\right) = \frac{N}{N} \times \left(p\left(b_t\right) + p\left(c_t\right)\right) = \frac{b+c}{N} \equiv \frac{N-a-d}{N} \equiv 1 - p\left(A_t <=> B_t\right) \tag{13}$$

The relationship before is expressed in the following 2 × 2 table (**Table 12**).

Table 7. Conjunction. *A* and *B*.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	a	$b = 0$	$a+b=A$
	No	$c = 0$	$d = 0$	$c+d=\underline{A}$
		$a+c=B$	$b+d=\underline{B}$	N

Table 8. Exclusion. *A* excludes *B* and *vice versa*.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	$a = 0$	b	$a+b=A$
	No	c	d	$c+d=\underline{A}$
		$a+c=B$	$b+d=\underline{B}$	N

Table 9. Disjunction. *A* or *B*.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	a	b	$a+b=A$
	No	c	$d = 0$	$c+d=\underline{A}$
		$a+c=B$	$b+d=\underline{B}$	N

Table 10. Neither *A* or *B* relationship.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	$a = 0$	$b = 0$	$a + b = A$
	No	$c = 0$	d	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

Table 11. Equivalence of *A* and *B*.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	a	$b = 0$	$a + b = A$
	No	$c = 0$	d	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

Table 12. Either *A* or *B* relationship.

		Conditioned *B*		
		Yes	No	
Condition *A*	Yes	$a = 0$	b	$a + b = A$
	No	c	$d = 0$	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

2.2. Material

Patients and Samples

Data and material for this re-analysis were published by Wandinger [11] *et al.*, a study specifically designed to investigate the relation between Multiple Sclerosis (MS) and viral infections. Wandinger *et al.* examined sera from a large cohort of 163 healthy control subjects (control group) and 108 patients with a diagnosis of clinically definite multiple sclerosis for the presence of human herpes viruses type 1 (HSV-1), HSV-2, cytomegalovirus (CMV) and EBV by the presence of IgG antibodies. In addition, other investigations (*i.e.* the detection of EBV DNA in all serum samples) were performed. Some of the data of Wandinger *et al.* data about the prevalence of IgG antibodies in serum samples from Multiple Sclerosis (MS) patients and healthy control subjects are summarized in the table shown below (**Table 13**).

The data of the prevalence of IgG antibodies in serum samples from multiple sclerosis (MS) patients and healthy control subjects are viewed in the following 2 × 2 table (**Table 14**).

2.3. Methods

2.3.1. The Chi-Squared Distribution

The properties of the chi-squared distribution were first investigated by Karl Pearson [12] in 1900. The chi-squared distribution is a widely used probability distributions in hypothesis testing [14], inferential statistics (**Table 15**) or in construction of confidence intervals.

In last consequence, the Chi Square with one degree of freedom is nothing but the distribution of a single normal deviate squared.

2.3.2. The Binomial Proportion Confidence Interval

The statistical significance of deviations from a theoretically expected distribution of observations can be tested

Table 13. Prevalence of IgG antibodies in MS patients and healthy control subjects.

Parameter	Multiple Sclerosis (MS)	Healthy control subjects
anti-EBNA-1 IgG	108	147
Sample size	108	163

Table 14. EBV and Multiple Sclerosis (MS).

		Multiple sclerosis		
		Yes	No	
EBV anti-EBNA-1 IgG	Yes	108	147	255
	No	0	16	16
		108	163	271

Table 15. Chi square distribution for degree of freedom *d.f.* = 1.

Critical values of chi-square distribution		
p-value	One sided X^2	Two sided X^2
0.1000000000	1.642374415	2.705543454
0.0500000000	**2.705543454**	**3.841458821**
0.0400000000	3.06490172	4.217884588
0.0300000000	3.537384596	4.709292247
0.0200000000	4.217884588	5.411894431
0.0100000000	**5.411894431**	**6.634896601**
0.0010000000	9.549535706	10.82756617
0.0001000000	13.83108362	15.13670523
0.0000100000	18.18929348	19.51142096
0.0000010000	**22.59504266**	**23.92812698**
0.0000001000	27.03311129	28.37398736
0.0000000100	31.49455797	32.84125335
0.0000000010	35.97368894	37.32489311
0.0000000001	40.46665791	41.8214562

by a *binomial test*. For large samples, the binomial distribution is well approximated by convenient Pearson's chi-squared test. The above relationships are grounded on the assumption, that the number of successes X out of a sample of n observations is equal to $X = N$. In general, let $df1_1$ denote the degrees of freedom 1 of the *f*-distribution for the lower confidence bound. Thus far, it is $df1_{\text{lower}} = 2(N - X + 1)$. Under conditions where $N = X$ the proportion of success is $p(X/N) = 1$, the is then $df1_{\text{lower}} = 2$. Let $df2_{\text{lower}}$ denote the degrees of freedom 2 of the *f*-distribution for the lower confidence bound. In particular, we obtain $df2_{\text{lower}} = 2 \times X$. Under conditions where $N = X$ the proportion of success is $p(X/N) = 1$ and $df2_{\text{lower}} = 2 \times N$. The exact *one-sided* lower confidence interval with confidence level 1 − *alpha* for the proportion of successes $p(X/N) = 1$ can be calculated [13] as

$$p_{\text{Lower}} = \frac{N}{N + F_{(df1_{\text{lower}}, df2_{\text{lower}}, Alpha)}} \tag{14}$$

Example.

Given a sample proportion p and sample size N we can test claims about the population proportion p_0. Different hypothesis tests and test methods (binomial test, one-sample z-test, the t statistic et cetera) can be used to determine whether a hypothesized population proportion p_0 differs significantly from an observed sample proportionp. A hypothesis test requires that a null hypothesis and an alternative hypothesis are mutually exclusive. That is, if a null hypothesis is true, the alternative hypothesis must be false and *vice versa*. How can we conduct a hypothesis test of a proportion. Especially under conditions, where an observed *sample proportion p is equal to* 1, the F distribution [13] is of use for these purposes. Thus far, the proportion of successes of our sample above is equal to $p(X/N) = p(271/271) = 1$. Assuming an *alpha* = 0.05 level of significance the F-value should be calculated as provided above. The F-value for $X = N = 271$ (*Alpha* = 0.05) is $F_{df\,1=2,\ df\,2=542,\ Alpha=0.05} = 3.01235141$. The exact *one-sided* lower confidence bound for the proportion of successes $p(X/N) = p(271/271) = 1$ follows as

$$p_{\text{Lower}} = \frac{N}{N + F_{(df\,1_{\text{lower}}, df\,2_{\text{lower}}, Alpha)}} = \frac{271}{271 + 3.01235141} = 0.989006512 \tag{15}$$

In other words, we assume that the p in the population is greater or equal to 0.989006512. Furthermore, the *one-sided* lower confidence interval with confidence level 1 − *alpha* for the proportion of successes $p(X/N) = 1$, reflects a significance level of *i.e. alpha* = 0.05, and can be calculated for $N > 50$ approximately [13] as

$$p_{\text{Lower}} \approx 1 - \frac{3}{N} \tag{16}$$

A 100 × (1 − *alpha*)% confidence interval consists of all those values $p(X/N)$ for which a test of the hypothesis $p(X/N) = 1$ is not rejected at a significance level of 100 × (*alpha*)%.

2.3.3. Causal Relationship *k*

The mathematical formula of the causal relationship k was used to determine the cause-effect relationship between Epstein-Barr Virus (EBV) infections and Multiple Sclerosis (MS). According to Barukčić [14], the causal relationship k is calculated as

$$k(A_t, B_t) \equiv \frac{p(a_t) - p(A_t) \times p(B_t)}{\sqrt[2]{p(A_t) \times (1 - p(A_t)) \times p(B_t) \times (1 - p(B_t))}} \equiv \frac{(N \times a) - (A \times B)}{\sqrt[2]{A \times \underline{A} \times B \times \underline{B}}} \tag{17}$$

The relationship before is expressed in the following 2 × 2 table (**Table 16**).

Pearson's chi-squared test X^2

$$\chi^2 \equiv \frac{N \times ((a \times d) - (b \times c)) \times ((a \times d) - (b \times c))}{(a + b) \times (c + d) \times (a + c) \times (b + d)} \tag{18}$$

is used to evaluate how likely it is that the observed causal relationship k arose by chance. The 2 × 2 contingency table is dichotomous while the statistical X^2 distribution is continuous. Thus far, Pearson's chi-square test tends to make results larger than they should be and is biased upwards on this account. This upwards bias of Pearson's chi-square test can be corrected by using Yates correction.

Scholium.

As a response to Yules association of two attributes Karl Pearson introduced *the mean square contingency* [15] into statistics as

Table 16. The causal relationship *k*.

		Effect *B*		
		Yes	No	
Cause *A*	Yes	a	b	$a + b = A$
	No	c	d	$c + d = \underline{A}$
		$a + c = B$	$b + d = \underline{B}$	N

$$\phi^2 \equiv \frac{((a \times d) - (b \times c)) \times ((a \times d) - (b \times c))}{(a+b) \times (c+d) \times (a+c) \times (b+d)} \tag{19}$$

Still, Pearson failed to derive a mathematical formula of the causal relationship *k* and much more than this. Pearson himself exterminated any kind causation from statistics ultimately. Following Pearson, "We are now in a position, I think, to appreciate the scientific value of the word cause. Scientifically, cause… is meaningless…" [14]. According to Pearson, the words cause and effect belong strictly to the sphere of sense-impressions. Thus far, "there is… no true cause and effect" [14]. The reader can hardly fail to have been impressed that Pearson himself denies any kind of causality. In the first place, there is no causation at all. "No phenomena are causal" [14]. Finally, "The wider view of the universe sees all phenomena as correlated, but not causally related" [14]. Consequently, Pearson demands that "… there is association but not causation" [14]. We have now reached some very important conclusions about Pearson's account for causality. Due to Pearson, there is no causation at all. Thus far, neither Pearson's correlation coefficient nor his mean square contingency can be regarded as the mathematical formula of the causal relationship *k*. In particular, Pearson failed to derive and to provide a self-consistent mathematical proof of a mathematical formula of the causal relationship.

2.3.4. Statistical Analysis

Data were analyzed using Microsoft Excel version 14.0.7166.5000 (32-Bit) software (Microsoft GmbH, Munich, Germany). The mathematical formula of the causal relationship *k* [14] and the chi-square distribution [12] were applied to determine the significance of causal relationship between EBV and multiple sclerosis (MS). A *p* value of <0.05 was considered significant.

3. Results

3.1. Clinical Characteristics

Wandinger [11] *et al.* examined 108 MS patients from the Department of Neurology (University of Lübeck School of Medicine). All patients were examined independently by two neurologists and had a diagnosis of clinically definite MS. Kurtzke's functional systems and Expanded Disability Status Scale (EDSS) were used to grade the physical disability.

3.2. EBV Seropositivity

The viral status was classified by following serologic definitions. Wandinger [11] *et al.* defined *primary EBV infection* by positivity of anti-EA-IgG and/or anti-EA-IgM in the absence of anti-EBNA-1 antibodies. A *latent or past EBV infection* was defined by positivity of anti-EBNA-1 antibodies. A *reactivation of a latent EBV infection* was defined by EBNA-1-IgG-positive individuals by additional positive anti-EA-IgG and anti-EA-IgM or additional high anti-EA-IgM. The marker for *latent EBV infection* was defined by an anti-EBV nuclear antigen type 1 (anti-EBNA-1) immunoglobulin (Ig)G antibodies.

3.3. Epstein Barr Virus (EBV) Is a *Conditio Sine Qua Non* of Multiple Sclerosis (MS)

A hypothesis test is used to distinguish between the null hypothesis and the alternative hypothesis.

Theorem 1.

Null hypothesis: EBV is *a conditio sine qua non* of multiple sclerosis (MS) ($p_0 \geq p$).

Alternative hypothesis: EBV is *not a conditio sine qua non* of multiple sclerosis (MS) ($p_0 < p$).

Significance level (Alpha) below which the null hypothesis will be rejected: 0.05.

Proof by a statistical hypothesis test.

The data of the prevalence of IgG antibodies in serum samples from Multiple Sclerosis (MS) patients and healthy control subjects are viewed in the following 2 × 2 table (**Table 17**).

The proportion of successes $p(A_t \leftarrow B_t)$ of *the condition sine qua non relationship* in the sample or the test statistic can be calculated defined before as

$$p(A_t \leftarrow B_t) \equiv \frac{a+b+d}{N} \equiv \frac{A+d}{N} \equiv \frac{a+\underline{B}}{N} \equiv \frac{108+147+16}{271} \equiv \frac{247}{247} \equiv 1 \tag{20}$$

The critical value p_{lower} is calculated approximately as

$$p_{\text{Lower}} \approx 1 - \frac{3}{N} = 1 - \frac{3}{247} = 0.988929889 \tag{21}$$

The critical value p_{lower} = 0.989006512 and is less than the proportion of successes $p(A_t \leftarrow B_t) = 1$ as obtained from the observations (significance level alpha = 0.05).

Conclusio.

We cannot reject the null hypothesis in favor of the alternative hypotheses. The sample data do support the Null hypothesis that *Epstein Barr Virus* (*EBV*) *is a conditio sine qua non of Multiple Sclerosis* (*MS*).

In other words, *without* an infection with Epstein Barr Virus (EBV) *no* development of multiple sclerosis (MS).

Quod erat demonstrandum.

3.4. Epstein Barr Virus (EBV) Is the Cause of Multiple Sclerosis (MS)

Theorem 2.

Conditions.

Alpha level = 5%.

The two tailed critical Chi square value (degrees of freedom = 1) for alpha level 5% is 3.841458821.

Claims.

Null hypothesis (H_0): $k = 0$ (No causal relationship).

There is no causal relationship between Epstein Barr virus (EBV) and multiple sclerosis (MS).

Alternative hypothesis (H_A): $k \neq 0$ (Causal relationship).

There is a significant causal relationship between Epstein Barr virus (EBV) and multiple sclerosis (MS).

Proof by two sided hypothesis test.

Based on the data (**Table 18**) of Wandinger *et al.*, we compute the causal relationship $k(\text{EBV, MS})_{\text{Obtained}}$ (our test statistic) as

$$k(\text{EBV}, \text{MS})_{\text{Obtained}} \equiv \frac{(N \times a) - (A \times B)}{\sqrt[2]{A \times \underline{A} \times B \times \underline{B}}} = \frac{271 \times 108 - 255 \times 108}{\sqrt[2]{108 \times 163 \times 255 \times 16}} = +0.2038956576. \tag{22}$$

Following Barukčić, the test statistics obtained is equivalent with a X^2 value of

$$\chi^2 \equiv k(\text{EBV}, \text{MS})_{\text{Obtained}} \times k(\text{EBV}, \text{MS})_{\text{Obtained}} \times N = 271 \times (0.2038956576)^2 = 11.2664020209. \tag{23}$$

A two tailed Chi square of 11.2664020209 is equivalent to a p-value of 0.0004251570.

Table 17. Without EBV no Multiple Sclerosis (MS).

		Multiple sclerosis		
		Yes	No	
EBV anti-EBNA-1 IgG	Yes	108	147	255
	No	0	16	16
		108	108	271

Table 18. EBV and Multiple Sclerosis (MS).

		Multiple sclerosis		
		Yes	No	
EBV anti-EBNA-1 IgG	Yes	108	147	255
	No	0	16	16
		108	163	271

Conclusio.

The value of the test statistic (k obtained or Chi square calculated) is 11.2664020209 and exceeds the critical Chi square value of 3.841458821. Consequently, we reject the null hypothesis (H_0) and accept the alternative hypothesis (H_A).

There is a highly significant causal relationship between Epstein Barr virus (EBV) and multiple sclerosis (k = +0.2038956576, p-value 0.0004251570).

Quod erat demonstrandum.

4. Discussion

Today, the etiology of Multiple Sclerosis (MS) is largely unknown but Multiple Sclerosis (MS) is rare among individuals without serum EBV antibodies. Thus far, there is an accumulating literature for a role of Epstein-Barr Virus (EBV) infections in the pathogenesis of Multiple Sclerosis (MS). Especially, several epidemiological studies suggested an association between infection with Epstein-Barr Virus (EBV) and the occurrence of Multiple Sclerosis (MS) disease. In particular, a recent large prospective epidemiological study showed a relationship between an increase of serum antibody titres against EBV before onset of MS. Acherio *et al.* [16] conducted a prospective, nested case-control study of 62,439 women participating in the Nurses' Health Study to determine whether elevation in serum antibody titers to EBV precede the occurrence of Multiple Sclerosis (MS). Acherio *et al.* concluded that EBV is associated with the etiology of multiple sclerosis. Recently, Levin *et al.* [17] conducted a study among more than 3 million US military personnel and found a relationship between EBV infection and development of MS. Apart from these and other studies aiming at the aetiology of multiple sclerosis (MS), the cause of Multiple Sclerosis (MS) has still not been identified.

We conducted a re-analysis of the study of Wandinger [11] *et al.* to re-investigate the role between EBV infection and MS disease. Using some of the data obtained by the study of Wandinger *et al.*, we questioned whether Epstein-Barr Virus (EBV) is the cause or a cause of multiple sclerosis (MS). The study of Wandinger *et al.* was properly constructed. In accordance with previous studies, Wandinger *et al.* found an unexpectedly high seropositivity rate in MS patients for EBV compared with control subjects. Wandinger *et al.* observed an association of the EBV with MS but failed to detect the true meaning of Epstein-Barr virusin the pathogenesis of multiple sclerosis (MS).

In addition, our study confirms *a conditio sine qua non relationship* between EBV infection and Multiple Sclerosis (MS). In other words, *without* an infection with Epstein-Barr Virus (EBV) *no* development of Multiple Sclerosis (MS) (significance level alpha = 0.05). We observed a highly significant causal relationship between Epstein-Barr Virus (EBV) and multiple sclerosis (k = +0.203895658, p value = 0.000425157). A particular aspect of our study is the identification of Epstein-Barr Virus (EBV) as *the cause* of multiple sclerosis. Since *without* an infection by Epstein-Barr Virus (EBV) *no* multiple sclerosis develops and due to the fact that there is a highly significant causal relationship between Epstein-Barr Virus (EBV) and multiple sclerosis, we are allowed to deduce that Epstein-Barr Virus (EBV) is not only a cause but the cause of Multiple Sclerosis (MS).

5. Conclusion

A particular aspect of our study is the identification of Epstein-Barr Virus (EBV) as *the cause* of Multiple Sclerosis (MS). Finally, the cause of multiple sclerosis is identified. Consequently, it is more than necessary to develop a low-cost and highly effective vaccine against Epstein-Barr Virus (EBV).

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An Accurate Numerical Solution for the Modified Equal Width Wave Equation Using the Fourier Pseudo-Spectral Method

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Abstract

In this study, the numerical solution for the Modified Equal Width Wave (MEW) equation is presented using Fourier spectral method that use to discretize the space variable and Leap-frog method scheme for time dependence. Test problems including the single soliton wave motion, interaction of two solitary waves and interaction of three solitary waves will use to validate the proposed method. The three invariants of the motion are evaluated to determine the conservation properties of the generated scheme. Finally, a Maxwellian initial condition pulse is then studied. The L_2 and L_∞ error norms are computed to study the accuracy and the simplicity of the presented method.

Keywords

The Modified Equal Width Wave Equation, Fourier Pseudo-Spectral Method, Solitary Waves, Fast Fourier Transform

1. Introduction

Discretization using finite differences in time and spectral methods in space has proved to be very useful in solving numerically non-linear Partial Differential Equations (PDEs) describing wave propagation. The Korteweg de Vries (KdV) equation is one famous example to which such combined schemes have been applied efficiently to analyze efficiently unidirectional solitary wave propagation in one dimension [1]-[3]. In [4] [5] the combination of spectral methods and finite differences is applied to well-known nonlinear PDE of the Boussinesq type which admits bidirectional wave propagation, has closed form solitary wave solutions and like the KdV is completely integrable in one space dimension. Also, the combination of spectral methods and leap frog

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is applied to the Regularized Long Wave (RLW) equation [6]. In this paper, a combination of spectral method and leap frog is applied to the modified equal width wave equation. The modified equal width wave equation based upon the Equal Width Wave (EW) equation [7] [8] which was suggested by Morrison *et al*. [9] is used as a model partial differential equation for the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes. This equation is related with the Modified Regularized Long Wave (MRLW) equation [10] and modified Korteweg-de Vries (MKdV) equation [11]. All the modified equations are nonlinear wave equations with cubic nonlinearities and all of them have solitary wave solutions, which are wave packets or pulses. These waves propagate in non-linear media by keeping wave forms and velocity even after interaction occurs. Few analytical solutions of the MEW equation are known. Thus numerical solutions of the MEW equation can be important and comparison between analytic solutions can be made. Geyikli and Battal Gazi Karakoc, [12] [13] solved the MEW equation by a collocation method using septic B-spline finite elements and using a Petrov-Galerkin finite element method with weight functions quadratic and element shape functions which are cubic B-splines. Esen applied a lumped Galerkin method based on quadratic B-spline finite elements which have been used for solving the EW and MEW equations [14] [15]. Saka proposed algorithms for the numerical solution of the MEW equation using quintic B-spline collocation method [16]. Zaki considered the solitary wave interactions for the MEW equation by collocation method using quintic B-spline finite elements [17] and obtained the numerical solution of the EW equation by using least-squares method [18]. Wazwaz investigated the MEW equation and two of its variants by the tanh and the sine-cosine methods [19]. A solution based on a collocation method incorporated cubic B-splines is investigated by and Saka and Dag [20]. Variational iteration method is introduced to solve the MEW equation by Lu [21]. Evans and Raslan [22] studied the generalized EW equation by using collocation method based on quadratic B-splines to obtain the numerical solutions of a single solitary waves and the birth of solitons. Hamdi *et al.* [23] derived exact solitary wave solutions of the generalized EW equation using Maple software. Esen and Kutluay studied a linearized implicit finite difference method in solving the MEW equation [24]. Karakoç and Geyikli [25] solved the MEW equation numerically by a lumped Galerkin method using cubic B-spline finite elements. The modified equal width wave equation has the normalized form [9]

$$U_t + 3U^2 U_x - \mu U_{xxt} = 0, \tag{1}$$

where μ is a positive parameter and the subscripts x and t denote differentiation, when solved analytically, within an infinite region with physical boundary conditions $U \to 0$ as $|x| \to \infty$. In this study, boundary conditions are chosen from

$$U(a,t) = 0, U(b,t) = 0, t > 0 \tag{2}$$

and the initial condition

$$U(x,0) = f(x), a \le x \le b \tag{3}$$

where $f(x)$ is a localized disturbance inside the considered interval. We investigate the numerical solution of the MEW equation using the Fourier Leap-Frog methods. The proposed method is validated by studying the motion of a single solitary wave, development of interaction of two positive solitary waves and development of three positive solitary waves interaction for the MEW Equation (1).

2. Analysis the Proposed Method

For the numerical treatment, the spatial variable x of Equation (1) is restricted over an interval $a \le x \le b$. In this study, consider the MEW Equation (1) with the boundary conditions in Equation (2). A numerical method is developed for the periodic initial value problem in which U is a prescribed function of x at $t = 0$ and the solution is periodic in x outside a basic interval $a \le x \le b$. For most of the problems considered, interval may be chosen large enough so the boundaries do not affect the wave interactions being studied. Equation (1) can be written as

$$V_t = -3U^2 U_x, \tag{4}$$

where

$$V = U - \mu U_{xx}. \tag{5}$$

For ease of presentation the spatial period $[a, b]$ is normalized to $[0, 2\pi]$, with the change of variable

$$x \to \frac{2\pi}{b-a}(x-a).$$

Let $L = b - a$. Thus, Equations (4) and (5) become

$$V = U - \left(\frac{2\pi}{L}\right)^2 \mu U_{xx}, \tag{6}$$

$$V_t = -3\left(\frac{2\pi}{L}\right) U^2 U_x, \tag{7}$$

$U(x,t)$ is transformed into Fourier space with respect to x, and derivatives (or other operators) with respect to x. This operation can be done with the Fast Fourier transform (FFT). Applying the inverse Fourier transform $\frac{\partial^n U}{\partial x^n} = F^{-1}\left\{(ik)^n F(U)\right\}, n = 1, 2, \cdots$ with $n = 1$ and $n = 2$. The Equations (6) and (7) become

$$V = U - \left(\frac{2\pi}{L}\right)^2 \mu F^{-1}\left\{-k^2 F(U)\right\}, \tag{8}$$

$$V_t = -3\left(\frac{2\pi}{L}\right) U^2 F^{-1}\left\{ik F(U)\right\}. \tag{9}$$

In practice, we need to discretize Equations (6) and (7). For any integer $N > 0$, consider $x_j = j\Delta x = \frac{2\pi j}{N}$, $j = 0, 1, \cdots, N-1$. Let $U(x,t)$ be the solution of Equations (8) and (9). Then, we transform it into the discrete Fourier space as

$$\hat{U}(k,t) = F(U) = \frac{1}{N}\sum_{J=0}^{N-1} U(x_j,t)\,\mathrm{e}^{-ikx_j}, -\frac{N}{2} \le k \le \frac{N}{2} - 1. \tag{10}$$

From this, using the inversion formula, we get

$$U(x_j,t) = F^{-1}(\hat{U}) = \sum_{k=-N/2}^{N/2-1} \hat{U}(k,t)\,\mathrm{e}^{ikx_j}, 0 \le j \le N-1. \tag{11}$$

Replacing F and F^{-1} by their discrete counterparts, and discretizing Equations (8) and (9) give

$$V(x_j,t) = U(x_j,t) - \left(\frac{2\pi}{L}\right)^2 \mu F^{-1}\left\{-k^2 F(U)\right\}, \tag{12}$$

$$\frac{\partial V(x_j,t)}{\partial t} = -3\left(\frac{2\pi}{L}\right) U^2(x_j,t) F^{-1}\left\{ik F(U)\right\}. \tag{13}$$

Letting $\boldsymbol{U} = \left[U(x_0,t), U(x_1,t), \cdots, U(x_{N-1},t)\right]^{\mathrm{T}}$. Equation (13) can be written in the vector form

$$\boldsymbol{V}_t = \boldsymbol{G}(\boldsymbol{U}) \tag{14}$$

where $\boldsymbol{G}(\boldsymbol{U})$ defines the right hand side of Equation (13).

3. Fourier Leap-Frog Method for MEW Equation

A time integration known as a Leap-Frog method (a two-step scheme) is given as

$$V_t = \frac{V(x,t+\Delta t) - V(x,t-\Delta t)}{2\Delta t} = \frac{V^{n+1} - V^{n-1}}{2\Delta t}$$

Use the Leap-Frog scheme to advance in time, we obtain $V(x,t+\Delta t) = V(x,t-\Delta t) + 2\Delta t G(U(t))$.

This is called the Fourier-Leap-Frog (FLF) scheme for the MEW Equation (14). FLF method needs two levels of initial data, we begin with $U(x,0)$ to get $V(x,0)$ from Equation (12), we get

$$V(x,n\Delta t)=F^{-1}\left(\left(F\left(U\left(x,n\Delta t\right)\right)\right)\left(1+\left(\frac{2\pi}{L}\right)^2\mu k^2\right)\right), \tag{15}$$

$$V(x,0)=F^{-1}\left(\left(F\left(U\left(x,0\right)\right)\right)\left(1+\left(\frac{2\pi}{L}\right)^2\mu k^2\right)\right). \tag{16}$$

Then evaluate second level of initial solution $V(x,\Delta t)$ by using a higher-order one-step method, for example, a fourth-order Runge-Kutta method (RK4).

$$\begin{aligned}
K_1 &= F\left(U(x,0),0\right)\\
K_2 &= F\left(U(x,0)+\frac{1}{2}\Delta t K_1,\frac{1}{2}\Delta t\right)\\
K_3 &= F\left(U(x,0)+\frac{1}{2}\Delta t K_2,\frac{1}{2}\Delta t\right)\\
K_4 &= F\left(U(x,0)+\Delta t K_3,\Delta t\right)\\
V(x,\Delta t) &= V(x,0)+\frac{\Delta t}{6}\left[K_1+2K_2+2K_3+K_4\right]
\end{aligned} \tag{17}$$

then substitute $V(x,\Delta t)$ in

$$U(x,n\Delta t)=F^{-1}\left(\frac{F\left(V\left(x,n\Delta t\right)\right)}{1+\left(\frac{2\pi}{L}\right)^2\mu k^2}\right) \tag{18}$$

to get $U(x,\Delta t)$. Thus, the time discretization for Equation (13) is given as

$$V(x,t+\Delta t)=V(x,t-\Delta t)-2\Delta t\left(3\left(\frac{2\pi}{L}\right)U^2(x,t)F^{-1}\left\{ik\,F\left(U\left(x,t\right)\right)\right\}\right). \tag{19}$$

We substitute $V(x, 0)$ and $U(x, \Delta t)$ in Equation (19) to evaluate $V(x, 2\Delta t)$ then substitute $V(x, 2\Delta t)$ in Equation (18) to evaluate $U(x, 2\Delta t)$, so we have $V(x, \Delta t)$ and $U(x, 2\Delta t)$, substitute in Equation (19) to evaluate $V(x, 3\Delta t)$ and evaluate $U(x, 3\Delta t)$ from Equation (18) and so on, until we evaluate $U(x, t)$ at time $t = n\Delta t$.

4. Cases Study and Results

In order to show how good the numerical solutions are in comparison with the exact ones, L_2 and L_∞ error norms will be computed by

$$\begin{aligned}
L_2 &= \left\|u^{exact}-u^{num}\right\|_2=\left[\Delta x\sum_{i=1}^{N}\left|u_i^{exact}-u_i^{num}\right|^2\right]^{1/2},\\
L_\infty &= \left\|u^{exact}-u^{num}\right\|_\infty=\max_i\left|u_i^{exact}-u_i^{num}\right|.
\end{aligned} \tag{20}$$

The conservation properties of the MEW equation will be examined by calculating the following three invariants, given as [17] which respectively correspond to mass, momentum, and energy

$$\begin{aligned}
C_1 &= \int_a^b U\mathrm{d}x=\Delta x\sum_{j=1}^{n}U\left(x_j,t\right),\\
C_2 &= \int_a^b\left[U^2+\mu\left(U_x\right)^2\right]\mathrm{d}x=\Delta x\sum_{j=1}^{n}\left[\left(U\left(x_j,t\right)\right)^2+\mu\left(U_x\left(x_j,t\right)\right)^2\right],\\
C_3 &= \int_a^b U^4\mathrm{d}x=\Delta x\sum_{j=1}^{n}\left(U\left(x_j,t\right)\right)^4.
\end{aligned} \tag{21}$$

For the computation of U_x in Equation (21), we used Fourier transform. To implement the performance of the method, three test problems will be considered: the motion of a single solitary wave, development of two positive solitary waves interaction, development of three positive solitary wave interaction.

4.1. The Motion of Single Solitary Wave

Consider Equation (1) with the boundary $U \to 0$ as $x \to \pm\infty$ and initial condition

$$U(x,0) = A\operatorname{sec}h\left[k(x-x_0)\right]. \tag{22}$$

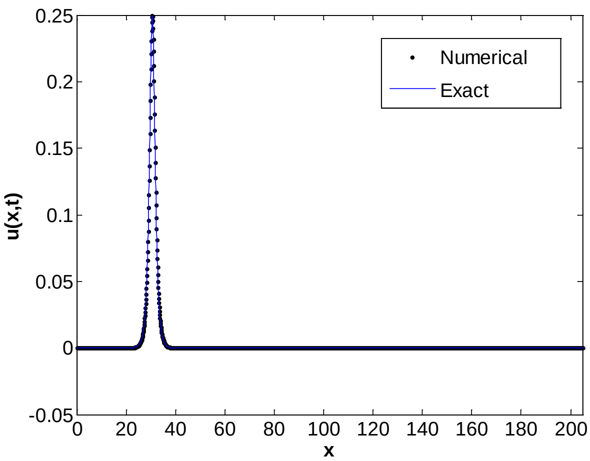
This problem has a solitary wave solution of the form

$$U(x,t) = A\operatorname{sec}h\left[k(x-x_0-\upsilon t)\right] \tag{23}$$

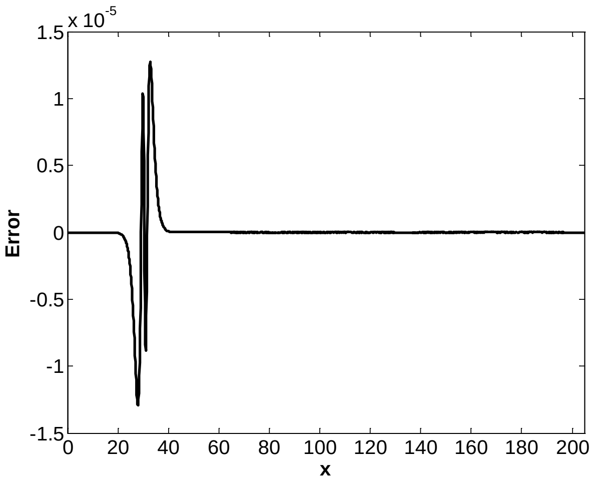
which represents the motion of a single solitary wave with amplitude A, where the wave velocity $\upsilon = A^2/2$ and $k = \sqrt{1/\mu}$. For this problem the analytical values of the invariants are [16]

$$C_1 = \frac{A\pi}{k}, C_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3}, C_3 = \frac{4A^4}{3k}. \tag{24}$$

For the numerical simulation of the motion of a single solitary wave, the parameters $\Delta x = 0.1$, $\Delta t = 0.001$, $\mu = 1$, $x_0 = 30$, $N = 2048$ and $A = 0.25$ are chosen. The analytical values for the invariants are $C_1 = 0.7853982$, $C_2 = 0.1666667$, and $C_3 = 0.0052083$. As it is seen from **Table 1**, the invariants C_1 and C_3 remain almost constant during the computer run at times $t = 0$ to $t = 100$ (changes of the invariants C_1 and C_3 approach zero), where C_2 changes from its initial value by less than 1×10^{-9}. The error norms L_2 and L_∞ at different various times are shown in **Table 1**. It is shown that the numerical values very close to the exact values. **Figure 1(a)** shows that the proposed method performs the motion of propagation of a solitary wave satisfactorily, which moved to the right at a constant speed and preserved its amplitude and shape with increasing time as expected. Amplitude is 0.25 at $t = 0$ which is located at $x = 30$, while it is 0.249985 at $t = 20$ which is located at $x = 30.6149$. The absolute difference in amplitudes at times $t = 0$ and $t = 20$ is only 1.5×10^{-5}. Error distribution at time $t = 20$ is drawn in **Figure 1(b)**, from which it can be seen that maximum errors happened just around the peak position of the solitary wave. **Table 2** displays the values of the error norms and numerical invariants obtained at different values of N with $\Delta x = 0.1$, $\Delta t = 0.001$, $\mu = 1$, $x_0 = 30$ and $A = 0.25$. As it is seen from **Table 2**, the error norms decrease (halved) when N increases (doubled) and numerical invariants C_1, C_2 and C_3 closed to the analytical values when N increases. The comparison between the results obtained by the present with those in the other studies [15] [22] [24] [25] also documented in **Table 2**.



(a)



(b)

Figure 1. (a) The motion of a single solitary wave and (b) the error distribution in FLF scheme for MEW equation with $A = 0.25$, $N = 2048$, $\Delta x = 0.1$ and $\Delta t = 0.001$ at $t = 20$.

Table 1. Invariants and error norms for the single soliton using FLF scheme with $A = 0.25$, $N = 2048$, $\Delta x = 0.1$ and $\Delta t = 0.001$.

t	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	0.785014668	0.166625987	0.005205790	0.0000000	0.0000000
5	0.785014668	0.166625987	0.005205790	0.0069317	0.0032278
10	0.785014668	0.166625986	0.005205790	0.0160855	0.0081248
15	0.785014668	0.166625986	0.005205790	0.0226029	0.0113299
20	0.785014668	0.166625986	0.005205790	0.0275859	0.0129414
25	0.785014668	0.166625986	0.005205790	0.0343768	0.0161599
30	0.785014668	0.166625986	0.005205790	0.0410980	0.0193344
35	0.785014668	0.166625986	0.005205790	0.0477366	0.0224764
40	0.785014668	0.166625986	0.005205790	0.0558158	0.0273561
45	0.785014668	0.166625986	0.005205790	0.0622846	0.0305792
50	0.785014668	0.166625986	0.005205790	0.0686557	0.0337939
100	0.785014668	0.166625986	0.005205790	0.1239589	0.0654669

Table 2. Invariants, error norms for the single soliton MEW equation using FLF scheme with $A = 0.25$, $\Delta x = 0.1$ and $\Delta t = 0.001$ at different values of N at $t = 20$ and comparison with different methods at $A = 0.25$, $\Delta t = 0.05$ and $\Delta x = 0.1$.

N	C_1	C_2	C_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
512	0.7838642	0.1665041	0.0051982	0.1103535	0.0517942
1024	0.7846312	0.1665853	0.0052033	0.0551734	0.0258918
2048	0.7850147	0.1666260	0.0052058	0.0275859	0.0129412
4096	0.7852064	0.1666463	0.0052071	0.0137927	0.0064692
8192	0.7853023	0.1666565	0.0052077	0.0068963	0.0032342
Ref [15]	0.7853898	0.1667614	0.0052082	0.0796940	0.0465523
Ref [22]	0.7849545	0.1664765	0.0051995	0.2905166	0.2498925
Ref [24]	0.7853977	0.1664735	0.0052083	0.2692812	0.2569972
Ref [25]	0.7853967	0.1666663	0.0052083	0.0800980	0.0460618

4.2. Interaction of Two Solitary Waves

The initial condition given by the linear sum of two separate solitary waves of various amplitudes

$$u(x,0)=\sum_{j=1}^{2} A_j \operatorname{sec}h\left(k\left(x-x_j\right)\right), \tag{25}$$

where $k = 1/\sqrt{\mu}$. Firstly the interaction of two positive solitary waves is study with the parameters $A_1 = 1$, $A_2 = 0.5$, $x_1 = 15$, $x_2 = 30$, $N = 8192$, $\Delta x = 0.1$ and $\Delta t = 0.01$. The analytic invariants are [25], $C_1 = \pi\left(A_1 + A_2\right) = 4.7123889$, $C_2 = (8/3)\left(A_1^2 + A_2^2\right) = 3.3333333$ and $C_3 = (4/3)\left(A_1^4 + A_2^4\right) = 1.4166667$. The initial function was placed on the left side of the region with the larger wave to the left of the smaller one as seen in **Figure 2(a)**. Both waves move to the right with velocities dependent upon their magnitudes. The larger wave catches up with the smaller one as time increase. Interaction started at about time $t = 25$, the overlapping process continues until the time $t = 40$, then two solitary waves emerge from the interaction and resume their former shapes and amplitudes as shown in **Figures 2(b)-(f)**. The magnitude of the smaller wave 0.510741 on reaching

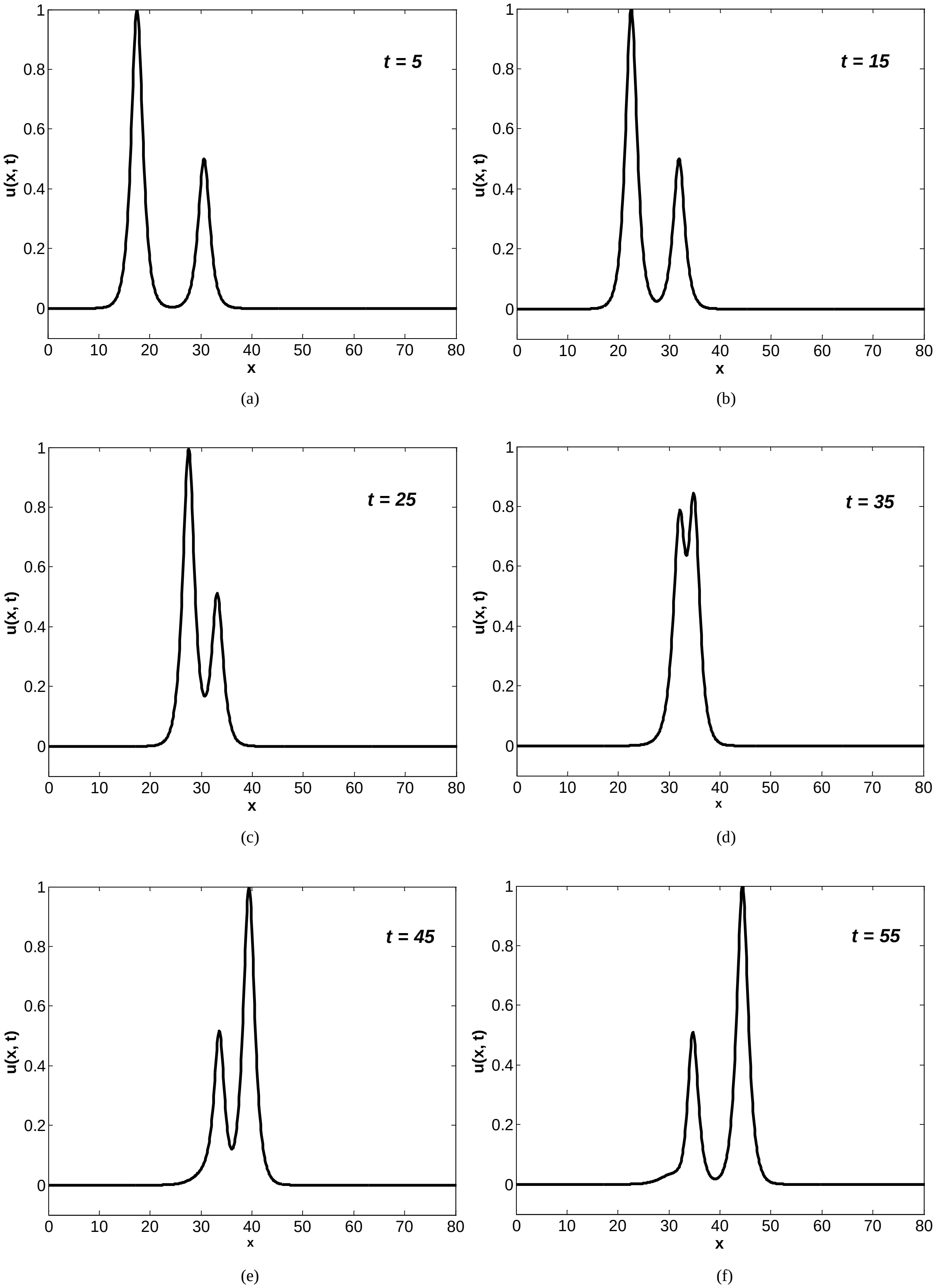


Figure 2. Interaction of two solitary waves at different times with $A_1 = 1$ and $A_2 = 0.5$.

position $x = 34.7$ and of the larger wave 1.000097 having the position $x = 44.4$ are measured at time $t = 55$ so that difference in amplitudes is 0.010741 for the smaller wave and 0.000097 for the larger wave. **Table 3** displays the values of the invariants obtained by the present method. It is observed that the obtained values of the invariants remain almost constant during the computer run. The change in C_2 is 6.11×10^{-5} and in C_3 is 5.68×10^{-5} and C_1 is exact up to the last recorded digit.

The intersection of two solitary waves was also studies with the following parameters: $\mu = 1$, $x_1 = 15$, $x_2 = 30$, $A_1 = -2$, $A_2 = 1$, $N = 8192$, $\Delta t = 0.01$ and $\Delta x = 0.1$ in the range $0 \le x \le 819.2$. The experiment was run from $t = 0$ to $t = 55$ to allow the interaction to take place. **Figure 3** shows the development of the solitary wave interaction. As is seen from **Figure 3**, at $t = 0$ a wave with the negative amplitude is on the left of another wave with the positive amplitude. The larger wave with the negative amplitude catches up with the smaller one with the positive amplitude as the time increases. At $t = 55$, the amplitude of the smaller wave is at the point 0.9741792 at the point 52.5064095 whereas the amplitude of the larger one is −2.0014682 at the point 123.6150897326334 It is found that the absolute difference in amplitudes is 0.025820781 for the smaller wave and 0.00146821 for the larger wave. The analytical invariants can be found as $C_1 = -3.1415927$, $C_2 = 13.3333333$ and $C_2 = 22.6666667$. It can be seen in **Table 3** that the values obtained for the invariants are satisfactorily constant during the computer run.

4.3. Interaction of Three Solitary Waves

Interaction of three solitary waves is studied by considering Equation (1) with the following initial condition:

$$u(x,0) = \sum_{j=1}^{3} A_j \operatorname{sec}h\left(k\left(x - x_j\right)\right), \tag{26}$$

where $= 1/\sqrt{\mu}$. The computations are carried out with parameters = 1, $A_1 = 1$, $A_2 = 0.5$, $A_3 = 0.25$, $x_1 = 15$, $x_2 = 30$, $x_3 = 45$, $N = 8192$, $\Delta x = 0.1$ and $\Delta t = 0.01$. Solitary wave having the largest amplitude is located to the left of the smaller ones. As is well known, solitary waves with larger amplitudes have a greater velocity than those with smaller amplitudes. Consequently, as time goes on the larger two solitary waves catches up with the smaller one, the overlapping process of the three solitary waves continues while the larger solitary waves have overtaken the smaller ones. Plot of the three solitary waves is depicted at various times in **Figure 4**. Interaction of three solitary

Table 3. Invariants for the interaction of two solitary waves with $\Delta t = 0.01$, $\Delta x = 0.1$ and $N = 8192$.

t	$A_1 = 1, A_2 = 0.5$			$A_1 = -2, A_2 = 1$		
	C_1	C_2	C_3	C_1	C_2	C_3
0	4.7118132	3.3331323	1.4164968	−3.1412080	13.3325098	22.6638508
5	4.7118132	3.3330876	1.4164524	−3.1412080	13.3293775	22.6514303
10	4.7118132	3.3330830	1.4164479	−3.1412080	13.3306037	22.6526691
15	4.7118132	3.3330828	1.4164480	−3.1412080	13.3293817	22.6514527
20	4.7118132	3.3330828	1.4164480	−3.1412080	13.3293781	22.6514501
25	4.7118132	3.3330822	1.4164476	−3.1412080	13.3293776	22.6514500
30	4.7118132	3.3330776	1.4164476	−3.1412080	13.3293774	22.6514500
35	4.7118132	3.3330712	1.4164400	−3.1412080	13.3293774	22.6514500
40	4.7118132	3.3330803	1.4164462	−3.1412080	13.3293773	22.6514500
45	4.7118132	3.3330826	1.4164478	−3.1412080	13.3293749	22.6514500
50	4.7118132	3.3330831	1.4164481	−3.1412080	13.3293750	22.6514500
55	4.7118132	3.3330831	1.4164481	−3.1412080	13.3293751	22.6514500

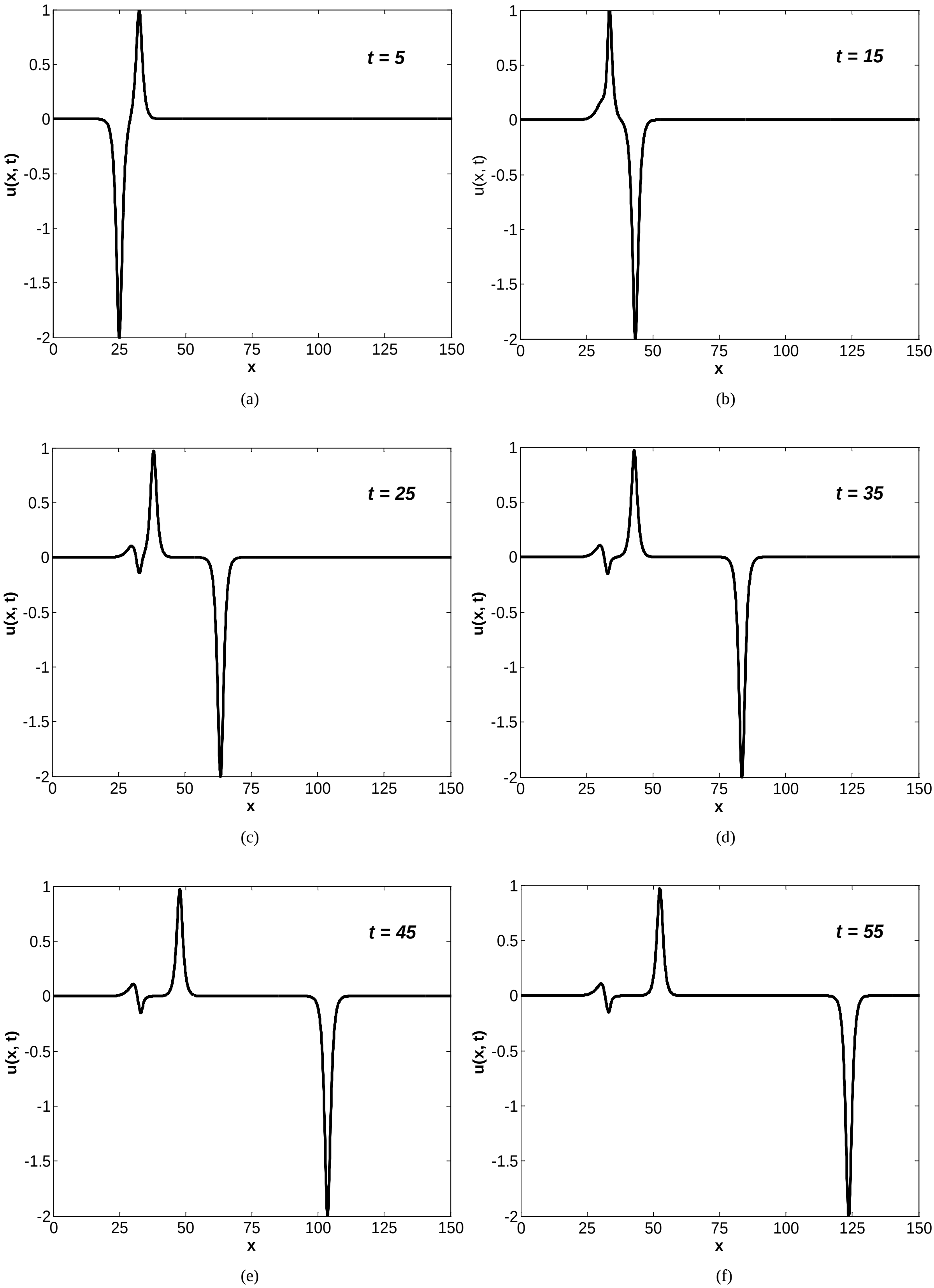


Figure 3. Interaction of two solitary waves at different times with $A_1 = -2$ and $A_2 = 1$.

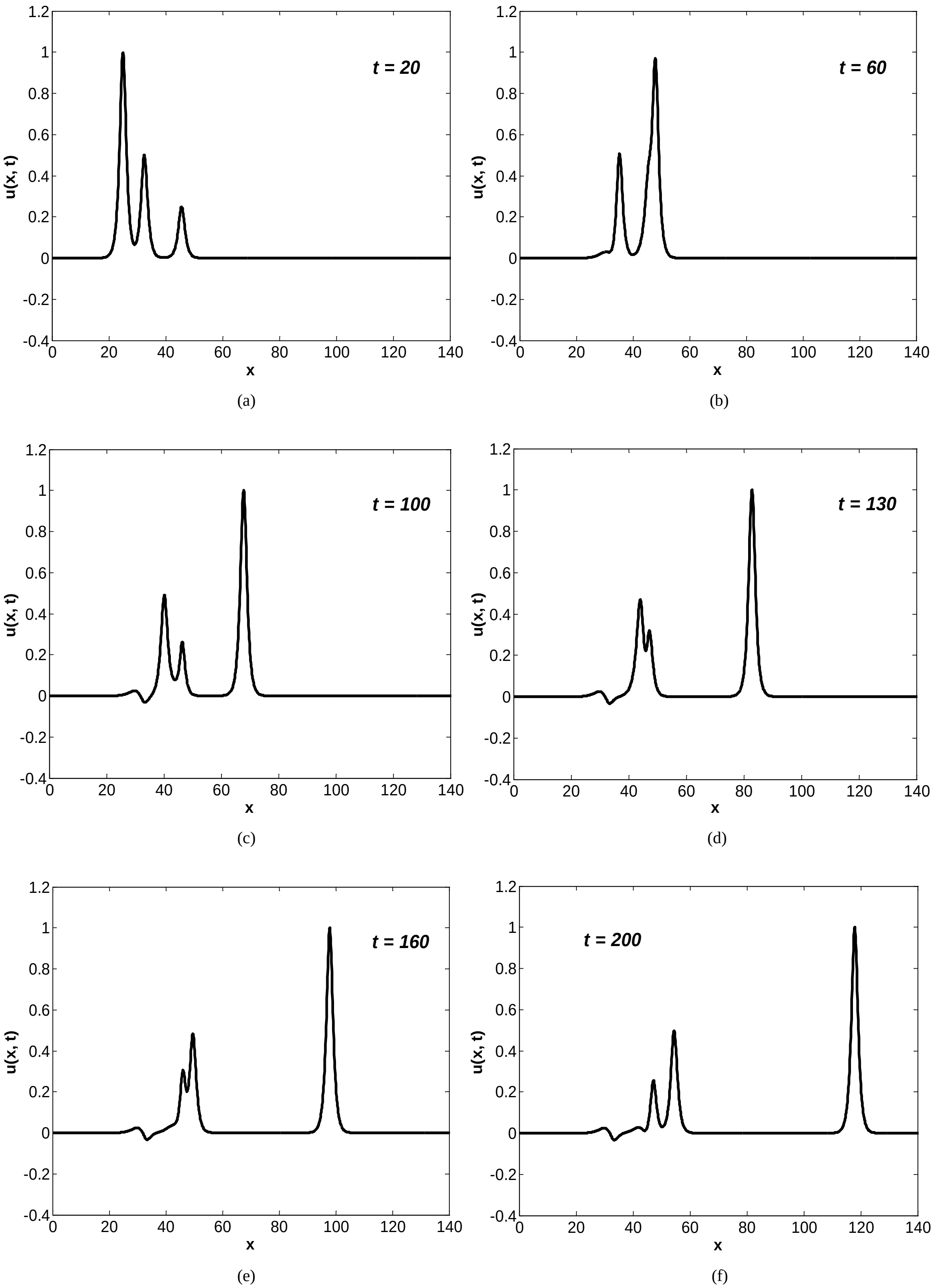


Figure 4. Interaction of three solitary waves at different times.

waves can be openly observed from the time-amplitude graph in **Figure 4** for the three algorithms. At t = 200, the amplitudes of the smaller waves are 0.25613 at the point x = 47.21 and 0.49672 at the point x = 54.41, whereas the amplitude of the larger one is 1.00032 at the point x = 117.91. **Table 4** displays the values of the invariants obtained by the present method. It is observed that the obtained values of the invariants remain almost constant during the computer run. The change in C_2 is 5.37 × 10^{-5} and in C_3 is 5.09 × 10^{-5} and C_1 is exact up to the last recorded digit. The analytical values can be found [25] as $C_1 = \pi\left(A_1 + A_2 + A_3\right) = 5.4977871$, $C_2 = (8/3)\left(A_1^2 + A_2^2 + A_3^2\right) = 3.5$ and $C_3 = (4/3)\left(A_1^4 + A_2^4 + A_3^4\right) = 1.421875$.

4.4. The Maxwellian Initial Condition

We consider here is the numerical solution of the Equation (1) with the Maxwellian initial condition

$$u(x,0) = \mathrm{e}^{-x^2}, \tag{27}$$

with the boundary conditions

$$u(-20,t) = u_x(-20,t) = u(20,t) = u_x(20,t) = 0.$$

As it is known, Maxwellian initial condition the behavior of the solution depends on the values of μ. The computations are carried out for the cases μ = 1, 0.5, 0.1, 0.05, 0.02 and 0.005 which are used in [12] [17] [21]. When μ = 1, 0.5 is used as shown **Figure 5(a)** and **Figure 5(b)** at time t = 12 the Maxwellian initial condition does not cause development into a clean solitary wave. However with smaller values of μ = 1, 0.1, 0.05, 0.02 and 0.005 Maxwellian initial condition breaks up into more solitary waves which drawn in **Figures 5(c)-(f)** at time t = 12. The numerical conserved quantities with μ = 1, 0.5, 0.1, 0.05, 0.02 and 0.005 are given in **Table 5**. It can be seen in **Table 4** that the values obtained for the invariants are satisfactorily constant during the computer run.

5. Conclusion

The Fourier Leap Frog method has been successfully applied to obtain the numerical solution of the modified equal width wave equation. Four test problems are worked out to examine the performance of the used method. The motion of a single solitary wave and its accuracy was shown by calculating error norms L_2 and L_∞ and shown in the figures and tables. The interaction of two solitary waves and its accuracy shown by compare with other numerical solutions. The interaction of three solitary waves and its accuracy shown by compare with other numerical solutions. A Maxwellian initial condition pulse is then studied at different values of μ. The invariants

Table 4. Invariants for the interaction of three solitary waves.

t	C_1	C_2	C_3
0	5.4971155	3.4997894	1.4217047
20	5.4971155	3.4997399	1.4216558
40	5.4971155	3.4997374	1.4216540
60	5.4971155	3.4997357	1.4216538
80	5.4971155	3.4997401	1.4216560
100	5.4971155	3.4997401	1.4216560
120	5.4971155	3.4997401	1.4216560
140	5.4971155	3.4997400	1.4216559
160	5.4971155	3.4997401	1.4216560
180	5.4971155	3.4997401	1.4216560
200	5.4971155	3.4997401	1.4216560

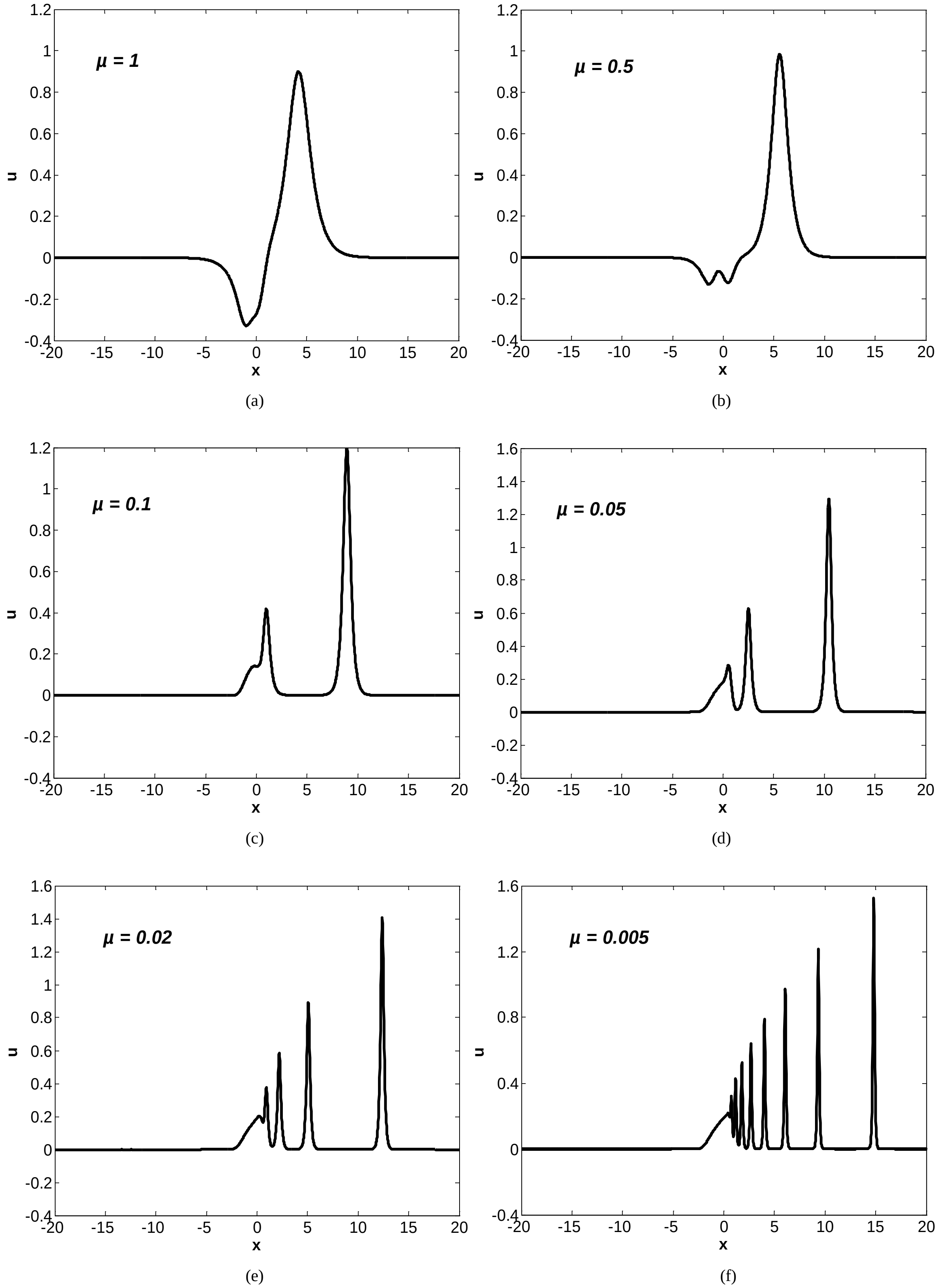


Figure 5. Maxwellian initial condition, state at $t = 12$ and different values of μ.

Table 5. Invariants for Maxwellian initial condition at different values of μ.

t	μ	C_1	C_2	C_3
0		1.7715884	2.5066286	0.8857942
3		1.7715884	2.5066279	0.8857938
6	1	1.7715884	2.5066276	0.8857936
9		1.7715884	2.5066277	0.8857937
12		1.7715884	2.5066275	0.8857937
0		1.7715884	1.8796654	0.8857942
3		1.7715884	1.8796645	0.8857934
6	0.5	1.7715884	1.8796646	0.8857934
9		1.7715884	1.8796646	0.8857935
12		1.7715884	1.8796645	0.8857935
0		1.7715884	1.3780948	0.8857942
3		1.7715884	1.3780956	0.8857942
6	0.1	1.7715884	1.3780957	0.8857942
9		1.7715884	1.3780955	0.8857941
12		1.7715884	1.3780955	0.8857941
0		1.7715884	1.3153985	0.8857942
3		1.7715884	1.3154016	0.8857967
6	0.05	1.7715884	1.3154017	0.8857967
9		1.7715884	1.3154017	0.8857967
12		1.7715884	1.3154016	0.8857967
0		1.7715884	1.2777807	0.8857942
3		1.7715884	1.2777913	0.8858065
6	0.02	1.7715884	1.2777914	0.8858066
9		1.7715884	1.2777913	0.8858066
12		1.7715884	1.2777914	0.8858066
0		1.7715884	1.2589718	0.8857942
3		1.7715884	1.2590204	0.8858617
6	0.005	1.7715884	1.2590208	0.8858619
9		1.7715884	1.2590209	0.8858619
12		1.7715884	1.2590209	0.8858619

are satisfactorily constant in computer run in all cases. The obtained results show that the present method is a remarkably successful numerical method and can also be efficiently applied to other types of non-linear problems.

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Solutions of Zhiber-Shabat and Related Equations Using a Modified tanh-coth Function Method

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Abstract

The modified tanh-coth function method is used to obtain new exact travelling wave solutions for Zhiber-Shabat equation and the related equations: Liouville equation, sinh-Gordon equation, Dodd-Bullough-Mikhailov equation, and Tzitzeica-Dodd-Bullough equation. Exact travelling wave solutions for each equation are derived and expressed in terms of hyperbolic functions, trigonometric functions and rational functions. The modified tanh-coth function method is easy to execute, brief, efficient, and can be used to solve many other nonlinear evolution equations.

Keywords

A Modified tanh-coth Function Method, Zhiber-Shabat Equation, Liouville Equation, sinh-Gordon Equation, Dodd-Bullough-Mikhailov Equation, Tzitzeica-Dodd-Bullough Equation, Travelling Wave Solutions, Solitary Wave Solutions

1. Introduction

In this study we will investigate the solution of the nonlinear Zhiber-Shabat equation [1]

$$u_{xt} + pe^{u} + qe^{-u} + re^{-2u} = 0 \tag{1}$$

where p, q and r are arbitrary constants. If $q = r = 0$, Equation (1) becomes the Liouville equation. If $r = 0$, Equation (1) becomes the sinh-Gordon equation. And for $q = 0$, Equation (1) reduces to the well-known Dodd-Bullough-Mikhailov equation. However, for $p = 0$, $q = -1$, $r = 1$, we get the Tzitzeica-Dodd-Bullough equation. These equations play an important role in many areas such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, nonlinear optics, dislocation in crystals, kink dynamics, and quantum

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filed theory [1]. The Zhiber-Shabat equation and other related equations were studied by some authors. Wazwaz in [2] and [3] applied the tanh method and the extended tanh method for handling the Zhiber-Shabat equation and other related equations: Liouville equation, sinh-Gordon, Dodd-Bullough-Mikhailov equation, and Tzitzeica-Dodd-Bullough equation. Fan and Hon in [4] have used the extended tanh method for handling Dodd-Bullough-Mikhailov equation. Wu and He in [5] solved the Dodd-Bullough-Mikhailov equation using the Exp-function method. Wazzan in [6] solved the Zhiber-Shabat equation and other related equations using the $\left(\frac{G'}{G}\right)$ expansion-method. Our intention in this work is to find new solitary wave solutions for the nonlinear Zhiber-Shabat equation. Since there is no unified method that can be used to handle all types of nonlinear problems, we will use a modified tanh-coth function method [7]-[10]. Moreover, we will carry out comparisons between solutions obtained by the modified tanh-coth function method and other aforementioned methods.

2. The Modified tanh-coth Function Method

2.1. Description of the Method

To illustrate the basic concepts of the modified tanh-coth function method, we consider a given PDE in two variables given by

$$P\left(u,u_t,u_x,u_{tt},u_{xt},u_{xx},\cdots\right)=0. \tag{2}$$

We first consider its travelling solutions $u=u\left(x,t\right)=u\left(\xi\right)$, where $\xi=x-\lambda t$ or $\xi=x+\lambda t$, then Equation (2) becomes an ordinary differential equation

$$Q\left(u,u',u'',u''',u'''',\cdots\right)=0. \tag{3}$$

The next crucial step is that the solution we are looking for is expressed in the form:

$$u\left(\xi\right)=a_0+\sum_{i=1}^{m}\left(a_i w^i+b_i w^{-i}\right) \tag{4}$$

and

$$w'=R+w^2, \tag{5}$$

where R is a parameter to be determined later, $w=w\left(\xi\right)$ and $w'=\frac{\mathrm{d}w}{\mathrm{d}\xi}$. The parameter m can be found by balancing the highest order linear term with the nonlinear terms. Inserting (4) and (5) into the ordinary differential Equation (3) will yield a system of algebraic equations with respect to a_0, a_i, b_i and R (where $i=1,\cdots,m$). Because all the coefficients of w^i have to vanish, and using any symbolic computation program such as Maple or Mathematica, one can determine a_0, a_i, b_i and R. The Riccati Equation (5) has the following general solutions:

1) If $R<0$,

$$w=-\sqrt{-R}\tanh\left[\sqrt{-R}\xi\right],$$

$$w=-\sqrt{-R}\coth\left[\sqrt{-R}\xi\right].$$

2) If $R=0$,

$$w=\frac{1}{\xi}.$$

3) If $R>0$,

$$w=\sqrt{R}\tan\left[\sqrt{R}\xi\right],$$

$$w=-\sqrt{R}\cot\left[\sqrt{R}\xi\right].$$

In the next section, five examples in mathematical physics are chosen to illustrate the modified tanh-coth function method.

2.2. Application

2.2.1. The Zhiber-Shabat Equation

As before, we use $u(x,t)=u(\xi)$ where $\xi = x-\lambda t$, this will carry out the Zhiber-Shabat Equation (1) into

$$-cu_{\xi\xi} + pe^{u} + qe^{-u} + re^{-2u} = 0. \tag{6}$$

We use the Painleve property:

$$v = e^{u},$$

or equivalently

$$u = \ln v, \tag{7}$$

from which we find

$$u' = \frac{1}{v}v', \tag{8}$$

$$u'' = \frac{1}{v}v'' - \frac{1}{v^2}(v')^2. \tag{8}$$

The transformations (7) and (8) carry out (6) into the ODE

$$-c\left(vv'' - (v')^2\right) + pv^3 + qv + r = 0. \tag{9}$$

Using the modified tanh-coth function method, balancing the term vv'' with v^3, gives $m = 2$, hence we set the modified tanh-coth function method assumption as follows:

$$v(x,t) = v(\xi) = a_0 + a_1 w + a_2 w^2 + \frac{b_1}{w} + \frac{b_2}{w^2} \tag{10}$$

where

$$w' = R + w^2, \tag{11}$$

and

$$w'' = 2Rw + 2w^3.$$

Without loss of generality, we set $p = q = r = 1$.

Substituting (10) into (9), and making use of Equation (11) collecting the coefficients of each power of w, and using Maple to solve the nonlinear system in a_0, a_1, a_2, b_1, b_2 and R, we obtain:

1) First set

$$a_0 = \frac{\gamma}{12\alpha},\ b_2 = 0,\ a_2 = 2\lambda,\ R = \frac{-1}{2\lambda}\left(\frac{\gamma^2 - 15\alpha\gamma - 36\alpha^2}{36\alpha^2}\right).$$

2) Second set

$$a_0 = \frac{\gamma}{12\alpha},\ a_2 = 0,\ b_2 = \frac{1}{2\lambda}\left(\frac{3\gamma^2 - 16\alpha\gamma - 144\alpha^2}{36\alpha^2}\right),$$

$$R = \frac{2b_2}{31}\left(\frac{11\gamma^2 - 78\alpha\gamma - 504\alpha^2}{36\alpha^2}\right).$$

3) Third set

$$a_0 = \frac{-11}{3596}\left(\frac{\overline{\gamma}^2 + 810\overline{\alpha\gamma} + 25668\overline{\alpha}^2}{36\overline{\alpha}^2}\right),\ a_2 = 2\lambda,\ b_2 = \frac{1}{128\lambda}\left(\frac{\overline{\gamma}}{6\overline{\alpha}}\right),$$

$$R=\frac{1}{496\lambda}\left(\frac{\bar{\gamma}^2-6\bar{\alpha}\bar{\gamma}-1116\bar{\alpha}^2}{36\bar{\alpha}^2}\right).$$

4) Fourth set

$$a_0=\frac{-11}{14384}\left(\frac{\bar{\beta}^2-1620\bar{\beta}\bar{\alpha}-115632\bar{\alpha}^2}{36\bar{\alpha}^2}\right),\ a_2=2\lambda,\ b_2=\frac{-\bar{\beta}}{1536\lambda\bar{\alpha}},$$

$$R=\frac{1}{1984\lambda}\left(\frac{\bar{\beta}^2+12\bar{\beta}\bar{\alpha}-4464\bar{\alpha}^2}{36\bar{\alpha}^2}\right).$$

5) Fifth set

$$a_0=\frac{-\beta}{24\alpha},\ a_2=2\lambda,\ b_2=0,\ R=\frac{-1}{8\lambda}\left(\frac{\beta^2+30\beta\alpha-144\alpha^2}{36\alpha^2}\right).$$

6) Sixth set

$$a_0=\frac{-\beta}{24\alpha},\ a_2=0,\ b_2=\frac{1}{32\lambda}\left(\frac{9\beta^2+96\beta\alpha-1720\alpha^2}{36\alpha^2}\right),$$

$$R=\frac{b_2}{62}\left(\frac{11\beta^2+156\beta\alpha-2016\alpha^2}{36\alpha^2}\right),$$

where,

$$\alpha=\left(188+12\sqrt{93}\right)^{\frac{1}{3}},$$

$$\gamma=\alpha^2+2\alpha+28,$$

$$\bar{\alpha}=\left(106028+10788\sqrt{93}\right)^{\frac{1}{3}},$$

$$\bar{\gamma}=\bar{\alpha}^2+2\bar{\alpha}+748,$$

$$\beta=(\gamma-6\alpha)\pm i\sqrt{3}\left(\alpha^2-28\right),$$

$$\bar{\beta}=(\bar{\gamma}-6\bar{\alpha})\pm i\sqrt{3}\left(\bar{\alpha}^2-748\right).$$

Note that, using the numerical value of α, we deduce $\gamma^2-15\alpha\gamma-36\alpha^2=\alpha^4-11\alpha^3-6\alpha^2-308\alpha+784<0$.

Recall that $u(x,t)=\ln\left(v(x,t)\right)$, hence we obtain:

According to the first set, for $R<0$, solutions for Equation (6) read

$$u_1(x,t)=\ln\left\{\frac{\gamma}{12\alpha}+\frac{\gamma^2-15\alpha\gamma-36\alpha^2}{36\alpha^2}\tanh^2\left[\frac{1}{12\alpha}\sqrt{\frac{2\left(\gamma^2-15\alpha\gamma-36\alpha^2\right)}{\lambda}}(x-\lambda t)\right]\right\},$$

and

$$u_2(x,t)=\ln\left\{\frac{\gamma}{12\alpha}+\frac{\gamma^2-15\alpha\gamma-36\alpha^2}{36\alpha^2}\coth^2\left[\frac{1}{12\alpha}\sqrt{\frac{2\left(\gamma^2-15\alpha\gamma-36\alpha^2\right)}{\lambda}}(x-\lambda t)\right]\right\}.$$

However, for $R>0$, the solutions are

$$u_3(x,t)=\ln\left\{\frac{\gamma}{12\alpha}-\frac{\gamma^2-15\alpha\gamma-36\alpha^2}{36\alpha^2}\tan^2\left[\frac{1}{12\alpha}\sqrt{-\frac{2\left(\alpha^4-11\alpha^3-6\alpha^2-308\alpha+784\right)}{\lambda}}(x-\lambda t)\right]\right\},$$

and

$$u_4(x,t)=\ln\left\{\frac{\gamma}{12\alpha}+\frac{\gamma^2-15\alpha\gamma-36\alpha^2}{36\alpha^2}\cot^2\left[\frac{1}{12\alpha}\sqrt{-\frac{2\left(\alpha^4-11\alpha^3-6\alpha^2-308\alpha+784\right)}{\lambda}}(x-\lambda t)\right]\right\}.$$

According to second set, notice that $11\gamma^2-78\alpha\gamma-504\alpha^2=11\alpha^4-34\alpha^3-952\alpha+8624>0$ and $3\gamma^2-16\alpha\gamma-144\alpha^2>0$, using the numerical value of α, and for $R<0$, we obtain the solutions:

$$u_5(x,t)=\ln\left\{\frac{\gamma}{12\alpha}-\frac{558\alpha^2}{11\gamma^2-78\alpha\gamma-504\alpha^2}\tanh^2\left[\frac{1}{6\alpha}\sqrt{-\frac{2\left(11\gamma^2-78\alpha\gamma-504\alpha^2\right)b_2}{31}}(x-\lambda t)\right]\right\},$$

$$u_6(x,t)=\ln\left\{\frac{\gamma}{12\alpha}-\frac{558\alpha^2}{11\gamma^2-78\alpha\gamma-504\alpha^2}\coth^2\left[\frac{1}{6\alpha}\sqrt{-\frac{2\left(11\gamma^2-78\alpha\gamma-504\alpha^2\right)b_2}{31}}(x-\lambda t)\right]\right\}.$$

However, for $R>0$, we obtain the travelling wave solutions:

$$u_7(x,t)=\ln\left\{\frac{\gamma}{12\alpha}+\frac{558\alpha^2}{11\gamma^2-78\alpha\gamma-504\alpha^2}\tan^2\left[\frac{1}{6\alpha}\sqrt{\frac{2\left(11\gamma^2-78\alpha\gamma-504\alpha^2\right)b_2}{31}}(x-\lambda t)\right]\right\},$$

and

$$u_8(x,t)=\ln\left\{\frac{\gamma}{12\alpha}+\frac{558\alpha^2}{11\gamma^2-78\alpha\gamma-504\alpha^2}\cot^2\left[\frac{1}{6\alpha}\sqrt{\frac{2\left(11\gamma^2-78\alpha\gamma-504\alpha^2\right)b_2}{31}}(x-\lambda t)\right]\right\},$$

where b_2 is given in 2).

According to The third set, notice that $\bar{\gamma}^2-6\overline{\alpha\gamma}-1116\bar{\alpha}^2>0$, using the numerical value of α, we obtain the soliton solutions, for $R<0$,

$$u_9(x,t)=\ln\left\{\frac{-11}{3596}\left(\frac{\bar{\gamma}^2+810\overline{\alpha\gamma}+25668\bar{\alpha}^2}{36\bar{\alpha}^2}\right)-\frac{1}{248}\left(\frac{\bar{\gamma}^2-6\overline{\alpha\gamma}-1116\bar{\alpha}^2}{36\bar{\alpha}^2}\right)\right.$$
$$\left.\times\tanh^2\left[\sqrt{-R}(x-\lambda t)\right]-\frac{93\overline{\alpha\gamma}}{4\left(\bar{\gamma}^2-6\overline{\alpha\gamma}-1116\bar{\alpha}^2\right)}\coth^2\left[\sqrt{-R}(x-\lambda t)\right]\right\}.$$

However, for $R>0$, we obtain the travelling wave solutions:

$$u_{10}(x,t)=\ln\left\{\frac{-11}{3596}\left(\frac{\bar{\gamma}^2+810\overline{\alpha\gamma}+25668\bar{\alpha}^2}{36\bar{\alpha}^2}\right)+\frac{1}{248}\left(\frac{\bar{\gamma}^2-6\overline{\alpha\gamma}-1116\bar{\alpha}^2}{36\bar{\alpha}^2}\right)\right.$$
$$\left.\times\tan^2\left[\sqrt{R}(x-\lambda t)\right]+\frac{93\overline{\alpha\gamma}}{4\left(\bar{\gamma}^2-6\overline{\alpha\gamma}-1116\bar{\alpha}^2\right)}\cot^2\left[\sqrt{R}(x-\lambda t)\right]\right\},$$

where, R is given in 3).

Note that, u_1, u_2, u_3 and u_4 are also obtained by Wazwaz using the tanh-function method in [2]. Other solutions are not reported in [2].

Sets of solutions in 4)-6) will give complex solutions.

2.2.2. The Liouville Equation

As stated before, if $q=r=0, p=1$ in the Zhiber-Shabat Equation (1), we obtain the Liouville equation:

$$u_{xt}+e^u=0.$$

Using the wave variable $\xi = x - \lambda t$, we get

$$-cu_{\xi\xi} + e^{u} = 0. \tag{12}$$

We again use the Painleve property:

$$u = \ln v,$$

to transform (12) into the ODE

$$-c\left(vv'' - (v')^2\right) + v^3 = 0. \tag{13}$$

Considering the homogeneous balance between vv'' and v^3 in Equation (13), gives $m = 2$, and using the modified tanh-coth function method, we suppose that the solution of Equation (13) is in the form:

$$v(x,t) = v(\xi) = a_0 + a_1 w + \frac{b_1}{w} + a_2 w^2 + \frac{b_2}{w^2}.$$

Proceeding as before we found:

1) First set

$$a_0 = 2R\lambda,\ a_1 = b_1 = b_2 = 0,\ a_2 = 2\lambda,\ R = R,$$

2) Second set

$$a_0 = a_1 = b_1 = b_2 = 0,\ a_2 = 2\lambda,\ R = 0,$$

3) Third set

$$a_0 = 2R\lambda,\ a_1 = b_1 = a_2 = 0,\ b_2 = 2\lambda R^2,\ R = R,$$

4) Fourth set

$$a_0 = 4R\lambda,\ a_1 = b_1 = 0,\ a_2 = 2\lambda,\ b_2 = 2\lambda R^2,\ R = R,$$

where R is free parameter. Recall that $u(x,t) = \ln v(x,t)$.

According to the first set we obtain the solutions:

$$u_1(x,t) = \ln\left\{2\lambda R\left(1 - \tanh^2\left[\sqrt{-R}(x - \lambda t)\right]\right)\right\},\ R < 0,$$

$$u_2(x,t) = \ln\left\{2\lambda R\left(1 - \coth^2\left[\sqrt{-R}(x - \lambda t)\right]\right)\right\},\ R < 0,$$

$$u_3(x,t) = \ln\left\{2\lambda R\left(1 + \tan^2\left[\sqrt{R}(x - \lambda t)\right]\right)\right\},\ R > 0,$$

$$u_4(x,t) = \ln\left\{2\lambda R\left(1 + \cot^2\left[\sqrt{R}(x - \lambda t)\right]\right)\right\},\ R > 0,$$

$$u_5(x,t) = \ln\left\{\frac{2\lambda}{(x - \lambda t)^2}\right\},\ R = 0.$$

According to the third set we obtain the similar to the solutions of the first set.

According to the fourth set we obtain the solutions, $R < 0$,

$$u_6(x,t) = \ln\left\{2\lambda R\left(1 - \tanh^2\left[\sqrt{-R}(x - \lambda t)\right] - \coth^2\left[\sqrt{-R}(x - \lambda t)\right]\right)\right\},$$

and if $R > 0$, then

$$u_7(x,t) = \ln\left\{2\lambda R\left(1 + \tan^2\left[\sqrt{R}(x - \lambda t)\right] + \cot^2\left[\sqrt{R}(x - \lambda t)\right]\right)\right\}.$$

Note that, u_1 and u_2 are also obtained by Wazwaz using the tanh-function method in [2]. Other solutions are not reported in [2].

2.2.3. The sinh-Gordon Equation

As stated before, if $r = 0$, $q = 1$, $p = 1$ in the Zhiber-Shabat Equation (1), we obtain the sinh-Gorden equation:

$$u_{xt} + \mathrm{e}^{u} + \mathrm{e}^{-u} = 0.$$

Using the wave variable $\xi = x - \lambda t$, we get

$$-cu_{\xi\xi} + \mathrm{e}^{u} - \mathrm{e}^{-u} = 0, \tag{14}$$

Using the Painleve property, Equation (14) is transformed into the ODE

$$-c\left(vv'' - (v')^{2}\right) + v^{3} - v = 0. \tag{15}$$

The balancing process gives $m = 2$. We can suppose that the solution of Equation (15) is the form:

$$v(x,t) = v(\xi) = a_0 + a_2 w^2 + \frac{b_2}{w^2}. \tag{16}$$

Following the same analysis presented above, we obtain:

1) First set

$$a_0 = 0,\ a_2 = 2\lambda = \frac{\pm 1}{R},\ b_2 = 0,\ R = \pm\frac{1}{2\lambda}.$$

2) Second set

$$a_0 = 0,\ a_2 = 0,\ b_2 = \frac{1}{2\lambda} = \pm R,\ R = \pm\frac{1}{2\lambda}.$$

3) Third set

$$a_0 = \frac{\pm 1}{2},\ a_2 = 2\lambda = \frac{\pm 1}{4R},\ b_2 = \frac{1}{32\lambda},\ R = \pm\frac{1}{8\lambda}.$$

According to the first set, and for $R < 0$, we obtain

$$u_1(x,t) = \ln\left\{\tanh^2\left[\sqrt{\frac{1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda > 0,$$

$$u_2(x,t) = \ln\left\{-\tanh^2\left[\sqrt{\frac{-1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda < 0,$$

$$u_3(x,t) = \ln\left\{\coth^2\left[\sqrt{\frac{1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda > 0,$$

$$u_4(x,t) = \ln\left\{-\coth^2\left[\sqrt{\frac{-1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda < 0,$$

for $R > 0$, we obtain

$$u_5(x,t) = \ln\left\{\tan^2\left[\sqrt{\frac{1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda > 0,$$

$$u_6(x,t) = \ln\left\{-\tan^2\left[\sqrt{\frac{-1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda < 0,$$

$$u_7(x,t) = \ln\left\{\cot^2\left[\sqrt{\frac{1}{2\lambda}}(x - \lambda t)\right]\right\},\ \lambda > 0,$$

$$u_8(x,t)=\ln\left\{-\cot^2\left[\sqrt{\frac{-1}{2\lambda}}(x-\lambda t)\right]\right\},\ \lambda<0.$$

According to the second set, we obtain similar solutions to the solutions of the first set.

According to the third set, we obtain, for $R<0$, $\lambda>0$,

$$u_{9,10}(x,t)=\ln\left\{\pm\frac{1}{2}\left(1+\frac{1}{2}\tanh^2\left[\frac{1}{2}\sqrt{\frac{1}{2\lambda}}(x-\lambda t)\right]+\frac{1}{2}\coth^2\left[\frac{1}{2}\sqrt{\frac{1}{2\lambda}}(x-\lambda t)\right]\right)\right\},$$

for $R<0$,

$$u_{11}(x,t)=\ln\left\{-\frac{1}{2}\left(1+\frac{1}{2}\tanh^2\left[\frac{1}{2}\sqrt{\frac{-1}{2\lambda}}(x-\lambda t)\right]+\frac{1}{2}\coth^2\left[\frac{1}{2}\sqrt{\frac{-1}{2\lambda}}(x-\lambda t)\right]\right)\right\},$$

for $R>0$,

$$u_{12}(x,t)=\ln\left(-\frac{1}{2}\left(1-\frac{1}{2}\tan^2\left[\frac{1}{2}\sqrt{\frac{1}{2\lambda}}(x-\lambda t)\right]-\frac{1}{2}\cot^2\left[\frac{1}{2}\sqrt{\frac{1}{2\lambda}}(x-\lambda t)\right]\right)\right),$$

and for $R>0$,

$$u_{13}(x,t)=\ln\left(\frac{1}{2}\left(1-\frac{1}{2}\tan^2\left[\frac{1}{2}\sqrt{\frac{-1}{2\lambda}}(x-\lambda t)\right]-\frac{1}{2}\cot^2\left[\frac{1}{2}\sqrt{\frac{-1}{2\lambda}}(x-\lambda t)\right]\right)\right).$$

Note that, u_1 and u_3 are also obtained by Wazwaz using the tanh-function method in [2]. Other solutions are not reported in [2].

2.2.4. The Dodd-Bullough-Mikhailov Equation

If $p=1$, $q=0$, $r=1$ in the Zhiber-Shabat Equation (1), we obtain the Dodd-Bullough-Mikhailov equation:

$$u_{xt}+\mathrm{e}^{u}+\mathrm{e}^{-2u}=0,$$

and by using the wave variable $\xi=x-\lambda t$, we get

$$-cu_{\xi\xi}+\mathrm{e}^{u}+\mathrm{e}^{-2u}=0. \tag{17}$$

We use the Painleve property:

$$u=\ln v, \tag{18}$$

to transform (17) into the ODE

$$-c\left(vv''-(v')^2\right)+v^3+1=0. \tag{19}$$

Considering the homogeneous balance between vv'' and v^3 in Equation (18), gives $m=2$, we can suppose that the solution of Equation (15) is the form

$$v(x,t)=v(\xi)=a_0+a_1w+\frac{b_1}{w}+a_2w^2+\frac{b_2}{w^2}. \tag{20}$$

Proceeding as before, we get

1) First set

$$a_0=\frac{1}{2},\ a_1=b_1=b_2=0,\ a_2=2\lambda,\ R=\frac{3}{4\lambda}.$$

2) Second set

$$a_0=\frac{1}{2}\left(-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}\right),\ a_1=b_1=b_2=0,\ a_2=2\lambda,\ R=\frac{3}{4}\frac{-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}}{\lambda}.$$

3) Third set

$$a_0 = \frac{1}{2},\ a_1 = b_1 = a_2 = 0,\ b_2 = \frac{9}{8\lambda},\ R = \frac{3}{4\lambda}.$$

4) Fourth set

$$a_0 = \frac{1}{2}\left(-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right),\ a_1 = b_1 = a_2 = 0,\ b_2 = -\frac{9}{8}\frac{\frac{1}{2} \mp \frac{i}{2}\sqrt{3}}{\lambda},\ R = \frac{3}{4}\frac{-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}}{\lambda}.$$

5) Fifth set

$$a_0 = -\frac{1}{4},\ a_1 = b_1 = 0,\ a_2 = 2\lambda,\ b_2 = \frac{9}{128\lambda},\ R = \frac{3}{16\lambda}.$$

6) Sixth set

$$a_0 = \frac{1}{4}\left(\frac{1}{2} \pm \frac{i}{2}\sqrt{3}\right),\ a_1 = b_1 = 0,\ a_2 = 2\lambda,\ b_2 = \frac{-9}{128}\frac{\frac{1}{2} \mp \frac{i}{2}\sqrt{3}}{\lambda},\ R = \frac{3}{16}\frac{-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}}{\lambda}.$$

According to the first set,we obtain the soliton solutions:

$$u_1(x,t) = \ln\left\{\frac{1}{2}\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{\frac{-3}{\lambda}}(x - \lambda t)\right]\right)\right\},\ \lambda < 0,$$

and

$$u_2(x,t) = \ln\left\{\frac{1}{2}\left(1 - 3\coth^2\left[\frac{1}{2}\sqrt{\frac{-3}{\lambda}}(x - \lambda t)\right]\right)\right\},\ \lambda < 0,$$

for $\lambda > 0$, we obtain the travelling wave solutions:

$$u_3(x,t) = \ln\left\{\frac{1}{2}\left(1 + 3\tan^2\left[\frac{1}{2}\sqrt{\frac{3}{\lambda}}(x - \lambda t)\right]\right)\right\},$$

and

$$u_4(x,t) = \ln\left\{\frac{1}{2}\left(1 + 3\cot^2\left[\frac{1}{2}\sqrt{\frac{3}{\lambda}}(x - \lambda t)\right]\right)\right\}.$$

According to the second set, we obtain the solutions

$$u_5(x,t) = \ln\left\{\frac{1}{2}\left(-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right)\left(1 - 3\tanh^2\left[\frac{1}{2}\sqrt{\frac{-3\left(-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right)}{\lambda}}(x - \lambda t)\right]\right)\right\},\ \lambda < 0,$$

and

$$u_6(x,t) = \ln\left\{\frac{1}{2}\left(-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right)\left(1 - 3\coth^2\left[\frac{1}{2}\sqrt{\frac{-3\left(-\frac{1}{2} \mp \frac{i}{2}\sqrt{3}\right)}{\lambda}}(x - \lambda t)\right]\right)\right\},\ \lambda < 0,$$

for $\lambda > 0$, we obtain the travelling wave solutions:

$$u_7(x,t)=\ln\left\{\frac{1}{2}\left(-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}\right)\left(1+3\tan^2\left[\frac{1}{2}\sqrt{\frac{3\left(-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]\right)\right\},$$

and

$$u_8(x,t)=\ln\left\{\frac{1}{2}\left(-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}\right)\left(1+3\cot^2\left[\frac{1}{2}\sqrt{\frac{3\left(-\frac{1}{2}\mp\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]\right)\right\}.$$

According to the third set and fourth set, we obtain similar solutions to the solutions of the first set and second set, respectively.

According to fifth set, we obtain, for $\lambda<0$, the following solutions

$$u_9=\ln\left\{\frac{-1}{4}\left(1+\frac{3}{2}\tanh^2\left[\frac{1}{4}\sqrt{\frac{-3}{\lambda}}(x-\lambda t)\right]+\frac{3}{2}\coth^2\left[\frac{1}{4}\sqrt{\frac{-3}{\lambda}}(x-\lambda t)\right]\right)\right\},$$

for $\lambda>0$, we obtain the travelling wave solutions

$$u_{10}=\ln\left\{\frac{-1}{4}\left(1-\frac{3}{2}\tan^2\left[\frac{1}{4}\sqrt{\frac{3}{\lambda}}(x-\lambda t)\right]-\frac{3}{2}\cot^2\left[\frac{1}{4}\sqrt{\frac{3}{\lambda}}(x-\lambda t)\right]\right)\right\}.$$

According to sixth set, for $\lambda>0$, this in turn gives the solitons solutions:

$$u_{11,12}=\ln\left\{\frac{1}{4}\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)\left(1+\frac{3}{2}\tanh^2\left[\frac{1}{4}\sqrt{\frac{3\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]+\frac{3}{2}\coth^2\left[\frac{1}{4}\sqrt{\frac{3\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]\right)\right\}$$

for $\lambda<0$, we obtain the travelling wave solutions:

$$u_{13,14}=\ln\left\{\frac{1}{4}\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)\left(1-\frac{3}{2}\tan^2\left[\frac{1}{4}\sqrt{\frac{-3\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]-\frac{3}{2}\cot^2\left[\frac{1}{4}\sqrt{\frac{-3\left(\frac{1}{2}\pm\frac{i}{2}\sqrt{3}\right)}{\lambda}}(x-\lambda t)\right]\right)\right\}$$

The solutions u_1, u_3, u_3, u_4, u_9, and u_{10} are also obtained by Wazwaz using the tanh-function method in [3]. Other solutions are not reported in [3].

2.2.5. The Tzitzeica-Dodd-Bullough Equation

This equation can be obtained if we set $p=0$, $q=-1$, $r=1$ in the Zhiber-Shabat Equation (1), and by using the wave variable $\xi=x-\lambda t$, we find the Tzitzeica-Dodd-Bullough equation:

$$-cu_{\xi\xi}-e^{-u}+e^{-2u}=0, \tag{21}$$

suppose that

$$v=e^{-u}, \tag{22}$$

or equivalently

$$u=\ln\left(\frac{1}{v}\right). \tag{23}$$

By using (23) we can transform Equation (21) to

$$c\left(vv''-(v')^2\right)-v^3+v^4=0. \tag{24}$$

Considering the homogeneous balance between vv'' and v^4 in Equation (24), gives m = 1, and by using the modified tanh-coth function method we can suppose that the solution of Equation (24) is the form:

$$v(x,t)=v(\xi)=a_0+a_1w+\frac{b_1}{w}. \tag{25}$$

Proceeding as before, we get the following set of solutions.

1) First set

$$a_0=\frac{1}{2},\ a_1=\pm\sqrt{\frac{-1}{4R}},\ b_1=0,\ \lambda=\frac{1}{4R}$$

2) Second set

$$a_0=\frac{1}{2},\ a_1=0,\ b_1=\pm\sqrt{\frac{-1}{4R}},\ \lambda=\frac{1}{4R}$$

3) Third set

$$a_0=\frac{1}{2},\ a_1=\sqrt{\frac{-1}{16R}},\ b_1=\pm\sqrt{\frac{-R}{16}},\ \lambda=\frac{1}{16R}$$

According to the first set, we obtain the solitons solutions

$$u_{1,2}(x,t)=\ln\left(\frac{2}{1\pm\tanh\left[\sqrt{-R}\left(x-\frac{1}{4R}t\right)\right]}\right),\ R<0,$$

and

$$u_{3,4}(x,t)=\ln\left(\frac{2}{1\pm\coth\left[\sqrt{-R}\left(x-\frac{1}{4R}t\right)\right]}\right),\ R<0,$$

for $R > 0$, we obtain the solutions:

$$u_{5,6}(x,t)=\ln\left(\frac{2}{1\pm i\tan\left[\sqrt{R}\left(x-\frac{1}{4R}t\right)\right]}\right),$$

and

$$u_{7,8}(x,t)=\ln\left(\frac{2}{1\pm i\cot\left[\sqrt{R}\left(x-\frac{1}{4R}t\right)\right]}\right).$$

According to the second set, we obtain the solutions

$$u_{9,10}(x,t)=\ln\left(\frac{2}{1\pm\frac{1}{2}\left(\tanh\left[\sqrt{-R}\left(x-\frac{1}{16R}t\right)\right]+\coth\left[\sqrt{-R}\left(x-\frac{1}{16R}t\right)\right]\right)}\right),\ R<0.$$

For $R > 0$, we obtain the solutions:

$$u_{11,12}\left(x,t\right)=\ln\left(\frac{2}{1\pm\frac{1}{2}\left(i\tan\left[\sqrt{-R}\left(x-\frac{1}{16R}t\right)\right]-i\cot\left[\sqrt{-R}\left(x-\frac{1}{16R}t\right)\right]\right)}\right).$$

According to the third set, we obtain the solutions

$$u_{13,14}=u_{9,10},u_{15,16}=u_{11,12}.$$

The solutions $u_{1,2},u_{3,4},u_{5,6}$ and $u_{7,8}$ are also obtained by Wazwaz using the tanh-function method in [3]. Other solutions are not reported in [3].

3. Conclusion

The Zhiber-Shabat equation, and the related equations: Liouville equation, sinh-Gordon equation Dodd-Bullough-Mikhailov equation, and the Tzitzeica-Dodd-Bullough equation were investigated using a modified tanh-coth method. New travelling wave solutions were established. The modified tanh-coth function method is a robust computational tool for obtaining exact solutions for the nonlinear Zhiber-Shabat equation, and the related equations. It is also an encouraging method to solve other nonlinear evolution equations.

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On the Oscillation of Second-Order Nonlinear Neutral Delay Dynamic Equations on Time Scales

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Abstract

Based on Riccati transformation and the inequality technique, we establish some new sufficient conditions for oscillation of the second-order neutral delay dynamic equations on time scales. Our results not only extend and improve some known theorems, but also unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation on time scales. At the end of this paper, we give an example to illustrate the main results.

Keywords

Oscillation, Dynamic Equations, Neutral, Time Scale

1. Introduction

The theory of time scales was first proposed by Hilger [1] in order to unify continuous and discrete analysis. Several researchers have made greater contributions to various aspects of this new theory; see [2]-[4]. The new theory of dynamic equations on time scales not only unifies the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to so-called q-difference equations where $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^t : t \in \mathbb{N}_0 \text{ for } q > 1\right\}$.

In recent years, there has been much research involving the oscillation and nonoscillation of solutions of various equations on time scales such as [5]-[18]. In this paper we study and give the sufficient conditions for oscillation of the second-order neutral delay dynamic equation

$$\left(a(t)\left(\left[x(t)+r(t)x(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + q(t)\,f\left(x(\tau(t))\right) = 0,\quad t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

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where $t_0 > 0$ and $\mathbb{T}$ is unbounded time scale. Besides that, we will have hypotheses as follows throughout the paper:

(H_1) γ is the ratio of two positive odd integers and $\gamma \geq 1$.

(H_2) $a, r, q : \mathbb{T} \to \mathbb{R}$ are positive rd-continuous functions with $r(t)$ satisfying $0 \leq r(t) < 1$.

(H_3) $\tau \in C^1_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{T}\right)$ is a strictly increasing and differentiable function such that $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$ and $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$.

(H_4) $f \in C(\mathbb{R}, \mathbb{R})$ is a continuous function which satisfies $\dfrac{f(x)}{x^\gamma} \geq L$ for all $x \neq 0$, where L is a positive constant.

In addition, for the sake of clearness and convenience,we will use the notation

$$z(t) := x(t) + r(t) x(\tau(t))$$

in the following narrative.

It is well known by reserchers in this field that an dynamic equation is called oscillatory in case all its solutions are oscillatory, and a solution of the equation is said to be oscillatory if it is neither eventually positive nor eventually negative. We only discuss those solutions x of Equation (1.1) that are not eventually zero in this paper. Moreover we refer to [3] [4] for general basic background, ideas and more details on dynamic equations.

Because of $a(t) > 0$, we shall consider Equation (1.1) respectively based on the case

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{1.2}$$

and the other case

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \Delta t < \infty. \tag{1.3}$$

2. Several Lemmas

In this section, we present and prove three lemmas which play important roles in the proofs of the main results.

Lemma 1. ([16]) *Assume that* $\tau : \mathbb{T} \to \mathbb{R}$ *is strictly increasing,* $\tilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ *is a time scale and* $\tau(\sigma(t)) = \sigma(\tau(t))$. *Let* $x : \tilde{\mathbb{T}} \to \mathbb{R}$. *If* $\tau^\Delta(t)$ *and* $x^\Delta(\tau(t))$ *exist for* $t \in \mathbb{T}^k$, *then* $\left(x(\tau(t))\right)^\Delta$ *exists, and*

$$\left(x(\tau(t))\right)^\Delta = x^\Delta(\tau(t)) \tau^\Delta(t). \tag{2.1}$$

Lemma 2. ([3]) *Assume that x is delta-differentiable and eventually positive or eventually negative, then*

$$\left(\left(x(t)\right)^\gamma\right)^\Delta = \gamma \int_0^1 \left[h x(\sigma(t)) + (1-h) x(t)\right]^{\gamma-1} x^\Delta(t) \mathrm{d}h. \tag{2.2}$$

We give the below lemma and prove it similar to that of Q. Zhang and X. Song ([17], Lemma 3.5).

Lemma 3. *Based on* (1.2), *assume that* (H_1)-(H_4) *hold. If x is an eventually positive solution of* (1.1), *there exists* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$z^\Delta(t) = \left[x(t) + r(t) x(\tau(t))\right]^\Delta > 0,\ \left(a(t)\left(z^\Delta(t)\right)^\gamma\right)^\Delta < 0,\ t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.3}$$

Proof. Assume $x(t)$ is an eventually positive solution of (1.1). That is, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Because of $z(t) = x(t) + r(t) x(\tau(t))$ and $0 \leq r(t) < 1$, we get $z(t) > 0$ esaily for $t \in [t_1, \infty)_{\mathbb{T}}$. At the same time for $t \in [t_1, \infty)_{\mathbb{T}}$, from equation (1.1) we obtain that

$$\left(a(t)\left(z^\Delta(t)\right)^\gamma\right)^\Delta < 0, \tag{2.4}$$

so $a(t)\left(z^\Delta(t)\right)^\gamma$ is decreasing. From (2.4), we know that $z^\Delta(t)$ is either eventually positive or eventually negative. Now we assert that $z^{\Delta}(t)>0$.

Suppose to the contrary that there exits $t_2\in[t_1,\infty)_{\mathbb{T}}$ such that $z^{\Delta}(t)<0$ for all $t\in[t_2,\infty)_{\mathbb{T}}$. Because $a(t)\left(z^{\Delta}(t)\right)^{\gamma}$ is decreasing,

$$a(t)\left(z^{\Delta}(t)\right)^{\gamma}\le a(t_2)\left(z^{\Delta}(t_2)\right)^{\gamma}=-M^{\gamma} \tag{2.5}$$

for $t\in[t_2,\infty)_{\mathbb{T}}$, where $M=\left[a(t_2)\right]^{\frac{1}{\gamma}}\left|z^{\Delta}(t_2)\right|>0$. Based on the above inequality (2.5), we get

$$z^{\Delta}(t)\le -M\left[\frac{1}{a(t)}\right]^{\frac{1}{\gamma}},\quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{2.6}$$

After integrating the two sides of inequality (2.6) from t_2 to $t\in[t_2,\infty)_{\mathbb{T}}$, we have

$$z(t)\le z(t_2)-M\int_{t_2}^{t}\left[\frac{1}{a(s)}\right]^{\frac{1}{\gamma}}\Delta s,\quad t\in[t_2,\infty)_{\mathbb{T}}. \tag{2.7}$$

When $t\to\infty$, we get $\lim_{t\to\infty} z(t)=-\infty$ from (1.2) and the above (2.7), which is contradictory to $z(t)>0$. So the above hypothesis of $z^{\Delta}(t)<0$ is false. In other words, we get $z^{\Delta}(t)>0$ for $t_1\in[t_0,\infty)_{\mathbb{T}}$. This completes the proof. □

3. Main Results

Now we state and prove our main results in this section.

Theorem 1. *Based on* (1.2), *assume that the conditions* (H_1)-(H_4) *hold. If there exists a positive nondecreasing* Δ*-differentiable function* $\delta\in C_{rd}^{1}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that for every* $T\in[t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty}\int_{T}^{t}\left[L\delta(s)q(s)\left(1-r(\tau(s))\right)^{\gamma}-\eta^{\gamma}(\tau(s))\delta^{\Delta}(s)\right]\Delta s=\infty, \tag{3.1}$$

where

$$\eta(\tau(t))=\left(\int_{T}^{t}\left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}}\tau^{\Delta}(s)\Delta s\right)^{-1}, \tag{3.2}$$

then (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. We may assume that $x(t)>0$ and $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}, t_1\in[t_0,\infty)_{\mathbb{T}}$. By the definition of $z(t)$, it follows $z(t)>0$. From (H_3) we know $\tau(t)\le t$, by Lemma 3 we have $z^{\Delta}(t)>0$, so $z(\tau(t))\le z(t)$, and $0\le r(t)<1$, we obtain

$$x(t)=z(t)-r(t)x(\tau(t))\ge z(t)-r(t)z(\tau(t))\ge(1-r(t))z(t),\quad t\in[t_1,\infty)_{\mathbb{T}}.$$

The proof that x is eventually negative is similar. By Lemma 3 we have $z^{\Delta}(t)>0$ for all $t\in[t_2,\infty)_{\mathbb{T}}$, $t_2\in[t_1,\infty)_{\mathbb{T}}$, and by Lemma 1 and ($H_3$), there exists $T\in[t_2,\infty)_{\mathbb{T}}$ such that $\left(z(\tau(t))\right)^{\Delta}>0$ for all $t\in[T,\infty)_{\mathbb{T}}$. Using (2.2) and (2.3), we have

$$\begin{aligned}\left(\left(z(\tau(t))\right)^{\gamma}\right)^{\Delta}&=\gamma\int_0^1\left[h\left(z(\sigma(\tau(t)))\right)+(1-h)z(\tau(t))\right]^{\gamma-1}\left(z(\tau(t))\right)^{\Delta}\mathrm{d}h\\&\ge\gamma\int_0^1\left[h\left(z(\tau(t))\right)+(1-h)z(\tau(t))\right]^{\gamma-1}\left(z(\tau(t))\right)^{\Delta}\mathrm{d}h\\&=\gamma\left(z(\tau(t))\right)^{\gamma-1}\left(z(\tau(t))\right)^{\Delta}.\end{aligned}$$

$\mathbb{T}$ is unbounded above, which implies $\mathbb{T}^{k}=\mathbb{T}$. Furthermore, from Lemma 1 we get

$$\left(z(\tau(t))\right)^{\Delta}=z^{\Delta}(\tau(t))\tau^{\Delta}(t).$$

Thus, by (H_3),

$$\left(\left(z\left(\tau(t)\right)\right)^{\gamma}\right)^{\Delta} \ge \gamma\left(z\left(\tau(t)\right)\right)^{\gamma-1} z^{\Delta}\left(\tau(t)\right)\tau^{\Delta}(t) > 0. \tag{3.3}$$

Next we define the function $W(t)$ by

$$W(t) = \delta(t)\frac{a(t)\left(z^{\Delta}(t)\right)^{\gamma}}{\left(z\left(\tau(t)\right)\right)^{\gamma}},\ t \in [T,\infty)_{\mathbb{T}}. \tag{3.4}$$

Then on $[T,\infty)_{\mathbb{T}}$, we have $W(t) > 0$. From the basic knowledge of the time scale calculus that you can see in [3], we obtain

$$W^{\Delta}(t) = \frac{\delta(t)}{\left(z\left(\tau(t)\right)\right)^{\gamma}}\left(a(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + a\left(\sigma(t)\right)\left(z^{\Delta}\left(\sigma(t)\right)\right)^{\gamma}\frac{\left(z\left(\tau(t)\right)\right)^{\gamma}\delta^{\Delta}(t) - \delta(t)\left(\left(z\left(\tau(t)\right)\right)^{\gamma}\right)^{\Delta}}{\left(z\left(\tau(t)\right)\right)^{\gamma}\left(z\left(\tau\left(\sigma(t)\right)\right)\right)^{\gamma}}.$$

From (1.1) and (H_4), we get

$$\left(a(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} = \left(a(t)\left(\left[x(t) + r(t)x\left(\tau(t)\right)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} \le -q(t)f\left(x\left(\tau(t)\right)\right),$$

i.e.,

$$\begin{aligned} W^{\Delta}(t) &\le -Lq(t)\delta(t)\left(\frac{x\left(\tau(t)\right)}{z\left(\tau(t)\right)}\right)^{\gamma} + \frac{a\left(\sigma(t)\right)\left(z^{\Delta}\left(\sigma(t)\right)\right)^{\gamma}\delta^{\Delta}(t)}{\left(z\left(\tau\left(\sigma(t)\right)\right)\right)^{\gamma}} \\ &\quad - \frac{\delta(t)a\left(\sigma(t)\right)\left(z^{\Delta}\left(\sigma(t)\right)\right)^{\gamma}\left(\left(z\left(\tau(t)\right)\right)^{\gamma}\right)^{\Delta}}{\left(z\left(\tau(t)\right)\right)^{\gamma}\left(z\left(\tau\left(\sigma(t)\right)\right)\right)^{\gamma}} \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\le -Lq(t)\delta(t)\left(1 - r\left(\tau(t)\right)\right)^{\gamma} + \frac{a\left(\sigma(t)\right)\left(z^{\Delta}\left(\sigma(t)\right)\right)^{\gamma}\delta^{\Delta}(t)}{\left(z\left(\tau\left(\sigma(t)\right)\right)\right)^{\gamma}} \\ &\le -Lq(t)\delta(t)\left(1 - r\left(\tau(t)\right)\right)^{\gamma} + \frac{a\left(\tau(t)\right)\left(z^{\Delta}\left(\tau(t)\right)\right)^{\gamma}\delta^{\Delta}(t)}{\left(z\left(\tau(t)\right)\right)^{\gamma}} \\ &= -Lq(t)\delta(t)\left(1 - r\left(\tau(t)\right)\right)^{\gamma} + a\left(\tau(t)\right)\delta^{\Delta}(t)\left(\frac{z^{\Delta}\left(\tau(t)\right)}{z\left(\tau(t)\right)}\right)^{\gamma}. \end{aligned} \tag{3.6}$$

On the other hand, because

$$\begin{aligned} z\left(\tau(t)\right) &= z\left(\tau(T)\right) + \int_T^t z^{\Delta}\left(\tau(s)\right)\tau^{\Delta}(s)\Delta s \\ &= z\left(\tau(T)\right) + \int_T^t \left(\frac{1}{a\left(\tau(s)\right)}\right)^{\frac{1}{\gamma}}\left(a\left(\tau(s)\right)\right)^{\frac{1}{\gamma}} z^{\Delta}\left(\tau(s)\right)\tau^{\Delta}(s)\Delta s \\ &\ge \left(\int_T^t \left(\frac{1}{a\left(\tau(s)\right)}\right)^{\frac{1}{\gamma}}\tau^{\Delta}(s)\Delta s\right)\left(a\left(\tau(t)\right)\left(z^{\Delta}\left(\tau(t)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}, \end{aligned}$$

we get

$$\left(\frac{z^{\Delta}(\tau(t))}{z(\tau(t))}\right)^{\gamma} \le \frac{1}{a(\tau(t))}\left(\int_T^t\left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}}\tau^{\Delta}(s)\Delta s\right)^{-\gamma} = \frac{\eta^{\gamma}(\tau(t))}{a(\tau(t))},\ t\in[T,\infty)_{\mathbb{T}}. \tag{3.7}$$

Using (3.7) in (3.6), we have

$$W^{\Delta}(t) \le -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \eta^{\gamma}(\tau(t))\delta^{\Delta}(t),\ [T,\infty)_{\mathbb{T}}. \tag{3.8}$$

At last, integrating (3.8) from T to t, we obtain

$$0 < W(t) \le W(T) - \int_T^t\left[L\delta(s)q(s)\left(1-r(\tau(s))\right)^{\gamma} - \eta^{\gamma}(\tau(s))\delta^{\Delta}(s)\right]\Delta s,$$

which creates a contradiction to (3.1). This completes the proof. □

Remark 1. *From Theorem* 1, *we can obtain different conditions for oscillation of all solutions of* (1.1) *with different choices of* $\delta(t)$.

Next, we give the conditions that guarantee every solution of (1.1) oscillates when (1.3) holds.

Theorem 2. *Based on* (1.3), *assume that the conditions* (H_1)-(H_4), (3.1) *and* (3.2) *hold*. *If for every* $T\in[t_0,\infty)_{\mathbb{T}}$

$$\int_T^{\infty}\left[\frac{1}{a(s)}\int_T^s\theta^{\gamma}(u)\left(1-r(\tau(u))\right)^{\gamma}q(u)\Delta u\right]^{\frac{1}{\gamma}}\Delta s = \infty \tag{3.9}$$

where

$$\theta(t) = \int_t^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\Delta s, \tag{3.10}$$

then (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$, then it is neither eventually positive nor eventually negative. Without loss of generality, we may assume that $x(t)>0$, then $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$, $t_1\in[t_0,\infty)_{\mathbb{T}}$, it follows $z(t)=x(t)+r(t)x(\tau(t))>0$ and

$$x(t) = z(t)-r(t)x(\tau(t)) \ge z(t)-r(t)z(\tau(t)) \ge (1-r(t))z(t),\ t\in[t_1,\infty)_{\mathbb{T}}.$$

The proof is similar when x is eventually negative. Since $a(t)\left(z^{\Delta}(t)\right)^{\gamma}$ is decreasing for all $t\in[T,\infty)_{\mathbb{T}}$ and $T\in[t_1,\infty)_{\mathbb{T}}$, it is eventually of one sign and hence $z^{\Delta}(t)$ is eventually of one sign. So we shall distinguish the following two cases to discuss:

(I) $z^{\Delta}(t)>0$ for $t\ge T$; and

(II) $z^{\Delta}(t)<0$ for $t\ge T$.

Case (*I*). The proof that $z^{\Delta}(t)$ is eventually positive is similar to that in Theorem 1, so it is omitted here.

Case (*II*). For $s\ge t\ge T$, we have

$$a(s)\left(-z^{\Delta}(s)\right)^{\gamma} \ge a(t)\left(-z^{\Delta}(t)\right)^{\gamma},$$

then

$$-z^{\Delta}(s) \ge \left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\gamma}}\left(-z^{\Delta}(t)\right). \tag{3.11}$$

Integrating (3.11) from t ($t\ge T$) to u ($u\ge t$) and letting $u\to\infty$, we have

$$z(t) \ge \left[\int_t^{\infty}\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\Delta s\right]\left(a(t)\right)^{\frac{1}{\gamma}}\left(-z^{\Delta}(t)\right) = -\theta(t)a^{\frac{1}{\gamma}}(t)z^{\Delta}(t),\quad t\in[T,\infty)_{\mathbb{T}},$$

and thus

$$\left(z(t)\right)^{\gamma} \geq -\left(\theta(t)\right)^{\gamma} a(t)\left(z^{\Delta}(t)\right)^{\gamma} \geq -\left(\theta(t)\right)^{\gamma} a(T)\left(z^{\Delta}(T)\right)^{\gamma} = b\left(\theta(t)\right)^{\gamma}, \quad t \in [T,\infty)_{\mathbb{T}}, \tag{3.12}$$

where $b = -a(T)\left(z^{\Delta}(T)\right)^{\gamma} > 0$. Applying (3.12) to Equation (1.1), we find

$$\begin{aligned} -\left(a(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} &\geq Lq(t)\left(x(\tau(t))\right)^{\gamma} \geq Lq(t)\left(1-r(\tau(t))\right)^{\gamma}\left(z(\tau(t))\right)^{\gamma} \\ &\geq Lq(t)\left(1-r(\tau(t))\right)^{\gamma}\left(z(t)\right)^{\gamma} \geq bL\theta^{\gamma}(t)q(t)\left(1-r(\tau(t))\right)^{\gamma}, \quad t \in [T,\infty)_{\mathbb{T}}. \end{aligned} \tag{3.13}$$

Integrating (3.13) from T to t, we have

$$\begin{aligned} -a(t)\left(z^{\Delta}(t)\right)^{\gamma} &\geq -a(T)\left(z^{\Delta}(T)\right)^{\gamma} + bL\int_T^t \theta^{\gamma}(s)q(s)\left(1-r(\tau(s))\right)^{\gamma}\Delta s \\ &\geq bL\int_T^t \theta^{\gamma}(s)q(s)\left(1-r(\tau(s))\right)^{\gamma}\Delta s. \end{aligned}$$

Therefore,

$$-z^{\Delta}(t) \geq \left[\frac{bL}{a(t)}\int_T^t \theta^{\gamma}(s)\left(1-r(\tau(s))\right)^{\gamma} q(s)\Delta s\right]^{\frac{1}{\gamma}}. \tag{3.14}$$

Next integrating (3.14) from T to t, we obtain

$$-z(t)+z(T) \geq \int_T^t \left[\frac{bL}{a(s)}\int_T^s \theta^{\gamma}(u)\left(1-r(\tau(u))\right)^{\gamma} q(u)\Delta u\right]^{\frac{1}{\gamma}} \Delta s \to \infty, \quad t \to \infty.$$

By (3.9), we have $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. This completes the proof. □

Remark 2. *By Theorem* 2, *we get the sufficient condition of oscillation for Equation* (1.1) *when the condition* (1.3) *is satisfied*, *while the usual result existing is that the conditions* (1.3) *was established*, *then every solution of the Equation* (1.1) *is either oscillatory or converges to zero on* $[t_0,\infty)_{\mathbb{T}}$.

Theorem 3. *Based on* (1.2), *assume* (H_1)-(H_4) *hold and* $\tau(\sigma(t)) = \sigma(\tau(t))$. *If there exists a positive* Δ-*differentiable function* $\delta \in C^1_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}\right)$ *such that for every* $T \in [t_0,\infty)_{\mathbb{T}}$

$$\limsup_{t\to\infty}\int_T^t \left[Lq(s)\delta(s)\left(1-r(\tau(s))\right)^{\gamma} - \frac{\left(a(\tau(s))\right)^{\frac{1}{\gamma}}\left(\delta^{\Delta}(s)\right)^2}{4\gamma\delta(s)\tau^{\Delta}(s)}\left(\eta^{\sigma}(\tau(s))\right)^{\gamma-1}\right]\Delta s = \infty, \tag{3.15}$$

where η *is as the same as that in* (3.2), *then* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we can assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. By the definition of $z(t)$, it follows $z(t) > 0$. The proof when x is eventually negative is similar. Proceeding as the proof of Theorem 1, we obtained (3.3) and (3.5). Using (3.3) in (3.5), we have that on $[T,\infty)_{\mathbb{T}}$

$$W^{\Delta}(t) \leq -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)a(\sigma(t))\left(z^{\Delta}(\sigma(t))\right)^{\gamma} z^{\Delta}(\tau(t))\tau^{\Delta}(t)}{\left(z(\tau(\sigma(t)))\right)^{\gamma+1}}. \tag{3.16}$$

Also, since $\left(a(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0$, we have

$$a(\tau(t))\left(z^{\Delta}(\tau(t))\right)^{\gamma} \geq a(\sigma(t))\left(z^{\Delta}(\sigma(t))\right)^{\gamma},$$

i.e.,

$$z^{\Delta}(\tau(t)) \geq \frac{\left(a(\sigma(t))\right)^{\frac{1}{\gamma}}}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}} z^{\Delta}(\sigma(t)). \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain on $[T,\infty)_{\mathbb{T}}$

$$W^{\Delta}(t) \le -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta(\sigma(t))\right)^{\frac{\gamma+1}{\gamma}}}\left(W(\sigma(t))\right)^{\frac{\gamma+1}{\gamma}}.$$

i.e.,

$$\begin{aligned} W^{\Delta}(t) \le {} & -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ & - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta(\sigma(t))\right)^{\frac{\gamma+1}{\gamma}}}\left(W(\sigma(t))\right)^{\frac{1-\gamma}{\gamma}}\left(W(\sigma(t))\right)^{2}. \end{aligned} \tag{3.18}$$

Now using inequality (3.7), we get

$$\begin{aligned} z(\tau(t)) & \ge \left(\int_T^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^{\Delta}(s)\Delta s\right)\left(a(\tau(t))\left(z^{\Delta}(\tau(t))\right)^{\gamma}\right)^{\frac{1}{\gamma}}, \\ & \ge \left(\int_T^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^{\Delta}(s)\Delta s\right)\left(a(t)\left(z^{\Delta}(t)\right)^{\gamma}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

Hence, we have

$$\frac{z(\tau(t))}{z^{\Delta}(t)} \ge \frac{a^{\frac{1}{\gamma}}(t)}{\eta(\tau(t))}.$$

This implies that on $[T,\infty)_{\mathbb{T}}$

$$\left(W(t)\right)^{\frac{1-\gamma}{\gamma}} = \left(\delta(t)a(t)\right)^{\frac{1-\gamma}{\gamma}}\left(\frac{z(\tau(t))}{z^{\Delta}(t)}\right)^{\gamma-1} \ge \left(\delta(t)a(t)\right)^{\frac{1-\gamma}{\gamma}}\frac{a^{\frac{\gamma-1}{\gamma}}(t)}{\eta^{\gamma-1}(\tau(t))} = \delta^{\frac{1-\gamma}{\gamma}}(t)\eta^{1-\gamma}(\tau(t)). \tag{3.19}$$

Using (3.19) in (3.18), we have on $[T,\infty)_{\mathbb{T}}$ that

$$\begin{aligned} W^{\Delta}(t) \le {} & -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ & - \frac{\gamma\delta(t)\tau^{\Delta}(t)}{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta(\sigma(t))\right)^{2}}\left(\eta(\sigma(\tau(t)))\right)^{1-\gamma}\left(W(\sigma(t))\right)^{2} \\ = {} & -Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} + \frac{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta^{\Delta}(t)\right)^{2}}{4\gamma\delta(t)\tau^{\Delta}(t)}\left(\eta(\sigma(\tau(t)))\right)^{\gamma-1} \\ & - \left[\frac{\sqrt{\gamma\delta(t)\tau^{\Delta}(t)\left(\eta(\sigma(t))\right)^{1-\gamma}}}{\delta(\sigma(t))\sqrt{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}}}W(\sigma(t)) - \frac{\sqrt{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}}\,\delta^{\Delta}(t)}{2\sqrt{\gamma\delta(t)\tau^{\Delta}(t)\left(\eta(\sigma(\tau(t)))\right)^{1-\gamma}}}\right]^{2} \\ \le {} & -\left[Lq(t)\delta(t)\left(1-r(\tau(t))\right)^{\gamma} - \frac{\left(a(\tau(t))\right)^{\frac{1}{\gamma}}\left(\delta^{\Delta}(t)\right)^{2}}{4\gamma\delta(t)\tau^{\Delta}(t)}\left(\eta(\sigma(\tau(t)))\right)^{\gamma-1}\right]. \end{aligned}$$

Integrating both sides of this inequality from T to t, taking the limsup of the resulting inequality as $t\to\infty$ and applying condition (3.15), we obtain a contradiction to the fact that $W(t)>0$ for $t\in[T,\infty)_{\mathbb{T}}$. This completes the proof. □

Using the same ideas as in the proof of Theorem 2, we can now obtain the following result based on (1.3).

Theorem 4. *Under the condition* (1.3), *assume that the conditions* (H_1)-(H_4), (3.9) *and* (3.15) *hold, then* (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$. □

4. Application

Now we shall reformulate the above conditions which are sufficient for the oscillation of (1.1) when (1.2) holds on different time scales:

If $\mathbb{T}=\mathbb{R}$, Equation (1.1) becomes

$$\left(a(t)\left(z'(t)\right)^{\gamma}\right)'+q(t)f\left(x\left(\tau(t)\right)\right)=0,\ t\in[t_0,\infty), \tag{4.1}$$

and then conditions (3.1) and (3.15), respectively, become

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[L\delta(s)q(s)\left(1-r\left(\tau(s)\right)\right)^{\gamma}-\delta'(s)\left(\int_{\tau(t_1)}^{\tau(s)}\left(\frac{1}{a(u)}\right)^{\frac{1}{\gamma}}\mathrm{d}u\right)^{-\gamma}\right]\mathrm{d}s=\infty \tag{4.2}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[Lq(s)\left(1-r\left(\tau(s)\right)\right)^{\gamma}\delta(s)-\frac{\left(a\left(\tau(s)\right)\right)^{\frac{1}{\gamma}}\left(\delta'(s)\right)^{2}}{4\gamma\delta(s)\tau'(s)}\left(\int_{\tau(t_1)}^{\tau(s)}\left(\frac{1}{a(u)}\right)^{\frac{1}{\gamma}}\mathrm{d}u\right)^{1-\gamma}\right]\mathrm{d}s=\infty. \tag{4.3}$$

The conditions (4.2) and (4.3) are new.

If $\mathbb{T}=\mathbb{Z}$, Equation (1.1) becomes

$$\Delta\left(a_n\left(\Delta z_n\right)^{\gamma}\right)+q_n f\left(x_{n-\sigma}\right)=0,\ n=0,1,2,\cdots, \tag{4.4}$$

and conditions (3.1) and (3.15), respectively, become

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[Lq_l\delta_l\left(1-r_{l-\sigma}\right)^{\gamma}-\Delta\delta_l\left(\sum_{k=n_0-\sigma}^{l-\sigma-1}\left(\frac{1}{a_k}\right)^{\frac{1}{\gamma}}\right)^{-\gamma}\right]=\infty \tag{4.5}$$

and

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[Lq_l\delta_l\left(1-r_{l-\sigma}\right)^{\gamma}-\frac{\left(a_{l-\sigma}\right)^{\frac{1}{\gamma}}\left(\Delta\delta_l\right)^{2}}{4\gamma\delta_l}\left(\sum_{k=n_0-\sigma}^{l-\sigma}\left(\frac{1}{a_k}\right)^{\frac{1}{\gamma}}\right)^{1-\gamma}\right]=\infty. \tag{4.6}$$

At same time, the Theorems 1 and 3 are new for the case $\mathbb{T}=\mathbb{Z}$.

Example 1. *Consider the second-order nonlinear delay* 2*-difference equations*

$$\left(t^{\frac{2}{3}}\left(z^{\Delta}(t)\right)^{\frac{5}{3}}\right)^{\Delta}+t^{-\frac{8}{5}}\left(x\left(\frac{t}{2}\right)\right)^{\frac{5}{3}}\left(1+x^{2}\left(\frac{t}{2}\right)\right)=0,\ t\in\overline{2^{\mathbb{Z}}},\ t\ge t_0:=2, \tag{4.7}$$

where $z(t)=x(t)+\frac{1}{2}x\left(\frac{t}{2}\right)$. *This gives*

$$a(t)=t^{\frac{2}{3}},\ q(t)=t^{-\frac{8}{5}},\ f(x)=x^{\frac{5}{3}}\left(1+x^{2}\right),r(t)=\frac{1}{2},\tau(t)=\frac{t}{2},\ \gamma=\frac{5}{3}.$$

The conditions (H_1)-(H_3) are clearly satisfied, and (H_4) holds with $L = 1$. Because

$$\int_2^t \left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}} \Delta s = \int_2^t s^{-\frac{2}{5}} \Delta s = \frac{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}{2^{\frac{3}{5}} - 1} \to \infty \text{ as } t \to \infty, t \geq 2,$$

(1.2) is satisfied. Now let $\delta(t) = t^{\frac{3}{5}}$ for all $t > s \geq 2$, and then

$$\eta(\tau(t)) = \left(\int_2^t \left(\frac{1}{a(\tau(s))}\right)^{\frac{1}{\gamma}} \tau^{\Delta}(s) \Delta s\right)^{-1} = \left(\frac{1}{2^{\frac{3}{5}}} \int_2^t s^{-\frac{2}{5}} \Delta s\right)^{-1} = 2^{\frac{3}{5}} \frac{2^{\frac{3}{5}} - 1}{t^{\frac{3}{5}} - 2^{\frac{3}{5}}}.$$

Thus when $t \to \infty$, we have

$$\int_2^t \left[L\delta(s) q(s) \left(1 - r(\tau(s))\right)^{\gamma} - \eta^{\gamma}(\tau(s)) \delta^{\Delta}(s)\right] \Delta s = \int_2^t \left[\frac{1}{2^{\frac{5}{3}} s} - 2\left(\frac{2^{\frac{3}{5}} - 1}{s^{\frac{3}{5}} - 2^{\frac{3}{5}}}\right)^{\frac{5}{3}} \left(s^{\frac{3}{5}}\right)^{\Delta}\right] \Delta s \to \infty.$$

It is easy to see that (3.1) is satisfied as well. Altogether, the Equation (4.7) is oscillatory by Theorem 1.

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The Models of Investing Schools

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Abstract

In this paper, we build the Linear Programming (LP) model, factor analysis model and return on investment model to measure the investment amount and which year to invest of each selected schools. We firstly analyze the indicators from attached files, and select effective indexes to choose schools donated. Then we select 17 indexes out after preprocessing all the indices. Secondly, we extract 1064 schools by MATLAB which is the Potential Candidate Schools from the table of attached files; we extract 10 common factors of these schools by factor analysis. After calculation, we rank the universities and select the top 100. We calculate the Return on Investment (ROI) based on these 17 indexes. Thirdly, we figure out the investment amount by conducting LP model through MATLAB. According to the property of schools, we calculate the annual limit investment and the mount of investment of each school. Fourthly, we determine which year to invest by ROI model which is operated by LINGO. In order to achieve optimal investment strategy and not duplication of investment, for five years, starting July 2016, we assume that the time duration that the organization's money should be provided is one year, and the school return to the Good grant Foundation only one year. Then we can get the investment amount per school, the return on that investment, and which years to invest. Fifthly, by changing parameter, the sensitivity analysis is conducted for our models. The result indicates that our models are feasible and robust. Finally, we evaluate our models, and point out the strengths and weakness. Through previous analysis, we can find that our models can be applied to many fields, which have a relatively high generalization.

Keywords

Investment, Factors Model, LP Model, ROI Model

1. Introduction

1.1. Background

Education is the foundation of country building; it has become the key of countries to enhance their comprehen-

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sive strength and competitiveness by vigorously developing higher education. The cost of running a university is very large, and the cost of the former education is higher than the university funds. But recently this proportion has declined year by year. The shortage of funding for higher education has become a bottleneck restricting the universities' further development. The problem of relying solely on government funding has become increasingly serious.

Therefore, if colleges and universities want to achieve better development, on the one hand they actively seek governments' funding. On the other hand they must raise money from society initiatively. What's more, they should use funds scientifically and rationally.

Charity is the nuclear spiritual pillar in one country's cultural environment, while donating became the part of people's daily life. Many rich and powerful people also regard charities as their lifelong responsibilities. Due to the cultural atmosphere of whole people-charitable, social donations become the main driving force of development of education in the United States.

Good Foundation Inc. is a small private Canadian foundation, started in 1974 by the late Milton and Verna Good. It distributes funds to charitable organizations in the communities of which family members are a part [1]. It helps improve educational performance of undergraduates attending colleges and universities in the United States.

To do this, the foundation intends to donate a total of $100,000,000 (US 100 million) to an appropriate group of schools per year, for five years, starting July 2016. In doing so, they do not want to duplicate the investments and focus of other large grant organizations such as the Gates Foundation and Lumina Foundation.

1.2. Our Work

Firstly, we should determine how many schools we would donate. In order to decide, we selected the best indexes. After selection, we may boil down the tasks to the following questions:

1) Select indexes to choose schools donated.

2) Determine an optimal investment strategy that identifies the schools.

3) The investment we donate to each school.

4) The return we will acquire by investing these schools.

2. Assumptions and Justification

1) If Good grant Foundation has selected schools that it will invest, then the time duration that the organization's money should be provided is one year. And the school only return to the Good grant Foundation one year.

2) The funds which School return to the Good grant Foundation, the Good grant Foundation deposit the funds in bank, and the bank's interest rates remain stable.

3) Assume that, in the t year, after we have the year's funds (a) to be invested in schools, there is a surplus of funds (b), then the remaining funds (b) are deposited in the bank.

3. Date Analysis

3.1. The Evaluation Principles of Investment Efficiency

1) Subject to separate principles. That is carry out the subject to evaluation and the main concern. In general, the main investors or operators are the main subject. In this paper, our team want to develop a model to determine an optimal investment strategy that identifies the schools, the investment amount per school. The return on that investment, and the time duration that the organization's money should be provided to have the highest positive effect [2]. So we choose to investor, that is Good grant Foundation as the main study subject of our thesis.

2) Evaluation Content and University functions corresponding principles. Content evaluation should be based on the "production (the output of higher education)" as the core, which is the basis for the establishment of the object being evaluation index system. Products are effective, but also to that produced sell out, that the product must "marketable", the result of evaluation of the content must be structured to withstand market test. Therefore, when establishing the index system and evaluation of the content, it must be recognized by the market as a precondition to our evaluation of school effectiveness.

3) "Representation" principle of indicators. With the evaluation of the content, you can create content and index system to adapt. However, there are many indicators reflect the content, we can not all be used as an evalua-

tion index. Given the relative benefit indicator is based on the principle crawl main contradiction, we only choose the education benefit relatively large index weighting factor—Representative characteristics, also has a dominant feature. So this paper, we select only representative indicators of income.

4) Normalization principle. In selecting the evaluation index system, the type of factors different from each other, can not be quantified comparison. For example, to compare cultivate a talent and made a scientific research, it is difficult to determine the pros and cons, but we may also have to face the quality problem. In this paper, we selected indicators and factor analysis, the selected indicators belonging to quantifiable indicators, and the factors are same types.

3.2. Index Selection

1) Index analysis

a) Unable to get a specific address of schools, the schools does not provide data privacy and the schools has been closed do not participate in the subsequent analyzes.

b) In terms of SAT, we choose one index in its representative, such as Average SAT equivalent score of students admitted, to carry out our further research.

c) In terms of PCIP, the data of Percentage of degrees awarded in all aspects is available, then we sum them up, as you can earn a degree student ratio

d) Select the schools that received funding from other organizations by the two indexes: Percentage of undergraduates who receive a Pell Grant and Percent of all federal undergraduate students receiving a federal student loan.

e) We summed up the student retention rate at four-year institutions by the two indexes: First-time, full-time student retention rate at four-year institutions and First-time, part-time student retention rate at four-year institutions.

f) We summed up the student retention rate at less-than-four-year institutions by the two indexes: First-time, full-time student retention rate at less-than-four-year institutions and First-time, part-time student retention rate at less-than-four-year institutions.

g) We summed up the delay graduation rate at less-than-four-year institutions by the two indexes: 150% completion rate for four-year institutions, pooled in two-year rolling averages suppressed for small *n* size and 200% completion rate for less-than-four-year institutions, pooled in two-year rolling averages and suppressed for small.

h) We summed up the delay graduation rate at four-year institutions by the two indexes: 150% completion rate for four-year institutions, pooled in two-year rolling averages and suppressed for small n size and 200% completion rate for four-year institutions, pooled in two-year rolling averages and suppressed for small n size.

j) In terms of which we can not operate on, we maintain its original state.

After carrying out the above process, we selected 17 index (**Table 1**).

2) Through factor analysis to choose the school

First, we extract 1064 schools by MATLAB from the table of Most Recent Cohorts Data, which is the Potential Candidate Schools. Second, we analyze the factors of the 1064 schools.

a) The selected indicators set to x, and we enroll the selected indicators to factor analysis in the default condition. From the output, we can obtained four common factors, which can reflect 65% of the information, but x5, x6, x10 information can not be fully reflected.

b) We want to choose schools, should choose the school as much as possible to express benefits and cost information. So we set standard is 90%, to explain 90% of variable information. From the output table, we need to extract 10 common factors, extracting 10 common factors and rotation adjustments, get Rotated Component Matrix.

c) We can get factor model from the Rotated Component Matrix:

$$X_1 = 0.573F_1 + 0.147F_2 + 0.455F_3 + \cdots + 0.083F_{10} + \varepsilon_1 \quad (1)$$

$$X_2 = -0.3F_1 + 0.34F_2 - 0.421F_3 + 0.371F_4 + \cdots - 0.016F_{10} + \varepsilon_2 \quad (2)$$

$$\cdots$$

And get KMO is 0.752, so suitable for factor analysis. Common for all of the variables can get reactions (**Table 2**).

Table 1. Index.

x	Variable name	Meaning	x	Variable name	Meaning
x_1	PREDDEG	Main Degree	x_{10}	RET-FT	First time, student retention rate
x_2	Control of institution	Private for-profit education	x_{11}	PCTFLOAN	The proportion of undergraduates receive federal loans
x_3	LOCALE	Location	x_{12}	UG25abv	The proportion of undergraduates over the age of 25
x_4	PCIP	Degrees awarded Percentage	x_{13}	GRAD_DEBT_MDN_SUPP	The average debt
x_5	UGDS	Registered undergraduate enrollment	x_{14}	RPY_3YR_RT_SUPP	3-year repayment rate
x_6	PPTUG_EF	The proportion of part-time undergraduate students	x_{15}	C150+C200	The number of graduate on time
x_7	NPT4	The average net price Title IV institutions	x_{16}	md_earn_wne_p10	The median income students 10 years later
x_8	NPT-PUB+PRIV	Household income	x_{17}	gt_25k_p6	In more than six years after a student Income $ 25,000/year (earnings threshold)
x_9	PCTPELL	The percentage of undergraduates receiving Pell Grants			

Table 2. Rotated component matrix.

KMO and Bartlett's test		
Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		0.752
Bartlett's Test of Sphericity	Approx. Chi-Square	46,384.973
	df	136
	Sig.	0.000

d) The "% of Variance" as the weight, which from the Extraction Sums of Squared Loadings. First, we count the scores (z) of all schools, we are based on the results of the output of the SPSS software, in the "Variance Explained Total" table, according to the "% of Variance", we can get the mathematical formula: The scores (z) = 0.18702*FAC1_1 (the first factor) + 0.12450*FAC2_1 + 0.11387*FAC3_1 + 0.09866*FAC4_1 + 0.08038*FAC5_1 + 0.06665*FAC6_1 + 0.06357*FAC7_1 + 0.06111*FAC8_1 + 0.06095*FAC9_1 + 0.05922* FAC10_1. Second, we descending all the scores, so all schools are ranked. Last, we choose the top 100 of the schools.

We select the top 100 schools sorted by scores of models above this passage. The following table is partly data of whole date (**Table 3**).

e) Through factor analysis, we analysis the variables each factor represents and the actual meaning, then classify 17 indexes (**Table 4**).

Through the classification of the 17 indexes, we select x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}. Among them, we define F_1, F_8 and F_9 on behalf of the school's income, F_2, F_4 on behalf of the school cost, because we will be variable points to ten, using only to income and cost, so other classification represents significant need not defined. We depend on the two kind of types to calculate ROI. World Bank senior education economist Pless Saha Luo pointed out, that "the rate of return on investment in education" and the rate of return on investment is very similar to other meanings, "Rate of return" is "at different points in time to table a summary of the costs and benefits, with the rate of return on earnings (percentage)" [3].

Table 3. The top 100 schools.

UNITID	INSTNM	z
132,903	University of Central Florida	0.91041
214,777	Pennsylvania State University-Main Campus	0.82541
166,656	MCPHS University	0.815412
210,739	De Sales University	0.794155
104,586	Embry-Riddle Aeronautical University-Prescott	0.784176
183,026	Southern New Hampshire University	0.77248
190,044	Clarkson University	0.768426
123,554	Saint Mary's College of California	0.755716
161,299	Maine Maritime Academy	0.753896
212,054	Drexel University	0.7379

Table 4. The variables each factor represents.

F	x		
F_1	x_1, x_{13}, x_{16}, x_{17}	F_6	x_5
F_2	x_2, x_7, x_8	F_7	x_3
F_3	x_{14}	F_8	x_6, x_{12}
F_4	x_9, x_{11}	F_9	x_{10}
F_5	x_{15}	F_{10}	x_4

4. Models

4.1. Model One

In this model, we need determine the investment of each school, to get the maximum return on investment. To solve this question, According to the schools' assets to determine the coefficient of investment (m), we give it a value of 40, so the quantity of invest limit can be calculated. We define β_i as ROI. It can be calculated by same index mentioned above, such as aid, earnings and cost. We define that Maximum return on investment is F.

$$\mathrm{Max}F = \sum_{i=1}^{100} x_i \beta_i$$

$$s.t \begin{cases} \sum_{i=1}^{100} x_i \le 5\times 10^8 \\ x_i \le m\omega_i \end{cases}$$

The meaning of each symbol in **Table 5**.

4.2. Model Two

In the selected 100 schools, we need to determine which schools and years to invest, in order to achieve optimal investment and not duplication of investment, for five years, starting July 2016. We define that Maximum income on investment is G.

$$\mathrm{Max}G = \sum_{t=1}^{5}\sum_{i=1}^{100} x_i \beta_i x_{it} \left(1+k\right)^{5-t} - Z_5$$

$$
s.t\begin{cases}
\sum_{i=1}^{100} x_i x_{i1} + Z_1 = 1\times 10^8 \\
\sum_{i=1}^{100} x_i x_{it} + Z_t = 1\times 10^8 + (1+k) Z_{t-1};\ t = 2,3,4,5 \\
\sum_{t=1}^{5} x_{it} \le 1;\ i = 1,2,\cdots,100 \\
x_{it} = 0,1
\end{cases}
$$

The meaning of each symbol in **Table 6**.

5. Model Solution

1) We solve the model one by MATLAB to get the cost, income, ROI and investment of each school, the following table is partly data of the whole date (**Table 7**).

2) We solve the model two by LINGO to get the investment of the 100 schools and decide which year we invest. The following table is partly data of the whole date (**Table 8**).

6. Sensitivity Analysis

In this section, we conduct sensitivity analysis for our model. We select x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{16}, x_{17}, these indexes include income indexes and cost indexes, we depend on the two kind of types to calculate ROI. Specifically, we test the sensitivity of parameter ROI, which we define in calculating investment amount. **Tables 9-11** and **Figure 1** show the results when its value is changed. We get ROI2 by delete one index, then calculate investment amount. Second, we get ROI3 by delete the other index, then calculate investment amount. Third, the same to get ROI4 by delete the other index.

$$\mathrm{ROI} = \frac{\text{revenue}}{\cos t} * 100\%$$

As we can see, with the change of ROI, the change of investment is slightly, so that our model is optimal and feasible.

Table 5. Symbol explanation.

Notation	Explanation
F	The value of investment returned
x_i	The value of investment returned of *i*-th school
β_i	The return on investment (ROI) of *i*-th school
ω_i	The property of *i*-th school
m	The coefficient of invest according to the property of schools

Table 6. Symbol explanation.

Nomenclatures	Explanation
x_i	The value of investment returned of *i*-th school
β_i	The return on investment (ROI) of *i*-th school
x_{it}	The value of whether the *i*-th school was invested in *t*-th year
k	The rates of bank interest
t	Year
z_t	The remaining money after donating in *t*-th year

Table 7. Investment amount.

UNITID	PROFIT	COST	ROI	Investment amount
132,903	104,288.4	87,579	0.190793	4,171,537.97
166,656	240,463.8	203,165	0.183589	9,618,552.19
161,299	127,692.7	115,561	0.104981	5,107,707.95
150,987	64,340.11	63,341	0.015774	2.57E+06
133,951	101,928.9	77,343	0.317882	4,077,157.61
171,100	105,925.4	98,599	0.074305	4.24E+06
154,688	140,790.6	105,056	0.340148	5,631,622.1
204,796	115,441.4	100,264	0.151374	4,617,655.41
228,723	98,003.76	70,091	0.398236	3,920,150.54

Table 8. Schools of which year to invest.

	Year				
	1	2	3	4	5
UNITID	228,459	133,951	166,656	132,903	150,987
	111,966	154,688	163,286	161,299	171,100
	174,066	228,723	144,892	204,796	195,191
	153,603	159,391	204,185	243,780	219,356
	240,444	178,411	130,943	226,231	129,020
	104,151	215,266	166,513	216,694	209,542
	207,388	181,297	196,149	164,739	219,471
	153,001	139,931	240,462	134,097	170,976
	171,128	127,918	240,480	151,777	229,115
	166,692	216,010	151,351	214,643	240,189
	152,798	148,584	196,088	233,295	201,195
	239,318	110,565	203,368	145,691	234,827
	240,471	211,644	218,663	215,284	215,275
	165,264	153,658	142,285	137,351	209,108
	176,770	206,349	214,704	196,158	104,179
	110,583	110,662	115,409	183,910	197,869
	217,882	134,130	164,076	110,680	110,529
	165,024	230,764	163,268	220,631	214,768
		199,139	144,962	232,982	215,798
		110,361		196,246	198,464
				182,670	200,280
					196,097

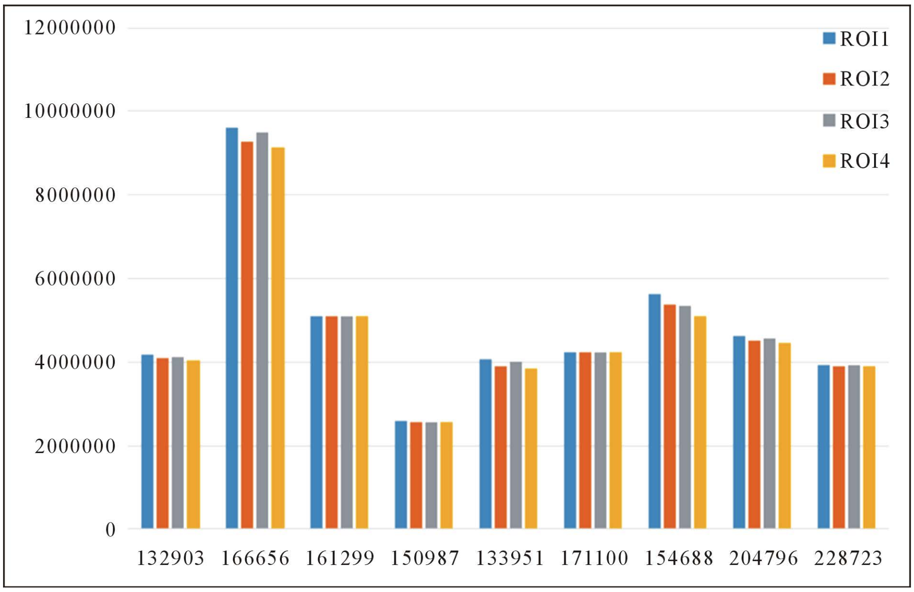

Figure 1. The investment amount in different ROI.

Table 9. The investment amount in ROI2.

UNITID	PROFIT	COST	ROI	Investment amount
132,903	102,205	87,579	0.167003186	4,088,198.88
166,656	231,830.4	203,165	0.141094098	9,273,215.3
161,299	127,520.2	115,561	0.103488091	5,100,807.49
150,987	64,165.63	63,341	0.013018822	2.57E+06
133,951	97,749.36	77,343	0.263842428	3,909,974.6
171,100	105,574	98,599	0.070741344	4.22E+06
154,688	134,360.8	105,056	0.278944934	5,374,433.56
204,796	112,524.4	100,264	0.12228105	4,500,975.49
228,723	97,712.54	70,091	0.394081154	3,908,501.69

Table 10. The investment amount in ROI3.

UNITID	PROFIT	COST	ROI	Investment amount
132,903	103,183.4	87,579	0.178175699	4,127,337.98
166,656	237,307.7	203,165	0.168053972	9,492,307.41
161,299	127,650.8	115,561	0.104618074	5,106,030.77
150,987	64,190.65	63,341	0.013413931	2.57E+06
133,951	100,058.2	77,343	0.293694845	4,002,329.62
171,100	105,881.2	98,599	0.073857059	4.24E+06
154,688	133,659	105,056	0.272264626	5,346,361.3
204,796	114,392.1	100,264	0.140909056	4,575,684.22
228,723	97,963.35	70,091	0.397659523	3,918,534.14

Table 11. The investment amount in ROI4.

UNITID	PROFIT	COST	ROI	Investment amount
132,903	101,100	87,579	0.154386006	4,043,998.88
166,656	228,674.3	203,165	0.125559336	9,146,970.5
161,299	127,478.3	115,561	0.103125252	5,099,130.29
150,987	64,016.17	63,341	0.010659213	2.56E+06
133,951	95,878.66	77,343	0.239655365	3,835,146.6
171,100	105,529.9	98,599	0.07029347	4.22E+06
154,688	127,229.3	105,056	0.2110619	5,089,172.76
204,796	111,475.1	100,264	0.111815878	4,459,004.29
228,723	97,672.13	70,091	0.393504618	3,906,885.29

7. Strengths and Weakness

7.1. Strengths

1) In model one, the model is simple and easy to understand, and we define the return-on-investment degree, making us to carry out a quantitative analysis of return-on-investment.

2) In model one and model two, we process the data and make a variety of charts, simple and intuitional.

3) In model two, we take the variable of Z_t into consideration so that the model is more realistic.

7.2. Weakness

1) In model one, the selection of the variable of m is subjectivity.

2) In model two, we assume that the time duration that the organization's money should be provided is one year.

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Mathematical Analysis of Nipah Virus Infections Using Optimal Control Theory

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Abstract

The optimal use of intervention strategies to mitigate the spread of Nipah Virus (NiV) using optimal control technique is studied in this paper. First of all we formulate a dynamic model of NiV infections with variable size population and two control strategies where creating awareness and treatment are considered as controls. We intend to find the optimal combination of these two control strategies that will minimize the cost of the two control measures and as a result the number of infectious individuals will decrease. We establish the existence for the optimal controls and Pontryagin's maximum principle is used to characterize the optimal controls. The numerical simulation suggests that optimal control technique is much more effective to minimize the infected individuals and the corresponding cost of the two controls. It is also monitored that in the case of high contact rate, controls have to work for longer period of time to get the desired result. Numerical simulation reveals that the spread of Nipah virus can be controlled effectively if we apply control strategy at early stage.

Keywords

Nipah Virus (NiV), Optimal Control, Existence of the State, Existence of the Objective Functional, Pontryagin's Maximum Principle, Transversality Condition, Optimality Condition, Hamiltonian (H)

1. Introduction

Mathematical modeling has become an important tool for analyzing the spread as well as control of infectious diseases. It is also a useful tool for the measurement of the effect of different strategies for controlling the spread of infectious diseases within a population. In recent years epidemiological modeling of infectious disease transmission has had an increasing influence on the theory and practice of disease management and control [1]. There

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are a number of different methods for calculating the optimal control for a specific mathematical model. For example, Pontryagin's maximum principle [2] allows the calculation of the optimal control for a system of ordinary differential equation with a given constraint. Here the optimal control strategy is used to minimize the infected individuals and to maximize the total number of recovered individuals.

Nipah virus, a member of the genus Henipavirus, a new class of virus in the Paramyxoviridae family, has drawn attention as an emerging zoonotic virus in south-east and south-Asian region [3]. This emerging infectious disease has become one of the most alarming threats of the public health mainly due to its periodic outbreaks and the high mortality rate [4]. Epidemiology is the study of the distribution and determinants of health related states or events in specified populations and the application of epidemiology is to control of health problems. The crucial point is that epidemiology concerns itself with populations or groups of population in contrast to clinical medicine, which deals with individuals (patients). Therefore, epidemiology describes health and disease in terms of frequencies and distributions of determinants and conditions in a population or in a specific group of a population. Although Nipah virus has caused only a few outbreaks, it infects a wide range of animals and causes severe disease and death in people, making it a public health concern [5]. Treatment is mostly symptomatic and supportive as the effect of antiviral drugs is not satisfactory. So the very high case fatality addresses the need for adequate and strict control and preventive measures.

This paper deals with application of optimal control to a dynamic model of Nipah Virus (NiV) infections and its possible control and preventive strategy with the help of optimal control technique. Our aim is to minimize the total number of infectious individuals and the cost which is related for creating awareness and treatment.

2. Formulation of Model

Nipah virus infection is a zoonotic virus and transmitted first from animal to human. Once it has been transmitted to human, then it continues to be transmitted through human to human (H2H) by the close contact of infected individuals due to its highly infectivity [6]. Let us suppose that $S(t)$, $I(t)$ and $R(t)$ denote the number of individuals in the susceptible, infectious and recovered classes at time t respectively. The total population at time t is represented by $N(t)=S(t)+I(t)+R(t)$.

We consider the following system of non-linear differential equation, is a type of standard SIR disease model, to describe the dynamics of Nipah Virus (NiV) infections in the community.

$$\begin{aligned}
S'(t) &= \nu N(t) - \beta S(t) I(t) - \mu S(t) \\
I'(t) &= \beta S(t) I(t) - (\gamma + \mu + \alpha) I(t) \\
R'(t) &= \gamma I(t) - \mu R(t) \\
N'(t) &= \nu N(t) - \alpha I(t) - \mu N(t),
\end{aligned} \tag{1}$$

with initial conditions,

$$S(0)=S_0 \geq 0, I(0)=I_0 \geq 0, R(0)=R_0 \geq 0, N(0)=N_0, \tag{2}$$

and where, the parameter β represents the effective contact rate, ν is the natural birth rate, μ is the natural mortality rate, γ is the recovery rate and α represents the disease induced death rate.

Since there is no proper vaccination program or appropriate drugs for NiV infections, so in the model we introduce two control strategies, namely, creating awareness (u_1) among the community about the risky areas before outbreak of the disease and the treatment (u_2). Here the control $u_1(t)$ measures the effort to be needed to increase awareness which results in the reduction of the transmission rate (β) and the control $u_2(t)$ measures the effort required for giving health cares for the infected people to reduce the infected individuals.

Now the NiV model with two control strategies is given below:

$$\begin{aligned}
S'(t) &= \nu N(t) - \beta S(t) I(t) - (\mu + u_1) S(t) \\
I'(t) &= \beta S(t) I(t) - (\gamma + \mu + \alpha) I(t) - u_2 I(t) \\
R'(t) &= \gamma I(t) - \mu R(t) + u_1 S(t) + u_2 I(t) \\
N'(t) &= \nu N(t) - \alpha I(t) - \mu N(t),
\end{aligned} \tag{3}$$

with initial conditions,

$$S(0)=S_0\ge 0,\ I(0)=I_0\ge 0,\ R(0)=R_0\ge 0,\ N(0)=N_0. \tag{4}$$

Here our main objective is to minimize the total number of infected individuals and to reduce the cost which is needed for creating awareness and treatment on a specified time interval. For the fulfillment of our purpose, we work with the following objective function which is similar as [7].

$$\text{Minimize } J(u_1(t),u_2(t))=\int_0^T A_1 I(t)+\frac{1}{2}\left(B_1u_1^2+B_2u_2^2\right)\mathrm{d}t$$

where, B_1 and B_2 are weight parameters that help to balance the corresponding costs. We define the control set as follows:

$$U=\left\{(u_1(t),u_2(t)):0\le u_1(t)\le 1,\ 0\le u_2(t)\le 1,\ t\in[0,T]\right\}.$$

In the objective function, A_1I represents the total number of infected individuals, $B_1\frac{u_1^2}{2}$ represents the cost for creating awareness and $B_2\frac{u_2^2}{2}$ represents the cost for treatment.

3. Existence of the Optimal Control for NiV Model

3.1. Existence of the State

Adding first three equations of the system (3) we get,

$$S'+I'+R'=\nu N(t)-\mu S(t)-\alpha I(t)-\mu I(t)-\mu R(t)$$

$$\Rightarrow S'+I'+R'=\nu N(t)-\mu(S+I+R)(t)-\alpha I(t)$$

$$\Rightarrow S'+I'+R'\le \nu N(t)$$

$$\Rightarrow S'+I'+R'\le \nu(S+I+R)(t)$$

$$\Rightarrow \frac{\mathrm{d}(S+I+R)}{S+I+R}\le \nu \mathrm{d}t.$$

On integrating we get,

$$S+I+R\le \mathrm{e}^{\nu T}\left[S_0+I_0+R_0\right]=M_1\in\mathbb{R}_+,\ t\in[0,T].$$

So we have

$$S(t)\le M_1,\ I(t)\le M_1 \text{ and } R(t)\le M_1.$$

From the fourth equation of (3) we have

$$N'(t)\le(\nu-\mu)N(t),$$

and then

$$N(t)\le N_0\mathrm{e}^{(\nu-\mu)T}=M_2\in\mathbb{R}_+,\ t\in[0,T].$$

So, finally we get $N(t)\le M_2$.

Since $S(t)$, $I(t)$, $R(t)$ and $N(t)$ are bounded above, so there exists solution for the system (3).

3.2. Existence of the Objective Functional

By proving the following theorem we can establish the existence of the objective functional:

Theorem 1. *Consider the control problem with system* (3). *Then there exists optimal controls* (u_1^*,u_2^*) *that minimize* $J(u_1,u_2)$ *over the control set U. i.e.,*

$$J\left(u_1^*, u_2^*\right) = \min_{u_1, u_2 \in U} J\left(u_1, u_2\right). \tag{5}$$

Proof: *To use an existence result in* [8], *we must check the following properties* [9].

1) *The set of controls and corresponding state variables is non-empty.*

2) *The control set U is convex and closed.*

3) *The right-hand side of the state system is bounded by a linear function in the state and control variables.*

4) *The integrand of the objective functional is convex on U and is bounded below by* $-k_2 + k_1 \left|\left(u_1, u_2\right)\right|^{\eta}$ *with* $k_1 > 0$, $k_2 > 0$ *and* $\eta > 1$.

To prove the above theorem we need to use the following theorem 2 and 3.

Theorem 2. *Let each of the functions* $F_1, \cdots, F_n$ *and the partial derivatives* $\frac{\partial F_1}{\partial x_1}, \cdots, \frac{\partial F_1}{\partial x_n}, \cdots, \frac{\partial F_n}{\partial x_1}, \cdots, \frac{\partial F_n}{\partial x_n}$ *be continuous in a region* $\mathcal{R}$ *of* $t, x_1, x_2, \cdots, x_n$ *space defined by* $\alpha < t < \beta, \alpha_1 < x_1 < \beta_1, \cdots, \alpha_n < x_1 < \beta_n$, *and let the point* $\left(t_0, x_1^0, x_2^0, \cdots, x_n^0\right)$ *be in* $\mathcal{R}$. *Then there is an interval* $\left[t - t_0\right] < h$ *in which there exists a unique solution (* $x_1 = \phi_1\left(t\right), \cdots, x_n = \phi_n\left(t\right)$ *) of the system of differential equations*

$$\begin{aligned} x_1' &= F_1\left(t, x_1, \cdots, x_n\right), \\ x_2' &= F_2\left(t, x_1, \cdots, x_n\right), \\ &\vdots \\ x_n' &= F_n\left(t, x_1, \cdots, x_n\right), \end{aligned}$$

that also satisfies the initial conditions

$$x_1\left(t_0\right) = x_1^0, x_2\left(t_0\right) = x_2^0, \cdots, x_n\left(t_0\right) = x_n^0.$$

Theorem 3. *Let* $x_i = F_i\left(t, x_1, \cdots, x_n\right)$ *for* $i \in \left[1, n\right]$ *be a system of n differential equations with initial conditions* $x_i\left(t_0\right) = x_i^0$ *for* $i \in \left[1, n\right]$. *If each of the functions* $F_1, \cdots, F_n$ *and the partial derivatives* $\frac{\partial F_1}{\partial x_1}, \cdots, \frac{\partial F_1}{\partial x_n}, \cdots, \frac{\partial F_n}{\partial x_1}, \cdots, \frac{\partial F_n}{\partial x_n}$ *are continuous in* $\mathcal{R}^{n+1}$ *space, then there exists a unique solution* $x_1 = \sigma_1\left(t\right), \cdots, x_n = \sigma_n\left(t\right)$ *that satisfies the initial conditions.*

Now with the help of the above two theorems we prove the four conditions of theorem (1).

Proof of theorem 1: To use an existence result in [8], we must check the following properties [9].

1) The set of controls and corresponding state variables is non-empty.

2) The control set U is convex and closed.

3) The right-hand side of the state system is bounded by a linear function in the state and control variables.

4) The integrand of the objective functional is convex on *U* and is bounded below by $-k_2 + k_1 \left|\left(u_1, u_2\right)\right|^{\eta}$ with $k_1 > 0$, $k_2 > 0$ and $\eta > 1$.

Proof of 1): Let us consider,

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= F_1\left(t, S, I, R, N\right) \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= F_2\left(t, S, I, R, N\right) \\ \frac{\mathrm{d}R}{\mathrm{d}t} &= F_3\left(t, S, I, R, N\right) \\ \frac{\mathrm{d}N}{\mathrm{d}t} &= F_4\left(t, S, I, R, N\right), \end{aligned}$$

where, F_1, F_2, F_3 and F_4 buildup the right hand side of the system (3). Let $u_1\left(t\right) = C_1$ and $u_2\left(t\right) = C_2$ for some constants C_1, C_2. F_1, F_2, F_3 and F_4 must be linear and they are also continuous everywhere. Moreover, the partial derivatives of F_1, F_2, F_3 and F_4 with respect to all state are constants and they are continuous everywhere.

So by following the theorem 3, we can say that there exists an unique solution $S = \sigma_1\left(t\right)$, $I = \sigma_2\left(t\right)$, $R = \sigma_3\left(t\right)$, $N = \sigma_4\left(t\right)$ which satisfies the initial conditions. Therefore, the set of controls and corresponding state variables is non-empty. Hence the condition 1) is satisfied.

Proof of 2): By definition, U is closed. Take any controls $u_1, u_2 \in U$ and $\theta \in [0,1]$. Then

$$0 \le \theta u_1 + (1-\theta) u_2.$$

Additionally, observe that $\theta u_1 \le \theta$ and $(1-\theta) u_2 \le (1-\theta)$. Then

$$\theta u_1 + (1-\theta) u_2 \le \theta + (1-\theta) = 1.$$

Hence, $0 \le \theta u_1 + (1-\theta) u_2 \le 1$ for all $u_1, u_2 \in U$ and $\theta \in [0,1]$. Therefore, U is convex, and condition 2) is satisfied.

Proof of 3):

We consider,

$$F_1 \le \nu N - u_1 S$$

$$F_2 \le K_1 I - u_2 I$$

$$F_3 \le u_1 S + u_2 I + \gamma I$$

$$F_4 \le \nu N.$$

The state system is given below:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = F_1(t, S, I, R, N)$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = F_2(t, S, I, R, N)$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = F_3(t, S, I, R, N)$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = F_4(t, S, I, R, N).$$

Now we rewrite the system in matrix form:

$$\bar{F}(t, \bar{X}, u) \le \bar{m}\left(t, \begin{bmatrix} S \\ I \\ R \\ N \end{bmatrix}\right) \bar{X}(t) + \bar{n}\left(t, \begin{bmatrix} S \\ I \\ R \\ N \end{bmatrix}\right) \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \tag{6}$$

where,

$$\bar{m}\left(t, \begin{bmatrix} S \\ I \\ R \\ N \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 & \nu \\ 0 & k_1 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \nu \end{bmatrix} \tag{7}$$

and

$$\bar{n}\left(t, \begin{bmatrix} S \\ I \\ R \\ N \end{bmatrix}\right) = \begin{bmatrix} -S \\ -I \\ S+I \\ 0 \end{bmatrix} \tag{8}$$

which gives a linear function of the controls u_1 and u_2 defined by time and state variables. Then we can find out the bound of the right hand side. It is noted that all parameters are constant and greater than or equal to zero. Therefore we can write,

$$\left|\bar{F}\left(t,\bar{X},u_1,u_2\right)\right| \le \|\bar{m}\|\left|\bar{X}\right| + \left|\bar{S}+\bar{I}\right|\left|\left(u_1(t),u_2(t)\right)\right|$$
$$\le p\left(\left|\bar{X}\right| + \left|\left(u_1(t),u_2(t)\right)\right|\right)$$

where $\bar{S}$, $\bar{I}$ are bounded and p includes the upper bound of the constant matrix. Hence we see that the right hand side is bounded by a sum of the state and the control. Therefore, condition 3) is satisfied.

Proof of 4):

For the proof of the condition 4) we use the result in [10] and [Fleming and Rishel (1975)]. The control and the state variables are non-negative values and are non-empty. In the minimization problem, the necessary convexity of the objective functional in u_1 is satisfied. The control variables $u_1, u_2 \in U$ is also convex and closed by definition. Furthermore, from [10] we see that the integrand in the objective functional which is $\left(AI(t) + A_2\frac{u_1^2}{2} + A_3\frac{u_2^2}{2}\right)$ is convex on the control set U.

Now we have to prove that $J(u_1,u_2) \ge -k_2 + k_1\left|(u_1,u_2)\right|^{\eta}$ with $k_1 > 0$, $k_2 > 0$ and $\eta > 1$.

Here,

$$J(u_1,u_2) = A_1 I(t) + B_1\frac{u_1^2}{2} + B_2\frac{u_2^2}{2}$$

$$J(u_1,u_2) \ge -A_1 I(t) + B_1\frac{u_1^2}{2} + B_2\frac{u_2^2}{2}$$

$$J(u_1,u_2) \ge -A_1 I(t) + \frac{1}{2}B\left(u_1^2+u_2^2\right) \quad \left[\text{let}, B_1 + B_2 = B\right]$$
$$= -k_2 + k_1\left(u_1,u_2\right)^2$$

where, $k_2 > 0$ that depends on the upper bounds of $I(t)$. We can also see that $\eta = 2 > 1$, $k_1 > 0$. Hence, condition 4) is satisfied.

4. Characterization of the Optimal Control

In order to derive the necessary condition for the optimal control, we use pontryagin's maximum principle [2]. This principle converts the system and the objective functional into a problem minimizing pointwise a Hamiltonian H with respect to u_1 and u_2. In the objective function, the value A is the balancing parameter, B_1 and B_2 are the weight parameters balancing the cost. Here we can see from the system (3) that R appears only in the recovered class. So, when we build up the optimality system, we will ignore the recovered class.

By using pantraygin's Maximum principle we first derive the Hamiltonian which is given below

$$\begin{aligned} H(S,I,N) = {} & A_1 I(t) + B_1\frac{u_1^2}{2} + B_2\frac{u_2^2}{2} + \lambda_s\left(\nu N - \beta SI - \mu S - u_1 S\right) \\ & + \lambda_I\left(\beta SI - (\gamma+\mu+\alpha)I - u_2 I\right) + \lambda_N\left((\nu-\mu)N - \alpha I\right) \end{aligned} \tag{9}$$

where, λ_S, λ_I, λ_N are the associated adjoints for the state S, I, N respectively. By differentiating the Hamiltonian (H) with respect to each state variable, we find the differential equation for the associated adjoint. Hence, the adjoint system is,

$$\begin{aligned} \lambda_S' &= \lambda_S\left(\mu + u_1 + \beta I\right) - \lambda_I \beta I \\ \lambda_I' &= \lambda_S \beta S + \lambda_I\left(\gamma+\mu+\alpha\right) - \lambda_I \beta S + \lambda_I u_2 + \lambda_N \alpha - A_1 \\ \lambda_N' &= \lambda_S \nu + \lambda_N\left(\nu - \mu\right) \end{aligned} \tag{10}$$

with the final conditions,

$$\lambda_S(T) = \lambda_I(T) = \lambda_N(T) = 0.$$

So by differentiating the Hamiltonian with respect to two controls u_1 and u_2 we obtain:

$$\left.\frac{\partial H}{\partial u_1}\right|_{u_1=u_1^*}=0$$

$$B_1 u_1^* - S\lambda_S = 0$$

$$u_1^* = \frac{S\lambda_S}{B_1}$$

and $\left.\frac{\partial H}{\partial u_2}\right|_{u_2=u_2^*}=0$

$$B_2 u_2^* - I\lambda_I = 0$$

$$u_2^* = \frac{I\lambda_I}{B_2}.$$

5. Optimality System

State equations:

$$\begin{aligned}
&S'(t) = \nu N(t) - \beta S(t) I(t) - (\mu + u_1) S(t),\\
&I'(t) = \beta S(t) I(t) - (\gamma + \mu + \alpha) I(t) - u_2 I(t),\\
&R'(t) = \gamma I(t) - \mu R(t) + u_1 S(t) + u_2 I(t),\\
&N'(t) = \nu N(t) - \alpha I(t) - \mu N(t),
\end{aligned} \tag{11}$$

with initial conditions,

$$S(0) = S_0 \ge 0,\ I(0) = I_0 \ge 0,\ R(0) = R_0 \ge 0,\ N(0) = N_0. \tag{12}$$

Adjoint equations:

$$\begin{aligned}
&\lambda_S' = \lambda_S(\mu + u_1 + \beta I) - \lambda_I \beta I.\\
&\lambda_I' = \lambda_S \beta S + \lambda_I(\gamma + \mu + \alpha) - \lambda_I \beta S + \lambda_I u_2 + \lambda_N \alpha - A_1.\\
&\lambda_N' = \lambda_S \nu + \lambda_N(\nu - \mu).
\end{aligned} \tag{13}$$

Transversality equations:

$$\lambda_S(T) = \lambda_I(T) = \lambda_N(T) = 0. \tag{14}$$

Characterization of the optimal control u_1^* and u_2^*:

$$u_1^* = \begin{cases} 0 & \text{if } \dfrac{S\lambda_S}{B_1} < 0,\\[2mm] \dfrac{S\lambda_S}{B_1} & \text{if } 0 \le \dfrac{S\lambda_S}{B_1} \le 1,\\[2mm] 1 & \text{if } \dfrac{S\lambda_S}{B_1} > 1. \end{cases} \tag{15}$$

and

$$u_2^* = \begin{cases} 0 & \text{if } \dfrac{I\lambda_I}{B_2} < 0,\\[2mm] \dfrac{I\lambda_I}{B_2} & \text{if } 0 \le \dfrac{I\lambda_I}{B_2} \le 1,\\[2mm] 1 & \text{if } \dfrac{I\lambda_I}{B_2} > 1. \end{cases} \tag{16}$$

In compact notion we can write,

$$u_1^* = \min\left[1, \max\left[0, \frac{S\lambda_S}{B_1}\right]\right] \tag{17}$$

and

$$u_2^* = \min\left[1, \max\left[0, \frac{I\lambda_I}{B_2}\right]\right]. \tag{18}$$

6. Numerical Results and Discussions

Numerical solutions to the optimal system are executed using MATLAB. The considered two controls (u_1, u_2) depend on the adjoints λ_S, λ_I and λ_N of the state variables S, I and N respectively. We simulate the model without control and with control and then we compare the results. We considered the numerical value of the controls u_1 and u_2 in between zero(0) and one(1) as they are not 100 percent effective. We also monitored the effectiveness of the weight parameter to see how the control is related to weight function. In this simulation we assumed the initial values of S, I and N as proportions instead of whole numbers.

The parameter values used in the simulations are presented in the following **Table 1**.

Figure 1 depicts the importance of the controls to the disease dynamics. From the graphs, we see that the control has a positive impact to reduce infection until the controls are effective enough. It is also clear from the figures that the disease can be controlled over finite period of time after imposing control strategies.

Figure 2 and **Figure 3** monitored the impact of the parameter (α) of disease induced death rate. Here we see if α is at a low rate ($\alpha = 0.3$) then the controls work effectively and as a result there is a significant reduction of the infections. When controls do not work it resulted the increase of infected individuals.

On the other hand for the higher rate of α (where awareness does not work, $u_1 = 0$ and the treatment u_2 works for a short period of time) there is a sharp decrease of infection due to death resulting the existence of fewer recovered people.

Figure 4 and **Figure 5** show the comparative situation of the disease dynamics for low and high contact rates. In the case of low contact rate ($\beta = 0.2$), the infectious individuals decrease until the controls work effectively and as a result there is a notable increment of recovered individual.

On the other hand, for the very high contact rate ($\beta = 2$), which resulted a severe disease burden, the controls work for a longer period of time to reduce the disease burden.

Figure 6 and **Figure 7** show the influence of the various weight parameters. Here we notice that for low

Table 1. Description and parameter values of the NiV model.

Variable	Description	Initial values
S_0	Initial susceptible individuals	0.90 [assumed]
I_0	Initial infected individuals	0.05 [assumed]
R_0	Initial recovered individuals	0.05 [assumed]
Parameters	**Description**	**Initial values**
ν	Birth rate	0.03 [assumed]
μ	Mortality rate	0.002 [assumed]
β	Contact rate	0.75 [7]
γ	Recovery rate	0.005 [assumed]
α	Disease induced death rate	0.01 [assumed]
A_1	Weight parameter	10 [7]
B_1	Weight parameter	1 [7]
B_2	Weight parameter	2 [7]
T	Number of years	6 [assumed]

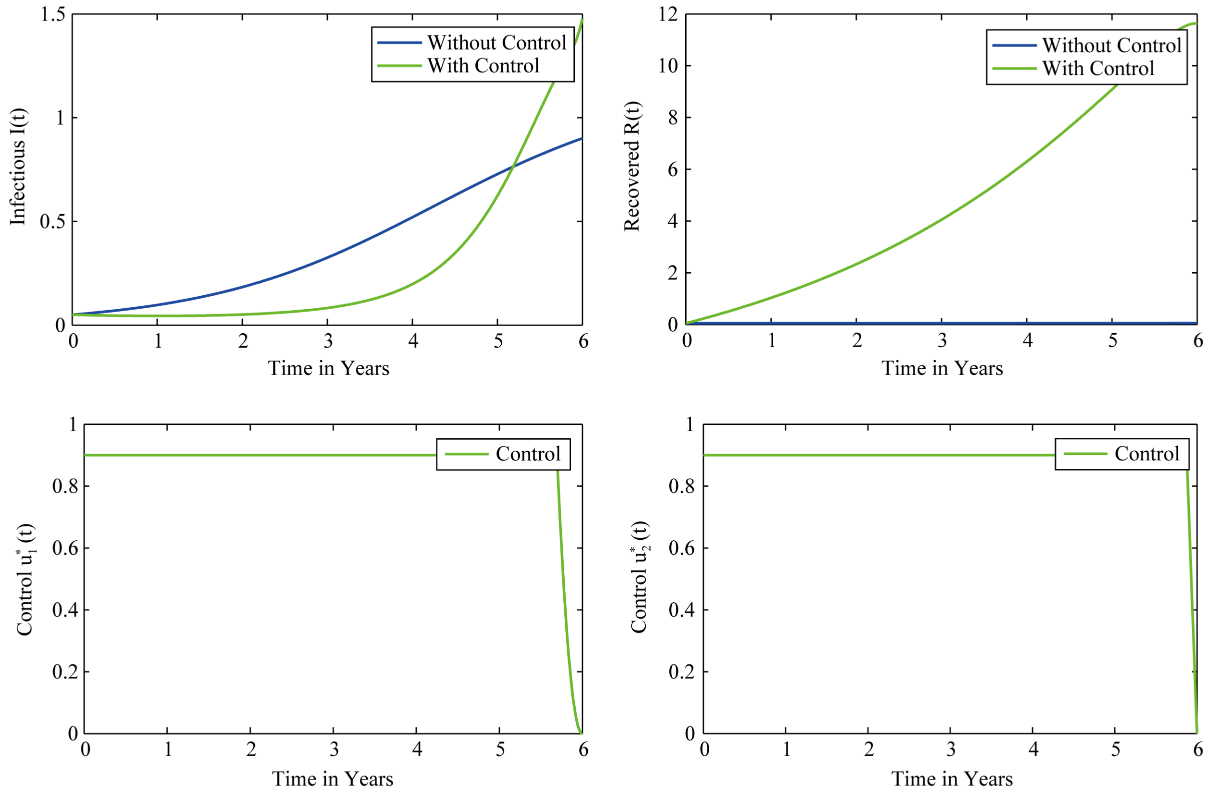

Figure 1. NIV model with control and without control, parameter values are taken from **Table 1**.

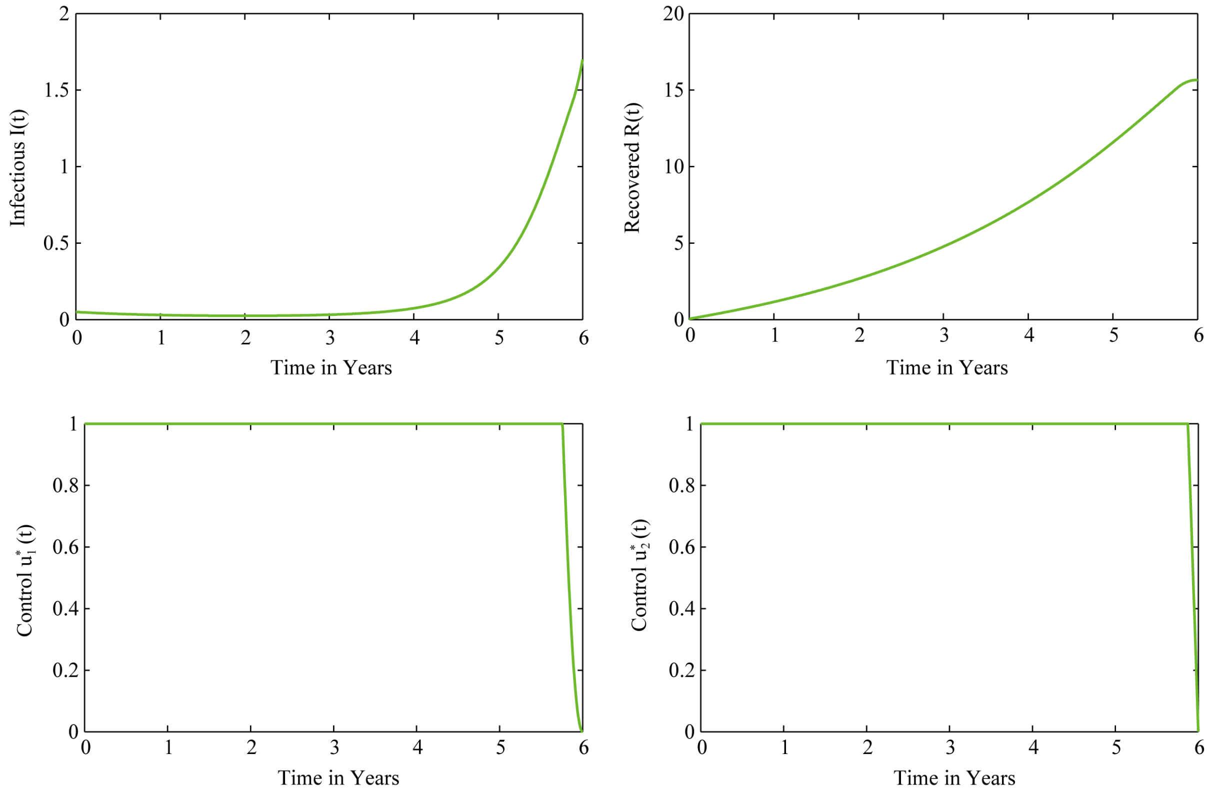

Figure 2. NiV model with low disease induced death rate, $\alpha = 0.3$ and other parameter values are taken from **Table 1**.

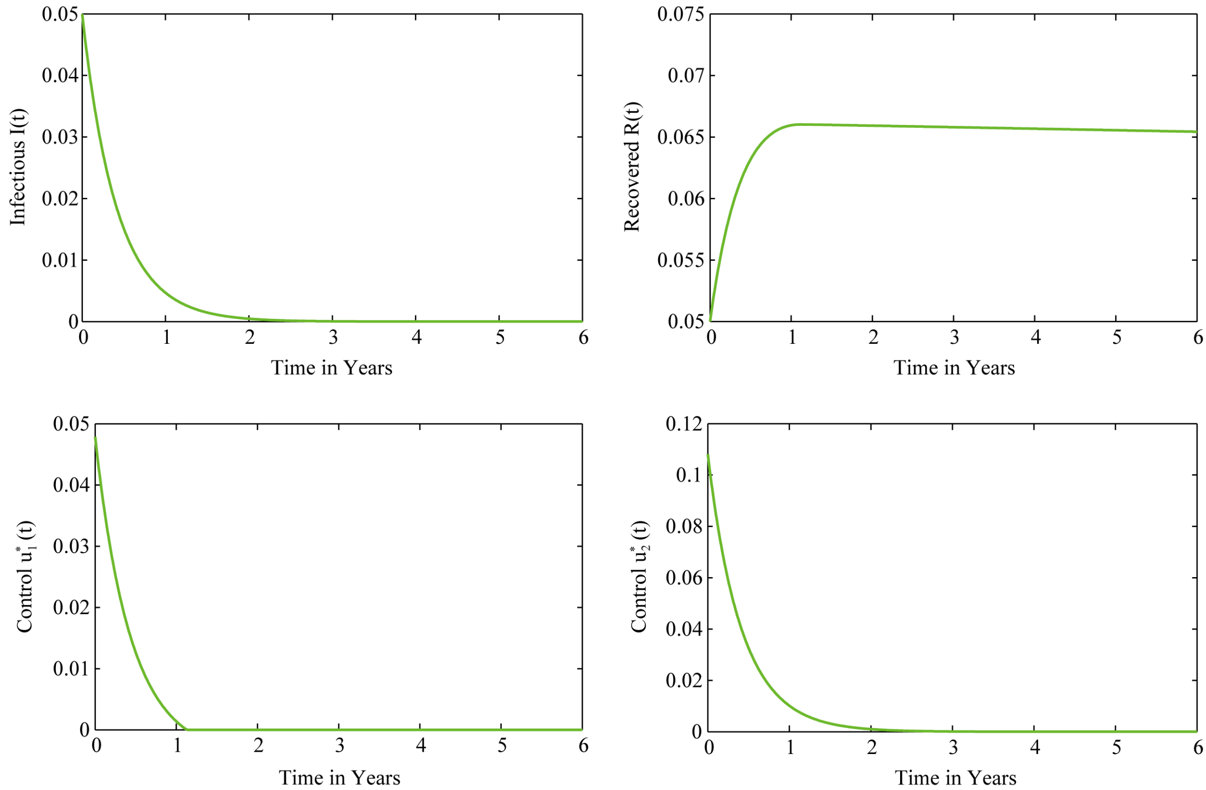

Figure 3. NiV model with high disease induced death rate, $\alpha = 3$ and other parameter values are taken from **Table 1**.

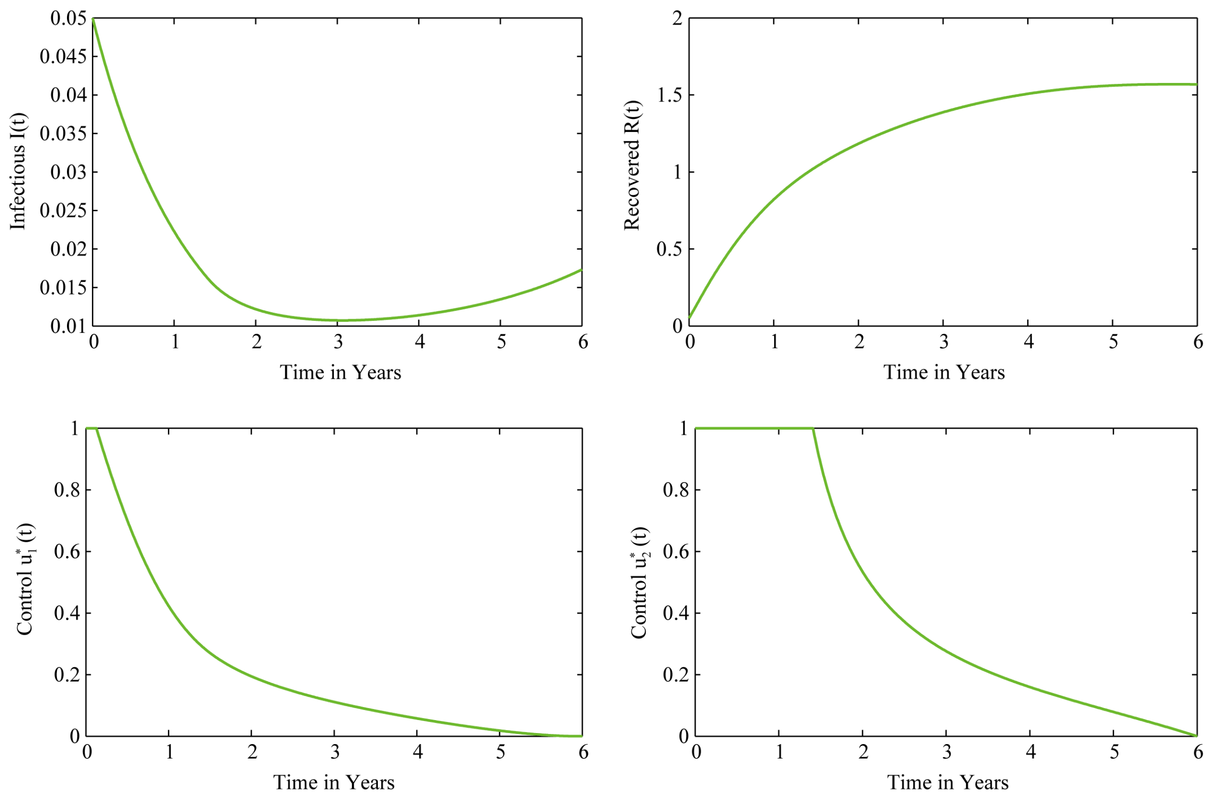

Figure 4. NiV model with low contact rate, $\beta = 0.2$ and other parameter values are taken from **Table 1**.

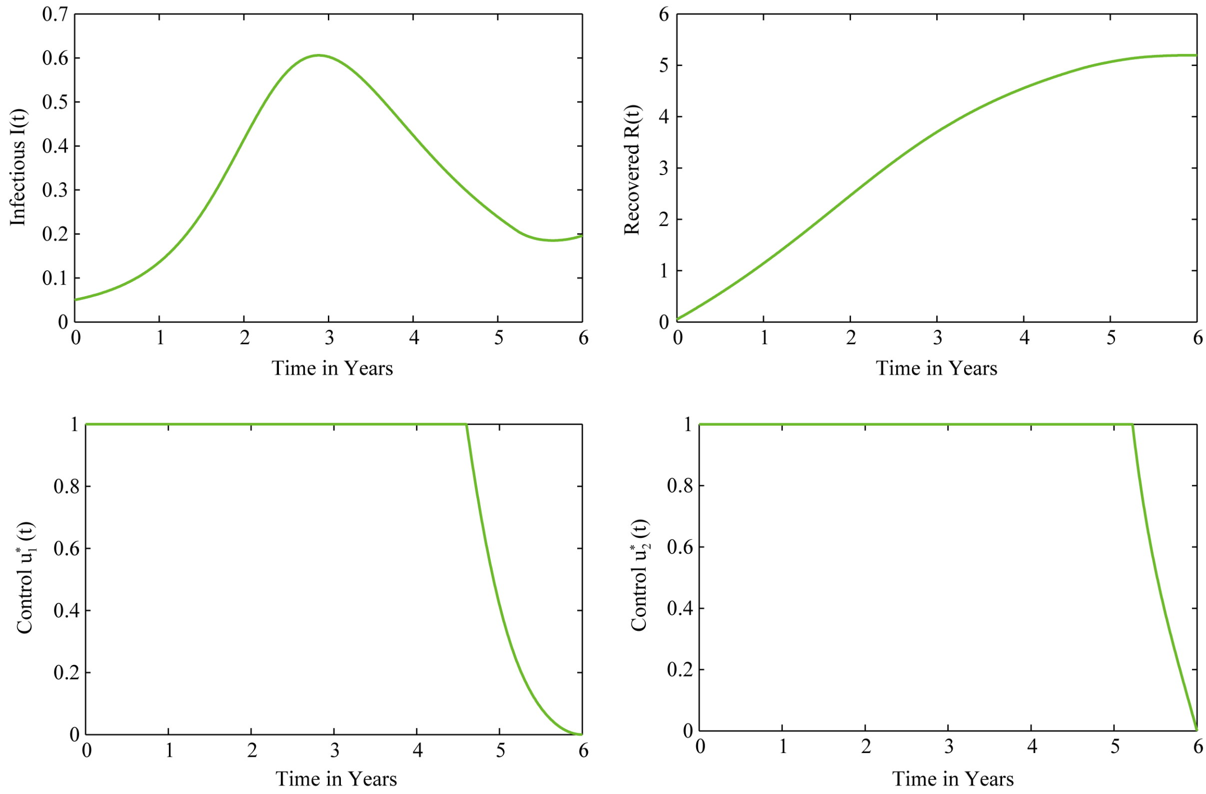

Figure 5. NiV model with high contact rate, $\beta = 2$ and other parameter values are taken from **Table 1**.

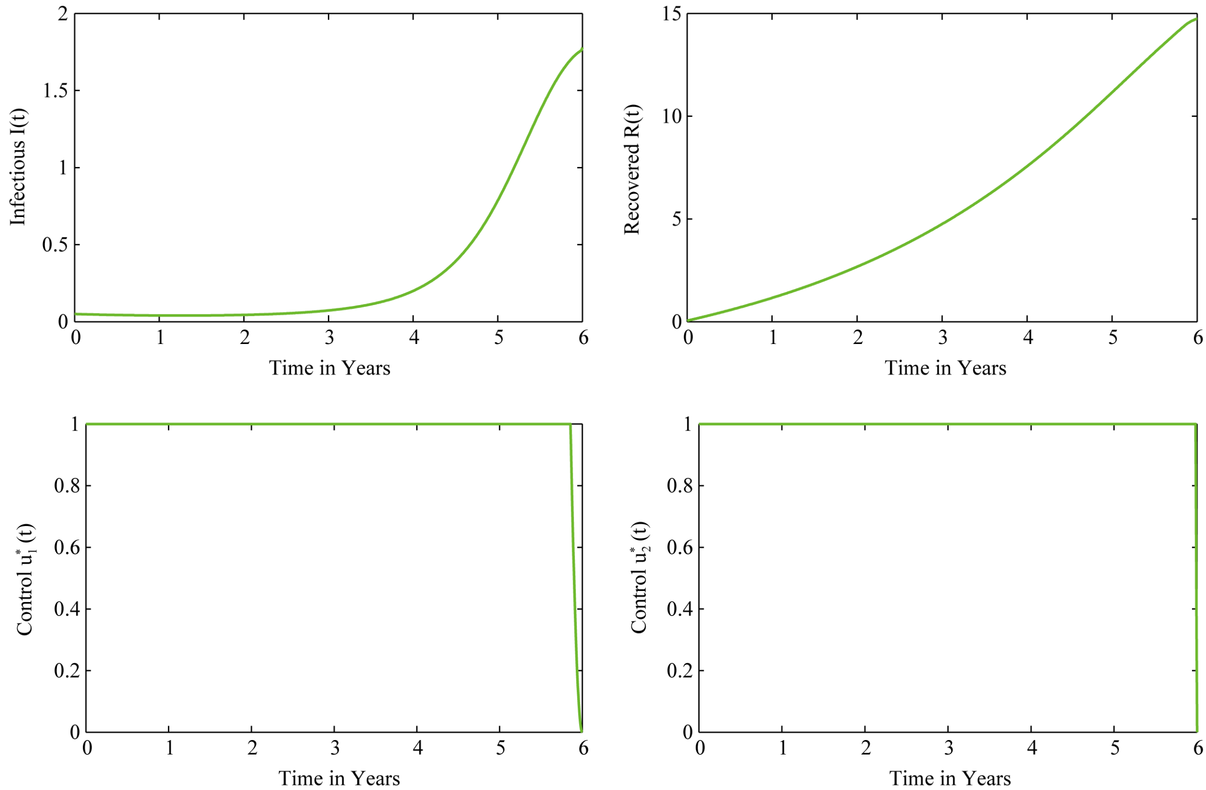

Figure 6. NiV model with low weight parameters, $B_1 = 0.2$, $B_2 = 0.3$ and other parameter values are taken from **Table 1**.

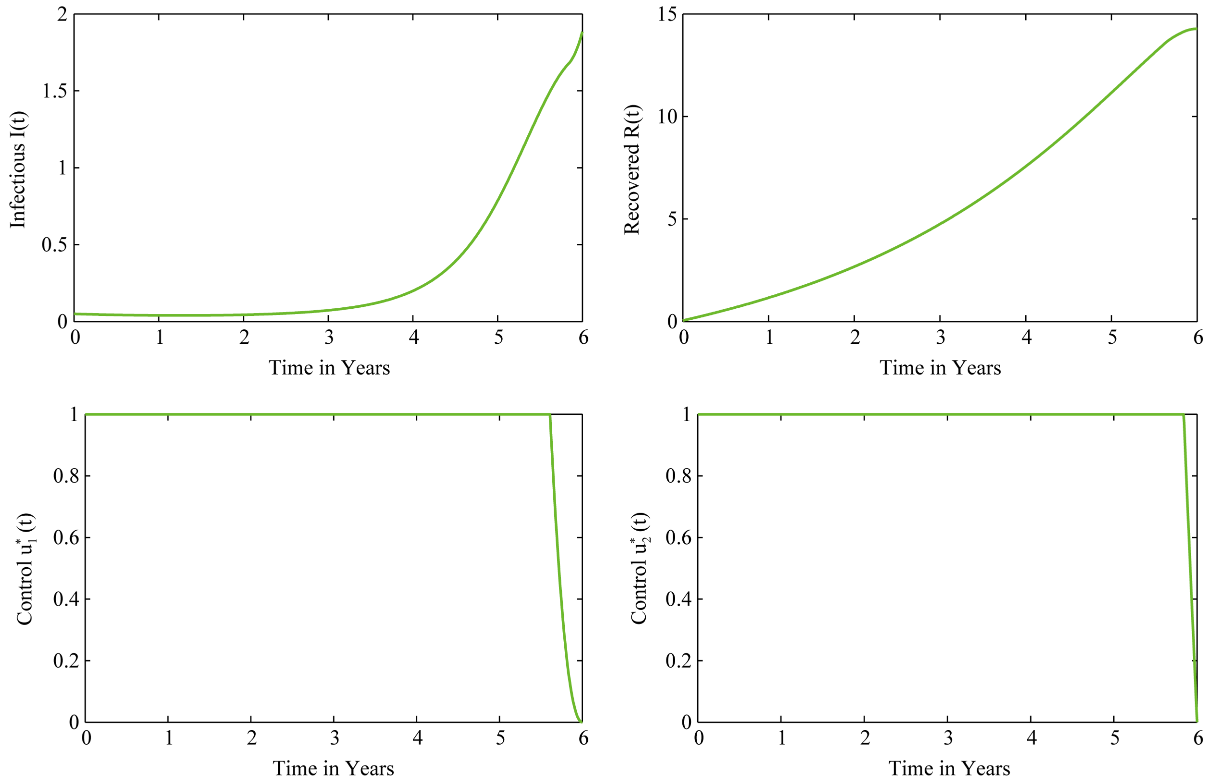

Figure 7. NiV model with high weight parameters, $B_1 = 2$, $B_2 = 3$ and other parameter values are taken from **Table 1**.

weight parameters ($B_1 = 0.2$, $B_2 = 0.3$) the infectious individuals decrease sharply for first few years (as the controls work at maximum level). It is also noticed that the infected individuals start to increase when the effectiveness of the controls start to decrease.

In the case of high weight parameter values ($B_1 = 2$, $B_2 = 3$) the high effectiveness of the controls are monitored and as a result there is a sharp reduction of infection during that effective level.

7. Conclusions

The important findings are given below:

- A comparison between with and without control strategy is monitored. The effect of control parameters is very much notable for reducing the infected individuals to control the disease dynamics.
- The controls need to be effective for longer period of time in case of high incidence.
- The optimal control is much more effective to minimize the infected individuals (as a result recovered individuals will be maximized) and also to minimize the cost of the two control measures.
- For low weight parameter values, the controls show their effectiveness at a maximum level.
- From the simulations it is monitored that the optimal combination of treatment and creating awareness is very prominent for disease elimination.

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Transformation Formulas for the First Kind of Lauricella's Function of Several Variables

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Abstract

Very recently Atash and Al-Gonah [1] derived two extension formulas for Lauricella's function of the second kind of several variables $F_B^{(2r+1)}$ and $F_B^{(2r)}$. Now in this research paper we derive two families of transformation formulas for the first kind of Lauricella's function of several variables $F_A^{(2r+1)}$ and $F_A^{(2r)}$ with the help of generalized Dixon's theorem on the sum of the series ${}_3F_2(1)$ obtained earlier by Lavoie *et al.* [2]. Some new and known results are also deduced as applications of our main formulas.

Keywords

Transformation Formulas, Lauricella's function, Dixon's Theorem, Kampé de Fériet Function

1. Introduction

In 1994, Lavoie *et al.* [2], obtained the following generalization of the classical Dixon's theorem for the series ${}_3F_2(1)$:

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$$
{}_3F_2\left[\begin{matrix} a,b,c; \\ 1+a-b+i,1+a-c+i+j; \end{matrix} 1\right]
$$

$$
= \frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma\left(b-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(c-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b)\Gamma(c)\Gamma(1+a-2c+i+j)\Gamma(1+a-b-c+i+j)}
$$

$$
\times\left\{A_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+1+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-b+1+\left[\frac{i}{2}\right]\right)}\right. \tag{1.1}
$$

$$
\left.+B_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+1+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{1}{2}a-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)}\right\},
$$

$$
\{\mathrm{Re}(a-2b-2c)>-2-2i-j;\ i=-3,-2,-1,0,1,2;\ j=0,1,2,3\},
$$

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients $A_{i,j}$ and $B_{i,j}$ are given respectively in [2]. When $i=j=0$, (1.1) reduces immediately to the classical Dixon's theorem [3], (see also [4])

$$
{}_3F_2\left[\begin{matrix} a,b,c; \\ 1+a-b,1+a-c; \end{matrix} 1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma(1+a)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma(1+a-b-c)} \tag{1.2}
$$

$$
\{\mathrm{Re}(a-2b-2c)>-2\}.
$$

We recall that the first kind of the Lauricella hypergeometric function of $(2r+1)$-variables $F_A^{(2r+1)}$ is defined as [5]:

$$
F_A^{(2r+1)}(a,b,b_1,c_1,\cdots,b_r,c_r;d,d_1,e_1,\cdots,d_r,e_r;x,x_1,y_1,\cdots,x_r,y_r)
$$

$$
= \sum_{m,m_1,n_1,\cdots,m_r,n_r=0}^{\infty} \frac{(a)_{m+(m_1+n_1)+\cdots+(m_r+n_r)}(b)_m(b_1)_{m_1}(c_1)_{n_1}\cdots(b_r)_{m_r}(c_r)_{n_r}}{(d)_m(d_1)_{m_1}(e_1)_{n_1}\cdots(d_r)_{m_r}(e_r)_{n_r}} \tag{1.3}
$$

$$
\times\frac{x^m}{m!}\frac{x_1^{m_1}}{m_1!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{x_r^{m_r}}{m_r!}\frac{y_r^{n_r}}{n_r!}
$$

$$
|x|+|x_1|+|y_1|+\cdots+|x_r|+|y_r|<1,
$$

where $(a)_n$ is the Pochhammer's symbol defined by [5]

$$
(a)_n=\begin{cases}1, & \text{if } n=0\\ a(a+1)(a+2)\cdots(a+n-1), & \text{if } n=1,2,3,\cdots\end{cases} \tag{1.4}
$$

When $x=0$, (1.3) reduces to the Lauricella function of $2r$-variables $F_A^{(2r)}$

$$
F_A^{(2r)}(a,b_1,c_1,\cdots,b_r,c_r;d_1,e_1,\cdots,d_r,e_r;x_1,y_1,\cdots,x_r,y_r)
$$

$$
= \sum_{m_1,n_1,\cdots,m_r,n_r=0}^{\infty} \frac{(a)_{(m_1+n_1)+\cdots+(m_r+n_r)}(b_1)_{m_1}(c_1)_{n_1}\cdots(b_r)_{m_r}(c_r)_{n_r}}{(d_1)_{m_1}(e_1)_{n_1}\cdots(d_r)_{m_r}(e_r)_{n_r}}\frac{x_1^{m_1}}{m_1!}\frac{y_1^{n_1}}{n_1!}\cdots\frac{x_r^{m_r}}{m_r!}\frac{y_r^{n_r}}{n_r!} \tag{1.5}
$$

$$|x_1|+|y_1|+\cdots+|x_r|+|y_r|<1.$$

Clearly, we have $F_A^{(2)}=F_2$, where F_2 is Appell's double hypergeometric function [5]

$$F_2(a,b,c;d,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.6}$$

Next, we recall that the generalized Lauricella function of several variables is defined as [5]:

$$\begin{aligned}
&F\begin{matrix}A:B';\cdots;B^{(n)}\\C:D';\cdots;D^{(n)}\end{matrix}[z_1,\cdots,z_n]\\
&\equiv F\begin{matrix}A:B';\cdots;B^{(n)}\\C:D';\cdots;D^{(n)}\end{matrix}\left(\begin{matrix}\left[(a):\theta',\cdots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\cdots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\cdots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\cdots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\; z_1,\cdots,z_n\right)\\
&=\sum_{m_1,\cdots,m_n=0}^{\infty}\Omega(m_1,\cdots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},
\end{aligned} \tag{1.7}$$

where

$$\Omega(m_1,\cdots,m_n)=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta'_j+\cdots+m_n\theta_j^{(n)}}\prod_{j=1}^{B'}(b'_j)_{m_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi'_j+\cdots+m_n\psi_j^{(n)}}\prod_{j=1}^{D'}(d'_j)_{m_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{m_n\delta_j^{(n)}}} \tag{1.8}$$

the coefficients $\theta_j^{(k)}$, $j=1,2\cdots,A;\phi_j^{(k)}$, $j=1,2,\cdots,B^{(k)}$; $\psi_j^{(k)}$, $j=1,2,\cdots,C$; $\delta_j^{(k)}, j=1,2,\cdots,D^{(k)}$; for all $k\in\{1,2,\cdots,n\}$ are real and positive; (a) abbreviates the array of A parameters; $a_1,\cdots,a_A,\left(b^{(k)}\right)$ abbreviates the array of $B^{(k)}$ parameters $b_j^{(k)}, j=1,2,\cdots,B^{(k)}$ for all $k\in\{1,2,\cdots,n\}$ with similar interpretations for (c) and $\left(d^{(k)}\right)$ $k\in1,2,\cdots,n$; *et cetera*. Note that, when the coefficients in Equation (1.7) equal to 1, the generalized Lauricella function (1.7) reduces to the following multivariable extension of the Kampé de Fériet function [5]:

$$\begin{aligned}
F\begin{matrix}p:q_1;\cdots;q_n\\l:m_1;\cdots;m_n\end{matrix}[z_1,\cdots,z_n]&\equiv F\begin{matrix}p:q_1;\cdots;q_n\\l:m_1;\cdots;m_n\end{matrix}\left(\begin{matrix}(a_p):(b'_{q_1});\cdots;\left(b_{q_n}^{(n)}\right);\\(c_l):(d'_{m_1});\cdots;\left(d_{m_n}^{(n)}\right);\end{matrix}\; z_1,\cdots,z_n\right)\\
&=\sum_{s_1,\cdots,s_n=0}^{\infty}\Omega(s_1,\cdots,s_n)\frac{z_1^{s_1}}{s_1!}\cdots\frac{z_n^{s_n}}{s_n!},
\end{aligned} \tag{1.9}$$

where

$$\Omega(s_1,\cdots,s_n)=\frac{\prod_{j=1}^{p}(a_j)_{s_1+\cdots+s_n}\prod_{j=1}^{q_1}(b'_j)_{s_1}\cdots\prod_{j=1}^{q_n}\left(b_j^{(n)}\right)_{s_n}}{\prod_{j=1}^{l}(c_j)_{s_1+\cdots+s_n}\prod_{j=1}^{m_1}(d'_j)_{s_1}\cdots\prod_{j=1}^{m_n}\left(d_j^{(n)}\right)_{s_n}}. \tag{1.10}$$

In our present investigation, we shall require the following results [5]:

$$(a)_{m+n}=(a)_m(a+m)_n \tag{1.11}$$

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\quad a\neq0,-1,-2,\cdots \tag{1.12}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma(1+a)=2^{a}\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)\Gamma\left(1+\frac{1}{2}a\right) \tag{1.13}$$

$$(a)_{2n}=2^{2n}\left(\frac{1}{2}a\right)_{n}\left(\frac{1}{2}a+\frac{1}{2}\right)_{n} \tag{1.14}$$

$$\frac{\Gamma(a-n)}{\Gamma(a)}=\frac{(-1)^{n}}{(1-a)_{n}} \tag{1.15}$$

$$(2n)!=2^{2n}\left(\frac{1}{2}\right)_{n}n! \tag{1.16}$$

2. Main Result

In this section, the following transformation formula will be established:

Theorem 2.1. For $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$, the following formula for Lauricella's function $F_A^{(2r+1)}$ holds true:

$$\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1-i,b_1,\cdots,b_r-i,b_r;c,c_1,c_1+i+j,\cdots,c_r,c_r+i+j;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\quad\times H_1(b_1,c_1,i,j,2m_1)\left\{A'_{i,j}A_1(b_1,c_1,i,j,2m_1)+B'_{i,j}B_1(b_1,c_1,i,j,2m_1)\right\}\times\cdots\\
&\quad\times H_r(b_r,c_r,i,j,2m_r)\left\{A'_{i,j}A_r(b_r,c_r,i,j,2m_r)+B'_{i,j}B_r(b_r,c_r,i,j,2m_r)\right\}+\cdots\\
&\quad+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}x^m x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c)_m(c_1)_{2m_1+1}\cdots(c_r)_{2m_r+1}m!(2m_1+1)!\cdots(2m_r+1)!}\\
&\quad\times H_1(b_1,c_1,i,j,2m_1+1)\left\{A''_{i,j}A_1(b_1,c_1,i,j,2m_1+1)+B''_{i,j}B_1(b_1,c_1,i,j,2m_1+1)\right\}\times\cdots\\
&\quad\times H_r(b_r,c_r,i,j,2m_r+1)\left\{A''_{i,j}A_r(b_r,c_r,i,j,2m_r+1)+B''_{i,j}B_r(b_r,c_r,i,j,2m_r+1)\right\}
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
H_r(b_r,c_r,i,j,m_r)&=2^{2(m_r+c_r-1)+i+j}\Gamma(1-b_r+i-m_r)\Gamma(c_r+i+j)\\
&\quad\times\frac{\Gamma\left(b_r-\frac{1}{2}|i|-\frac{1}{2}i\right)\Gamma\left(1-c_r-m_r-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b_r)\Gamma(1-c_r-m_r)\Gamma(2c_r-1+i+j+m_r)\Gamma(c_r-b_r+i+j)}
\end{aligned} \tag{2.2}$$

$$A_r(b_r,c_r,i,j,m_r)=\frac{\Gamma\left(\frac{1}{2}m_r+c_r-\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r-b_r+c_r+i+\left[\frac{j+1}{2}\right]\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{2}m_r\right)\Gamma\left(1-b_r-\frac{1}{2}m_r+\left[\frac{i}{2}\right]\right)} \tag{2.3}$$

$$B_r(b_r,c_r,i,j,m_r)=\frac{\Gamma\left(\frac{1}{2}m_r+c_r+\left[\frac{i+j}{2}\right]\right)\Gamma\left(\frac{1}{2}m_r-b_r+c_r+\frac{1}{2}+i+\left[\frac{j}{2}\right]\right)}{\Gamma\left(-\frac{1}{2}m_r\right)\Gamma\left(-\frac{1}{2}m_r-b_r+\frac{1}{2}+\left[\frac{i+1}{2}\right]\right)} \tag{2.4}$$

The coefficients $A'_{i,j}$ and $B'_{i,j}$ can be obtained from the tables of $A_{i,j}$ and $B_{i,j}$ given in [2] by replacing a and c by $-2m_r$ and $1-c_r-2m_r$, also the coefficients $A''_{i,j}$ and $B''_{i,j}$ can be obtained from the same tables of $A_{i,j}$ and $B_{i,j}$ by replacing a and c by $-2m_r-1$ and $-c_r-2m_r$ respectively.

Proofs.

In order to prove the Theorem 2.1, let us first prove the following result:

$$F_2(a,b,c;d,e;x,-x)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m x^m}{(d)_m m!}\,{}_3F_2\left[\begin{matrix}-m,b,1-d-m;\\1-b-m,d;\end{matrix}\;1\right] \tag{2.5}$$

To prove (2.5), denoting the left hand side of (2.5) by I, expanding $F_2(x,-x)$ in a power series as in (1.6) and using the result [5]:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n),$$

we have

$$I=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(a)_m(b)_{m-n}(c)_n(-1)^n x^m}{(d)_{m-n}(e)_n(m-n)!n!}.$$

Now, using the elementary identities [5]

$$(a)_{m-n}=\frac{(-1)^n(a)_m}{(1-a-m)_n},0\le n\le m$$

$$(m-n)!=\frac{(-1)^n m!}{(-m)_n},\ 0\le n\le m,$$

we have

$$I=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_m x^m}{(d)_m m!}\,{}_3F_2\left[\begin{matrix}-m,b,1-d-m;\\1-b-m,d;\end{matrix}\;1\right].$$

This completes the proof of (2.5).

***Proof of Theorem* 2.1.** Denoting the left hand side of (2.1) by S, expanding $F_A^{(2r+1)}$ in a power series as in (1.3), adjusting the parameters, using the results (1.11) and (2.5) and by repeating this procedure r-times, we have

$$S=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+m_1+\cdots+m_r}(b)_m(b_1-i)_{m_1}\cdots(b_r-i)_{m_r}x^m x_1^{m_1}\cdots x_r^{m_r}}{(c)_m(c_1)_{m_1}\cdots(c_r)_{m_r}m!m_1!\cdots m_r!}$$
$$\times f_1(b_1,c_1,i,j,m_1)\times\cdots\times f_r(b_r,c_r,i,j,m_r)$$

where

$$f_r(b_r,c_r,i,j,m_r)={}_3F_2\left[\begin{matrix}-m_r,b_r,1-c_r-m_r;\\1-b_r+i-m_r,c_r+i+j;\end{matrix}\;1\right]$$

Now, separating into even and odd powers of $(x_i,i=1,\cdots,r)$ by using the elementary identity [5]

$$\sum_{n=0}^{\infty}A(n)=\sum_{n=0}^{\infty}A(2n)+\sum_{n=0}^{\infty}A(2n+1),$$

we have

$$S=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r)$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+2m_2+\cdots+2m_r}(b)_m(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\cdots(b_r-i)_{2m_r}}{(c)_m(c_1)_{2m_1+1}(c_2)_{2m_2}\cdots(c_r)_{2m_r}}$$
$$\times\frac{x^m x_1^{2m_1+1}x_2^{2m_2}\cdots x_r^{2m_r}}{m!(2m_1+1)!(2m_2)!\cdots(2m_r)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1+1)\times f_2(b_2,c_2,i,j,2m_2)\times\cdots\times f_r(b_r,c_r,i,j,2m_r)+\cdots$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+2m_2+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\cdots(b_r-i)_{2m_r+1}}{(c)_m(c_1)_{2m_1}(c_2)_{2m_2+1}\cdots(c_r)_{2m_r+1}}$$
$$\times\frac{x^m x_1^{2m_1}x_2^{2m_2+1}\cdots x_r^{2m_r+1}}{m!(2m_1)!(2m_2+1)!\cdots(2m_r+1)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1)\times f_2(b_2,c_2,i,j,2m_2+1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r+1)$$
$$+\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+1+2m_2+1+\cdots+2m_r+1}(b)_m(b_1-i)_{2m_1+1}(b_2-i)_{2m_2+1}\cdots(b_r-i)_{2m_r+1}}{(c)_m(c_1)_{2m_1+1}(c_2)_{2m_2+1}\cdots(c_r)_{2m_r+1}}$$
$$\times\frac{x^m x_1^{2m_1+1}x_2^{2m_2+1}\cdots x_r^{2m_r+1}}{m!(2m_1+1)!(2m_2+1)!\cdots(2m_r+1)!}$$
$$\times f_1(b_1,c_1,i,j,2m_1+1)\times f_2(b_2,c_2,i,j,2m_2+1)\times\cdots\times f_r(b_r,c_r,i,j,2m_r+1).$$

Finally, if we use the result (1.1), then we obtain the right hand side of the Theorem 2.1. This completes the proof of the Theorem 2.1.

Remark. Taking $x = 0$ in (2.1), we deduce the following formulas:

Corollary 2.1. For $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$, the following formula for Lauricella's function $F_A^{(2r)}$ holds true:

$$F_A^{(2r)}(a,b_1-i,b_1,\cdots,b_r-i,b_r;c_1,c_1+i+j,\cdots,c_r,c_r+i+j;x_1,-x_1,\cdots,x_r,-x_r)$$
$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+\cdots+2m_r}(b_1-i)_{2m_1}\cdots(b_r-i)_{2m_r}x_1^{2m_1}\cdots x_r^{2m_r}}{(c_1)_{2m_1}\cdots(c_r)_{2m_r}(2m_1)!\cdots(2m_r)!}$$
$$\times H_1(b_1,c_1,i,j,2m_1)\left\{A'_{i,j}A_1(b_1,c_1,i,j,2m_1)+B'_{i,j}B_1(b_1,c_1,i,j,2m_1)\right\}\times\cdots$$
$$\times H_r(b_r,c_r,i,j,2m_r)\left\{A'_{i,j}A_r(b_r,c_r,i,j,2m_r)+B'_{i,j}B_r(b_r,c_r,i,j,2m_r)\right\}+\cdots \tag{2.6}$$
$$+\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{2m_1+1+\cdots+2m_r+1}(b_1-i)_{2m_1+1}\cdots(b_r-i)_{2m_r+1}x_1^{2m_1+1}\cdots x_r^{2m_r+1}}{(c_1)_{2m_1+1}\cdots(c_r)_{2m_r+1}(2m_1+1)!\cdots(2m_r+1)!}$$
$$\times H_1(b_1,c_1,i,j,2m_1+1)\left\{A''_{i,j}A_1(b_1,c_1,i,j,2m_1+1)+B''_{i,j}B_1(b_1,c_1,i,j,2m_1+1)\right\}\times\cdots$$
$$\times H_r(b_r,c_r,i,j,2m_r+1)\left\{A''_{i,j}A_r(b_r,c_r,i,j,2m_r+1)+B''_{i,j}B_r(b_r,c_r,i,j,2m_r+1)\right\}$$

3. Applications

1) In (2.1) if we take $r = 1$, then we get a known extension formulas [6] for Lauricella's function of three variables $F_A^{(3)}(a,b,b_1-i,b_1;c,c_1,c_1+i+j;x,x_1,-x_1)$ for $i=\{-3,-2,-1,0,1,2;\ j=0,1,2,3\}$.

2) In (2.1), if we take $i=j=0$, we have

$$
\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,c_1,c_1,\cdots,c_r,c_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1)_{2m_1}\cdots(b_r)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\quad\times H_1\left(b_1,c_1,2m_1\right)A_1\left(b_1,c_1,2m_1\right)\times\cdots\cdots\times H_r\left(b_r,c_r,2m_r\right)A_r\left(b_r,c_r,2m_r\right)\\
&=\sum_{m=0}^{\infty}\sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(a)_{m+2m_1+\cdots+2m_r}(b)_m(b_1)_{2m_1}\cdots(b_r)_{2m_r}x^m x_1^{2m_1}\cdots x_r^{2m_r}}{(c)_m(c_1)_{2m_1}\cdots(c_r)_{2m_r}m!(2m_1)!\cdots(2m_r)!}\\
&\quad\times\frac{2^{2m_1}\Gamma(c_1)\Gamma(1-b_1-2m_1)\Gamma(c_1-b_1+m_1)\Gamma\left(\frac{1}{2}\right)}{\Gamma(c_1-b_1)\Gamma(c_1+m_1)\Gamma(1-b_1-m_1)\Gamma\left(\frac{1}{2}-m_1\right)}\times\cdots\\
&\quad\times\frac{2^{2m_r}\Gamma(c_r)\Gamma(1-b_r-2m_r)\Gamma(c_r-b_r+m_r)\Gamma\left(\frac{1}{2}\right)}{\Gamma(c_r-b_r)\Gamma(c_r+m_r)\Gamma(1-b_r-m_r)\Gamma\left(\frac{1}{2}-m_r\right)}.
\end{aligned}
\tag{3.1}
$$

Now, in (3.1) if we use the results (1.12)-(1.16) and simplify, we obtain the following transformation formula:

$$
\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,c_1,c_1,\cdots,c_r,c_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{1:1;2;\cdots;2}_{0:1;3;\cdots;3}\left[\begin{matrix}(a:1,2,\cdots,2):(b:1); & (b_1:1),(c_1-b_1:1)\\ ---- \quad :(c:1); & (c_1:1),\left(\frac{1}{2}c_1:1\right),\left(\frac{1}{2}c_1+\frac{1}{2}:1\right)\end{matrix};\right.\\
&\qquad\left.\begin{matrix}\cdots; & (b_r:1),(c_r-b_r:1)\\ \cdots; & (c_r:1),\left(\frac{1}{2}c_r:1\right),\left(\frac{1}{2}c_r+\frac{1}{2}:1\right)\end{matrix};x,\frac{x_1^2}{4},\cdots,\frac{x_r^2}{4}\right]
\end{aligned}
\tag{3.2}
$$

which for $c_1=2b_1,c_2=2b_2,\cdots,c_r=2b_r$, reduces to

$$
\begin{aligned}
&F_A^{(2r+1)}\left(a,b,b_1,b_1,\cdots,b_r,b_r;c,2b_1,2b_1,\cdots,2b_r,2b_r;x,x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{1:1;1;\cdots;1}_{0:1;2;\cdots;2}\left[\begin{matrix}(a:1,2,\cdots,2):(b:1); & (b_1:1) & ;\cdots; & (b_r:1)\\ ---- \quad :(c:1); & (2b_1:1),\left(b_1+\frac{1}{2}:1\right) & ;\cdots; & (2b_r:1),\left(b_r+\frac{1}{2}:1\right)\end{matrix};x,\frac{x_1^2}{4},\cdots,\frac{x_r^2}{4}\right]
\end{aligned}
\tag{3.3}
$$

3) Similarly, in (2.6), if we take $i=j=0$, we have

$$
\begin{aligned}
&F_A^{(2r)}\left(a,b_1,b_1,\cdots,b_r,b_r;c_1,c_1,\cdots,c_r,c_r;x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{2:2;\cdots;2}_{0:3;\cdots;3}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1,c_1-b_1 & ;\cdots; & b_r,c_r-b_r\\ - \quad : & c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2} & ;\cdots; & c_r,\frac{1}{2}c_r,\frac{1}{2}c_r+\frac{1}{2}\end{matrix};x_1^2,\cdots,x_r^2\right]
\end{aligned}
\tag{3.4}
$$

which is a generalization of a known result of Bailey [7]

$$
F_2\left[a,b_1,b_1;c_1,c_1;x_1,-x_1\right]={}_4F_3\left[\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2},b_1,c_1-b_1;c_1,\frac{1}{2}c_1,\frac{1}{2}c_1+\frac{1}{2};x_1^2\right].
\tag{3.5}
$$

Further, in (3.4) if we take $c_1=2b_1,c_2=2b_2,\cdots,c_r=2b_r$, then we get

$$
\begin{aligned}
&F_A^{(2r)}\left(a,b_1,b_1,\cdots,b_r,b_r;2b_1,2b_1,\cdots,2b_r,2b_r;x_1,-x_1,\cdots,x_r,-x_r\right)\\
&=F^{2:1;\cdots;1}_{0:2;\cdots;2}\left[\begin{matrix}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2}: & b_1 & ;\cdots; & b_r\\ - \quad : & 2b_1,b_1+\frac{1}{2} & ;\cdots; & 2b_r,b_r+\frac{1}{2}\end{matrix};x_1^2,\cdots,x_r^2\right]
\end{aligned}
\tag{3.6}
$$

4. Conclusion

We conclude our present investigation by remarking that the main results established in this paper can be applied to obtain a large number of transformation formulas for the first kind of Lauricella's function of several variables $F_A^{(n)}$. Further, in the formulas (2.1) and (2.6), if we take $c_1 = 2b_1, c_2 = 2b_2, \cdots, c_r = 2b_r$, then we can obtain two new families of transformation formulas for Lauricella's functions of several variables

$$F_A^{(2r+1)}\left(a, b, b_1 - i, b_1, \cdots, b_r - i, b_r; c, 2b_1, 2b_1 + i + j, \cdots, 2b_r, 2b_r + i + j; x, x_1, -x_1, \cdots, x_r, -x_r\right)$$

and

$$F_A^{(2r)}\left(a, b_1 - i, b_1, \cdots, b_r - i, b_r; 2b_1, 2b_1 + i + j, \cdots, 2b_r, 2b_r + i + j; x_1, -x_1, \cdots, x_r, -x_r\right)$$

for $\{i = -3, -2, -1, 0, 1, 2; j = 0, 1, 2, 3\}$.

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The Field of Logistics Warehouse Layout Analysis and Research

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Abstract

Logistics warehouse layout problem in circulation is one of the issues the enterprise concerned; storage as a logistics hub for transport links plays a role in the protection of stored products. During storage planning, in order to design a reasonable layout of warehouses, companies must consider seasonal product, quantity demand characteristics and the treasury itself highly characteristic and so on. Handling the number of products and their performance, will certainly affect the design of the warehouse place; at the same time, with seasonal variations, warehouse receipt must not only be able to accommodate the capacity of the warehouse area and height for design, but also consider that forklift and artificial operation need some work area, and it cannot affect normal operation of the product by placeholder area.

Keywords

Warehouse Layout, Storage Area, Work Functioning

1. Introduction

In the field of logistics, the vendors need to provide goods; manufacturers and processors need to process products and distribution centers are for goods distribution. Goods will reach the final retail shops or consumers. This process is called logistics. In the range of logistics, there must be a cycling of goods in the circulation process. It is often that goods are not directly to reach the hands of consumers, or it is impossible in the process of transportation without interruption all the time, which is bound to have built warehouses for storage. The spatial layout's design is very important and which parts it can put goods will directly or indirectly affect the efficiency of commodity circulation stream, or affect the receipt time of goods, and even influence the service level of enterprise. Thus, it will affect the volatility of customer satisfaction. Therefore, it is very important to design warehousing and distribution; warehousing and internal displaying should be strictly carried out according to

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certain rules. It means that which goods need to be placed near the receiving area, which goods require stacking merchandise, companies should consider these issues as important factors. The field of logistics warehousing is a transport hub of logistics; if the storage space cannot be planned well, it would certainly have trouble in receiving and shipping. Thus it must affect the efficiency of logistics and transport or increase logistics costs. Finally, it will result in a downturn of the economy. Therefore, studying of the storage warehouse layout is very important in the field of logistics, such as the classification of warehousing, and the designing of goods.

2. Current Situation

Nowadays there are many scholars in the field of logistics warehouse layout in-depth study. Jiang and Feng Dingzhong cents (2013) two scholars from a wide range of point of view, the use of Matlab mathematical program run, calculate the warehouse layout most suitable angle for picking system to shorten the distance and time issues to provide a reliable theoretical proof [1]. Zhu Jianghong (2011) scholar for warehouse layout problems, from real-life examples of logistics park view, were the establishment of the index system, and ultimately the use of fuzzy mathematical evaluation proved the feasibility of the program [2]. Yang Xuechun and Mao Xiaofang two scholars (2014) for the storage of the problems, from the layout, process design, placement of goods warehouse in-depth study [3]. Xuan Ying and Chen (2013) for the Reserve for military supplies, the use of mathematical models of genetic algorithms, reasonable conclusions [4]. Li Yumin and Zhang Li (2012) for the rational distribution of pharmaceutical warehouse, logistics analysis using SLP, obtained important design warehouse layout [5]. Jiang Hua and Yang Maosheng (2008) for the external and internal environment of the warehouse, a comprehensive factor analysis, the rational distribution warehouse design [6]. According to many scholars research, we can know that is the importance of warehouse people can not be ignored. Warehousing logistics of agricultural products, aquatic products and other material storage and transportation, if can't good design, will bring items lost in the process of storage and transportation quality to drop, employees in the warehouse work efficiency could be greatly reduced. The purpose of this article and the above the breakthrough point is different, in this paper, the main innovation point is based on the size of the storage area, storage area computation formula, on the basis of the original study, joined the turnover of goods and factor analysis, make the calculation formula of storage area is more appropriate, and scholars differ from the point of view of the types of goods.

3. Warehouse Layout Considerations Analysis

Traditional companies often choose the storage layout PQRST layout analysis and research methods, PQRST method is the enterprise often select method, the method for design warehouse space, is in the enterprise, on the basis of the original data after large amounts of data statistics, a detailed classification and combination, for example, the demand for goods and goods handling times are combined to illustrate the location of the goods, etc., the factors considered in this method is not very comprehensive, PQRST method of storage of goods only from itself, just consider the number of moving goods demand, goods, the flow of goods, etc., this approach ignores the effective use of storage area of the warehouse, without the storage height and other factors into account and other channels, often for minor changes to the role of the warehouse. Therefore, this article from the storage area of this factor of view, the cargo area to classify optimization, taking into account the effective height of the Treasury, the effective channel, tray and shelf usage, etc., factors to consider more comprehensive, so that the internal layout of the warehouse more reasonable.

3.1. Auxiliary Facilities Planning Area

Storage area generally can be divided into two parts, one is the cargo storage area, area of this part is to meet footprint of goods, while the other part is that every business needs some fixing placeholder, for example: receiving areas for where to take delivery, delivery area, the place used to send goods, scan area, companies in order to record incoming and outgoing goods timely information system to facilitate the subsequent search, the scan area will be set up to facilitate the company's management. Set placeholder cargo area is also essential, because shipping and receiving, the company's personnel can not be placed directly on a shelf or storage location to the goods, because a large amount of goods, only the first set of goods stored in the zone, and then another part of the staff to go sorting goods. Work area where employees are managers doing things, often can not be ignored placeholder, such as supervision and management system, must have a fixed office space. Mentioned above it is essential for enterprise storage in the auxiliary facilities.

For auxiliary facilities in warehousing arrangement, under normal circumstances, we follow the process from starting, such as a business process after the storage of goods to be timely scan goods scanned are placed into the storage area for storage, storage after the classification of goods to be placed into their respective positions, waiting for demand comes after the timely collection of goods ready to ship cargo scanning. Under this process, the area of design aids in the process tend to be close to the department or footprint together to facilitate the management and shipping and receiving. For these regions, companies must be in the process of designing the warehouse, the shipping and receiving area, storage area, working out of such advance planning, so there is a macro to control, according to pre-planned area, the transceiver cargo area reasonable set according to the shape of warehousing. After the whole enterprise should be classified storage area, divided into equal units of the cell, and the cell area is clear how much, according to a work area required number of cells can be designed to meet their place.

3.2. Storage Area of Planning

In this paper, the above content is analyzed and summarized for auxiliary facilities in an area designed for the storage area of the goods, we often have to consider that the goods stacked way, because, if the goods are directly stacked on the spatial pattern of the Treasury, then enterprises only consider this product can be placed per unit area and commodity height and maximum load capacity of goods. If the goods pallet storage is required, then you need to consider how much a commodity tray to store a number of cargo storage tray can be factors in addition to the shelf area is to be considered. In addition, we have to consider whether you can put convergence between different goods, which requires an order of relevance systematic analysis, the correlation between the cargo and the cargo of a group of products is very high, you can consider whether put together, play a role in connecting and convenient shipping of goods.

For an area of storage, companies often have to calculate the storage area, for the same unit of the good, the size of the storage area and is often related to the annual purchase amount of goods, and is proportional to the relationship because of the size of the purchase amount determines the warehouse area size. In addition, the size and store the number of days the storage area of the product also has a relationship for a commodity, the logistics distribution process, if the number of days in a warehouse to store more, it will affect other goods into the warehouse storage area, turnover tends Days determine the degree of flexibility of the warehouse. At the same time, storage area and also the employees working hours usage per unit area and a storage area of the load-bearing capacity is related to and is inversely proportional relationship. After design and analysis, will be transformed storage area is calculated as follows:

$$S = \frac{4M * T}{T_0 * q * W * P}$$

Meaning the formula is as follows:

S: Requirements of storage area was the area (m^2);

M: Each quarter the amount of goods into the library (t);

T: The average number of days to store merchandise;

q: Average effective carrying capacity storage area (t/m^2);

W: effective storage area utilization coefficient;

T_0: Workers working days a year;

P: the average annual turnover of goods.

We can analyze the results that the number of goods are into the library stored in the warehouse a few days of each quarter, and are proportional to the storage area of the inevitable relationship between the goods placed in the warehouse longer, it will inevitably take a lot of cargo storage area the same each quarter for the same commodities greater the amount of the purchase, will inevitably increase the storage area. In addition, for the carrying capacity of the warehouse unit, if the unit area can withstand the weight of large quantities of goods, you can put more goods cargo at the same height, so that the storage area occupied by smaller. Employee time also affects the enterprise storage area, employee long time, high efficiency and speed will make shipping and receiving goods faster, can largely meet the business running time and customer satisfaction services. The average turnover rate of goods determines the goods in cargo on time spent, the higher the turnover rate, the faster turnover of goods in the cargo space on the residence time is shortened, it will stay out of storage to another product area.

3.3. Treasury Planning Height

In theory, the height of the Treasury is the higher the better, however, since the load carrying capacity per unit area, often can not be unlimited increase also have to consider the optimal distance shelf height, the height of the Treasury issues. In general, the height of the Treasury to be designed in two ways. The first is the optimum height of the cargo, that is, without considering the height of the warehouse, the goods can be placed at the highest level and maximum distance, in addition, is the maximum distance from the Treasury, to set aside part of the distance to the Treasury Internal space staff check, sometimes in order to protect the product quality problems, but also to the warehouse environment supervision and commissioning, it is bound to installation of air conditioning appliances category, so the Treasury is bound to consider the height of the presence and influence of these factors.

But in real life, we inevitably have to consider the reservation forklift operation or manual operation. If the business is a way to take manual jobs in the design process, we must consider the workers put goods or pickup time whether there is sufficient channel area can be placed ladders and other tools, if the area is very small, high enough, it can not be easily taken out goods, affecting the quality of work of employees. If the business takes the job way forklift truck must be to allow sufficient space and access to meet the normal operation efficiency of the truck, we must also consider the effective height and the height of the truck can not conflict with the goods, must design Treasury reasonable height for easy handling and delivery of goods pickup truck.

4. Summary

The good or fall of logistics in the field of warehouse planning and inventory analysis will inevitably affect the enterprise operation efficiency and quality in the process of the flow of goods in the logistics. Enterprises also start from accepting the goods orders with PQRST method in the layout of the warehouse. They are able to predict the future demand according to the original data and design the suitable warehouse. There will be a big risk and an uncertain result of this method, because sometimes there are many seasonal goods and it will exist some random effects. So according to this method, to design the warehouse layout is not reasonable. According to the effective areas of warehouse and classification of the area for standardization, enterprises accept the goods orders and need to calculate the goods per unit area. Enterprises should choose the display shelves, forklift truck or other tools instead of artificial operation to reduce the working intensity and improve work efficiency.

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The Pricing of Convertible Bonds with a Call Provision

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Abstract

This paper deals with the pricing of convertible bond with call provision based on the traditional B-S formula. By applying the principle of no arbitrage, the partial differential equation for the bond is established with identified boundary conditions, which solution results in the closed form of the pricing formula.

Keywords

Convertible Bonds, Call Provision, B-S Formula

1. Introduction

Convertible bonds are complicated and broadly used financial instruments combining the characteristics of stocks and bonds. In recent years, convertible bond has become a new investment product for the investors. The possibility to convert the bond into a predetermined number of stocks offers the chance to participate in rising stock prices with limited loss, given that the issuer does not default on its bond obligation. Convertible bonds often contain other embedded options such as call and put provisions. These options can be specified in various different ways, which make the products more complex. Especially, conversion and call opportunities may occur in case of certain restricted time periods with given stock price conditions, which results in the changing of the call price with time. The study of the convertible bond has a long history, firstly appearing in 1843; however, the pricing theory is relatively backward. With the work of Black and Scholes (1973) [1] and Merton (1973) [2] on the option pricing theory (B-S formula), Ingersoll (1977) [3] and Brennan and Schwartz (1980) [4] were the first to apply the B-S formula in pricing the convertible bonds. Tsiveriotis and Fernandes (1998) [5] innovatively divided the convertible bonds into two parts including the stock option and the pure cash flow. Davis and Lischka (1999) [6] considered the price of convertible bonds affected by the default risk, and a more complex three-factor model was proposed. Li (2008) [7] derived the stochastic interest rate model for the Vasicek model and

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the CIR model of the convertible bond pricing formula; the simulation results show that the results of the CIR model in the market are more reasonable than those by using the Vasicek model. In the pricing of convertible bonds, the most widely used numerical pricing methods include tree graph method (such as the Binomial tree model and triple tree model), finite difference method, finite element method, Monte Carlo method [8]-[16] and so on.

The purpose of this paper is to study the pricing of convertible bonds with call provision. By using the method for option pricing, the closed form of the price for convertible bond is presented.

The paper is organized as follows. In Section 2, we introduce the content of convertible bond with call provision briefly, and establish the pricing model for the convertible bonds with call provision. In Section 3, we solve the partial differential equation established for the convertible bonds, and give the closed form solution. Section 4 is a conclusion.

2. Convertible Bond with the Call Provision

Convertible bond has the characteristics of the implied stock option, the convertible bond holder once decided to execute the options, it becomes the shareholders of the company, and the right has no difference to the original company shareholder. The main impact for the holders to convert or not is the underlying asset price, the bond holders often choose to continuously hold or immediately convert into shares of the company based on the stock prices, when stock prices continued to slump or located in a special given regime, the bonds holder always held in hand to the maturity time or directly sell it to other investors; when stock prices continue to rise or the conversion is profitable, the holders usually choose convert the bonds into shares, and the amount of the trading profits depends of the specific stock price in the market.

However, there is possible that its stock price in the market continues to rise, its converted value far exceeds the profit obtained by holding to the maturity, which, to some extent, has serious impact on the interests of the previously-existing shareholders of the company. Therefore, the benefit issuers can reduce their cost of the issuance through establishing the call provision, to avoid the loss due to stock price soaring and market interest rates. The call provisioncan accelerate the conversion process and relieve the company's financial pressure**.** Redemption usually occurs in case that the stock market price is far higher than the conversion price. When the company announces the redemption, bond holders usually immediately opted conversion to avoid loss.

Here, we employ a standard assumption that the stock price movement S_t meet geometric Brownian motion,

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t,$$

where μ is expect return rate (constant), σ is the volatility of the stock price, $\mathrm{d}W_t$ is standard Brownian motion. Since the value of convertible bonds is related to the stock price and time, we use $V(S,t)$ to represent convertible bonds value with call provision. When the stock price of the company rises to the barrier fixed in advance ($S = S_B$), the issuer announced the redemption of bonds. At this point, investors immediately implement the contract to convert the convertible bonds into stocks in order to obtain a higher interest. If one chooses to continue to hold the bond, he will get the bond value. When the stock price S reaches barrier value S_B, the convertible bond will be executed. Let D a solution area as follows

$$D = \left\{ (S,t) \middle| 0 \le S \le S_B, 0 \le t \le T \right\}. \tag{1.1}$$

Then the final revenue function of the convertible bonds value is

$$(S_T - K)^+ I_{\{S_t < S_B, t \in [0,T]\}}, \tag{1.2}$$

where S_T is stock price at maturity date, K is transforming shares price stipulated in the contract, S_B is stock price redemptive threshold stipulated by the issuer. Here $I_\omega(S)$ is the indicator function of ω (abbreviated as I_ω),

$$I_\omega(S) = \begin{cases} 1, & S \in \omega, \\ 0, & S \notin \omega. \end{cases} \tag{1.3}$$

Clearly, $V(S_B,t) = 0.$ The termination conditions on $t = T$ is

$$V(S,T) = (S-K)^+, \; 0 \le S \le S_B. \tag{1.4}$$

In the given region $D=\{(S,t)|0\le S\le S_B,0\le t\le T\}$, by using the classical method [17], Black-Scholes equation for the convertible bonds value is

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0. \tag{1.5}$$

To sum up, the pricing of convertible bonds with call provision is a specific boundary value problem for the Black-Scholes equation, which has similar properties of the up-and-out options to some extent. Compared with the standard options, it has more boundary conditions.

3. Pricing Model and Its Solution

The pricing process of convertible bonds is, in the area $D=\{(S,t)|0\le S\le S_B,0\le t\le T\}$, to solve the partial differential equation

$$\begin{cases}\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}+(r-q)S\frac{\partial V}{\partial S}-rV=0, & (D);\\ V(S,T)=(S-K)^+, & (0\le S\le S_B);\\ V(S_B,t)=0. & (0\le t\le T).\end{cases} \tag{1.6}$$

Make the transformation

$$x=\ln\frac{S}{S_B},\ V=S_Bu. \tag{1.7}$$

Then

$$\begin{cases}\frac{\partial u}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2u}{\partial x^2}+\left(r-q-\frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x}-ru=0, & (x\in\mathbb{R}_-,0\le t\le T);\\ u(x,T)=\left(e^x-K_B\right)^+ & (-\infty<x<0);\\ u(0,t)=0 & (0\le t\le T),\end{cases} \tag{1.8}$$

where $K_B=\frac{K}{S_B}$.

Define

$$u=e^{\alpha x+\beta(T-t)}W, \tag{1.9}$$

with

$$\alpha=-\frac{1}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right), \tag{1.10}$$

$$\beta=-r-\frac{1}{2\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)^2. \tag{1.11}$$

Based on (1.9), W is suitable for definite solution problems on the area $\{x\in\mathbb{R}_-,0\le t\le T\}$, we have

$$\begin{cases}\frac{\partial W}{\partial t}+\frac{1}{2}\sigma^2\frac{\partial^2W}{\partial x^2}=0;\\ W(x,T)=e^{-\alpha x}\left(e^x-K_B\right)^+;\\ W(0,t)=0.\end{cases} \tag{1.12}$$

By using the mirror methods, defining

$$\varphi(x)=\begin{cases}\mathrm{e}^{-\alpha x}\left(\mathrm{e}^{x}-K_B\right) & x<0;\\ -\mathrm{e}^{\alpha x}\left(\mathrm{e}^{-x}-K_B\right) & x>0.\end{cases}\tag{1.13}$$

It is clearly $\varphi(x)=-\varphi(-x)$, that is, $\varphi(x)$ is an odd-functions. Considering the Cauchy problem on the $\{x\in\mathbb{R},0\le t\le T\}$, one gets

$$\begin{cases}\dfrac{\partial W}{\partial t}+\dfrac{1}{2}\sigma^2\dfrac{\partial^2 W}{\partial x^2}=0 & (x\in\mathbb{R},0\le t\le T);\\ W(x,T)=\varphi(x) & (x\in\mathbb{R}),\end{cases}\tag{1.14}$$

which is a odd function. The limitations on $D:\{x\in\mathbb{R}_-,0\le t\le T\}$ will be suitable for solution of the problem (1.12).

The solution of Cauchy problem (1.14) can be expressed as the Poisson equation

$$\begin{aligned}W(x,t)&=\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}\varphi(\xi)\mathrm{d}\xi\\&=\frac{1}{\sigma\sqrt{2\pi(T-t)}}\left[\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}}\varphi(\xi)\mathrm{d}\xi+\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}\varphi(-\xi)\mathrm{d}\xi\right]\\&=\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\left[\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}-\mathrm{e}^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}}\right]\mathrm{e}^{-\alpha\xi}\left(\mathrm{e}^{\xi}-K_B\right)^+\mathrm{d}\xi\\&=\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\left[\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}-\mathrm{e}^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}}\right]\mathrm{e}^{-\alpha\xi}\left[\left(\mathrm{e}^{\xi}-K_B\right)+\left(K_B-\mathrm{e}^{\xi}\right)^+\right]\mathrm{d}\xi.\end{aligned}$$

Then

$$\begin{aligned}W(x,t)=&\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}\mathrm{e}^{-\alpha\xi}\left(\mathrm{e}^{\xi}-K_B\right)\mathrm{d}\xi\\&-\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}}\mathrm{e}^{-\alpha\xi}\left(\mathrm{e}^{\xi}-K_B\right)\mathrm{d}\xi\\&+\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x-\xi)^2}{2\sigma^2(T-t)}}\mathrm{e}^{-\alpha\xi}\left(K_B-\mathrm{e}^{\xi}\right)^+\mathrm{d}\xi\\&-\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{(x+\xi)^2}{2\sigma^2(T-t)}}\mathrm{e}^{-\alpha\xi}\left(K_B-\mathrm{e}^{\xi}\right)^+\mathrm{d}\xi.\end{aligned}\tag{1.15}$$

From (1.9), inserting $W(x,t)$ into $u(x,t)$ shows

$$u(x,t)=(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})$$

where

$$(\mathrm{I})=\frac{\mathrm{e}^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\left(\mathrm{e}^{\xi}-K_B\right)\mathrm{d}\xi$$

$$=\frac{\mathrm{e}^{x-q(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi-\frac{K_B}{\sigma\sqrt{2\pi(T-t)}}\mathrm{e}^{-r(T-t)}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi,$$

then

$$(\mathrm{I})=\frac{K_B}{\sqrt{2\pi}}\mathrm{e}^{-r(T-t)}\int_{-\infty}^{\frac{x+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega-\frac{\mathrm{e}^{x-q(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega,$$

$$(\mathrm{II})=-\frac{\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x+\xi)-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\left(\mathrm{e}^{\xi}-K_B\right)\mathrm{d}\xi$$

$$=\frac{K_B\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x+\xi)-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi-\frac{\mathrm{e}^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{0}\mathrm{e}^{-\frac{\left[(x+\xi)-\left(r-q+\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi$$

$$=\frac{K_B\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega-\frac{\mathrm{e}^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega,$$

$$(\mathrm{III})=\frac{\mathrm{e}^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\left(K_B-\mathrm{e}^{\xi}\right)\mathrm{d}\xi$$

$$=\frac{K_B\mathrm{e}^{-r(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi-\frac{\mathrm{e}^{x-q(T-t)}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x-\xi)+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi$$

$$=\frac{\mathrm{e}^{x-q(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x-\ln K_B+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega-\frac{K_B}{\sqrt{2\pi}}\mathrm{e}^{-r(T-t)}\int_{-\infty}^{\frac{x-\ln K_B+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega,$$

and

$$(\mathrm{IV})=-\frac{\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x+\xi)-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\left(K_B-\mathrm{e}^{\xi}\right)\mathrm{d}\xi$$

$$=\frac{\mathrm{e}^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x+\xi)+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi-\frac{K_B\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\ln K_B}\mathrm{e}^{-\frac{\left[(x+\xi)-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}\mathrm{d}\xi,$$

then

$$(\mathrm{IV})=\frac{\mathrm{e}^{-q(T-t)-\frac{2}{\sigma^2}(r-q)x}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x+\ln K_B-\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega-\frac{K_B\mathrm{e}^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)x}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x+\ln K_B-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}\mathrm{e}^{-\frac{\omega^2}{2}}\mathrm{d}\omega.$$

This, together (1.7), yields

$$\begin{aligned}V(S,t)&=K\mathrm{e}^{-r(T-t)}\left(N(d_2)-N(d_5)\right)+S\mathrm{e}^{-q(T-t)}\left(N(d_6)-N(d_1)\right)\\&\quad+S_B\left(\frac{S}{S_B}\right)^{-\frac{2}{\sigma^2}(r-q)}\mathrm{e}^{-q(T-t)}\left(N(d_8)-N(d_3)\right)\\&\quad+K\left(\frac{S}{S_B}\right)^{-r(T-t)-\frac{2}{\sigma^2}\left(r-q-\frac{\sigma^2}{2}\right)}\mathrm{e}^{-r(T-t)}\left(N(d_4)-N(d_7)\right)\end{aligned}\tag{1.16}$$

with

$$d_1=\frac{\ln\frac{S}{S_B}+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};\quad d_2=\frac{\ln\frac{S}{S_B}+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_3=\frac{\ln\frac{S}{S_B}-\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};\quad d_4=\frac{\ln\frac{S}{S_B}-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_5=\frac{\ln\frac{S}{S_B}-\ln K_B+\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};\quad d_6=\frac{\ln\frac{S}{S_B}-\ln K_B+\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_7=\frac{\ln\frac{S}{S_B}+\ln K_B-\left(r-q-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}};\quad d_8=\frac{\ln\frac{S}{S_B}+\ln K_B-\left(r-q+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

4. Conclusions

In this paper, based on the analysis of the execution conditions of convertible bonds with call provision, by solving a certain boundary value problem of the Black-Scholes equation, the closed form of the pricing formula is obtained.

In reality, the holders of convertible bonds tend to be risk-averse; owners generally hold the convertible bonds until the maturity date to obtain the bond interest as income, but when stock prices rise to a certain level (as defined for redemption), in order to obtain more profits, the holder will convert the bonds to the related stock, and sell the stocks in the secondary market. This paper just provides a theoretical evaluation of the bond for the investors. In this sense, the studied model in the paper is interesting and realistic.

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A Method for the Solution of Educational Investment

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Abstract

In order to improve the performance of higher education in the United States, the Goodgrant Foundation intends to donate a total of $100,000,000 (US 100 million) to an appropriate group of schools per year, for five years, starting in July 2016. For this, our team puts forward upon an optimal investment strategy, which includes the schools to invest, the investment amount of each school, and the return due to investment, to solve this problem. Our main idea is as follows. First of all, we choose suitable investment school universities in the United States. Secondly, we use Analytic Hierarchy Process to get the rate of return on investment and venture capital. Thirdly, we establish a venture capital return model. Finally, solving the mathematical model ensures the investment amount of each school and the return due to investment. To implement this strategy, first of all, we obtain the candidate school based on students score card. Then, according to the factor analysis, we analyze the factors which mainly affect the choice of school. Secondly, we employ Analytic Hierarchy Process to get the rate of return on investment and capital risk. In the end, we establish a risk return model to get investment amount for each school, amount of risk and return. In order to ensure the minimum risk and the maximum return, we set up a multi objective programming model and solve it by using the constraint method. We get the result that includes the maximum net profit of the investment and risk loss rate. According to statistical analysis, we can get the overall return of net income within five years. Finally, we choose 320 candidate schools and get the investment amount of each school according to the principle of as many schools as possible. We have proved that the foundation will receive a return of more than 295.363 million in the next 5 years. After-verification, our strategy can be directly applied to the investment field and get good results.

Keywords

AHP, Multi-Objective Programming, Risk Investment Return

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1. Introduction

In the era of knowledge economy, higher education, as a social intelligence incubator, is more and more important. Every country develops vigorously its higher education to improve their comprehensive strength and competitiveness. Well higher education's development must have education funds, but the education funds from government drop year by year, and the shortage of education funds has restricted the development of higher education. Therefore, colleges and universities should actively raise funds from the society, use and manage the funds for better development.

The Goodgrant Foundation is a charitable organization that wants to help improve educational performance of undergraduates attending colleges and universities in the United States. It has a important goal that millions of students can get the best education resources across the country. The special fund can help build a bridge for the development of education, especially schools and students in poor areas. At the same time, it also helps the children who have dropped out of school in the poor families and develops more aid activities. Starting in July 2016, the foundation intends to donate a total of $100,000,000 (US 100 million) to an appropriate group of schools per year, for five years.

To provide high quality foundation with an optimal investment decisions, we are using the Analytic Hierarchy Process (AHP) to given weights for many factors influencing the investment returns under the predecessors' research. To establish the risk investment model, through the constraint method, we transform the multi-objective model to a single objective model.

2. Problem Analysis

By analyzing the data and casual working, we can get basic information about 2977 American universities, which include school location, size and nature, degree-granting conditions, undergraduate education enrollment, the students' proportion of each race, student retention and completion rates, students' race, family income situations, important subjects scores in school, the cases of scholarships and loan accepted, the average debt, repayment ability, the income after works and so on. In the big background of American education, we fully analyze the obtained data and establish a suitable model to help Goodgrant Foundation decide an optimal quality investment strategy, which include the school to invest, the money of investment in each school, the investment of time, and repay of investment returns.

3. Analysis Process

First of all, we analyze existing data. We selected 2936 schools from 7804 schools of most recent cohort data and analyze these date to find out the bigger influencing factors. Then by considering some factors, we determine 583 schools of investment in program ways. But in these 583 schools we selected, there are many null date in some schools. In order to avoid risks and reduce the risk of investment, we removed the more uncertainty schools. The end result that we choose is investment of schools.

Secondly, Considering Various factors of effecting return on investment, Determining influence level of these factors, selecting three greatest impact factors and applying for Analytic Hierarchy Process to determine repay of invest. Similarly, using the same method gets risk-free rate.

Next, according to the principle of minimum investment risk and maximum return on investment, Building risk investment return model by assigning to each school investment and total investment constraints.

Besides, by solving the risk return on investment model, obtaining the money of investment in each school and obtain repay on investment within a year. Suppose the return on investment every year is the same in five years, and then get to the expected return within five years.

In the end, Analysis of the results obtained, Evaluation of the model, the model summary and summary of outcome to analysis strength and weakness of model and give Valuable comments and suggestions.

4. Basic Assumptions

We make the following assumptions about all the process of solve the problem in this paper.

1) The influence of professional may be ignored, we only consider the investment of every school.

2) Every school of investment and return for every year are assumed to be the same.

3) We do not take into account interactions between factors.

Additional assumptions are made to simplify analysis for individual sections. These assumptions will be discussed at the appropriate locations.

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5. Model Development

5.1. The Data Processing

Before presenting our models, we describe the preprocessing work that we did with the data.

5.1.1. Screen out the Date of Investment School

1) Comparing the name of institution in Most Recent Cohorts Date (Scorecard Elements), than selecting out 2936 institutions from the currently certified in operating by the C Programming Language.

2) Using factor analysis compare the different criterion to influence The Goodgrant Foundation's choice of investment of institution by some subjective cognition, and using Statistical Product and Service Solutions to select some institutions from the step 1 through some much important criterion including HCM2, PREDDEG, CONTROL, LOCALE etc. As followers, **Figure 1** stands for the correlation of three variables.

Analyzing factors that influence the choice of investing school for the following reasons:

a) HCM2 indicates the school has special scholarship, which proves that the floating capital of the school is more. Hence, these schools don't need too much investment and this factor has greater impact on school choice.

b) PREDDEG indicates the situations of degree classification. If schools have more classification situations, the teachers of school are much stronger. We can't invest the school, so this factor has greater impact on school choice.

c) CONTROL indicates the nature of school, which is public school, private school, for-profit or non-profit. If the school is a public school, the government will invest much funding. If the school is a private for-profit school, it has additional income, so we would give more consideration to non-profit school. Therefore these factors have greater impact on school choice.

d) LOCALE indicates the size of city, if the school is located in the large scale city, which representative the cost of education is more. So we should give more investment. Therefore, this factor has much greater impact on school choice.

There are 18 factors like these above. We don't list one by one.

VAR00005 stands for UGDS_NHPI

VAR00006 stands for UGDS_2MOR

VAR00007 stands for UGDS_NRA

1) By Statistical Product and Service Solutions analyze 23 criterion to choose the target of selecting institutions. We analyze the date by the way of Frequency Analysis, and select 80% date of every criterion. By java programming, the output result that match the optimal integrated indicators are 583 educational institutions. (example, **Figure 2**, this criterion choose 4 and 5).

2) In order to calculate rate of return on investment return and risk rate and avoid investment risk, so deleting some missing data of criterion.

Correlation matrix

		VAR00005	VAR00006	VAR00007
Correlation	VAR00005	1.000	0.597	0.482
	VAR00006	0.597	1.000	0.320
	VAR00007	0.482	0.320	1.000
Correlation	VAR00005		0.000	0.000
	VAR00006	0.000		0.000
	VAR00007	0.000	0.000	

Figure 1. Correlation of three much important criterion.

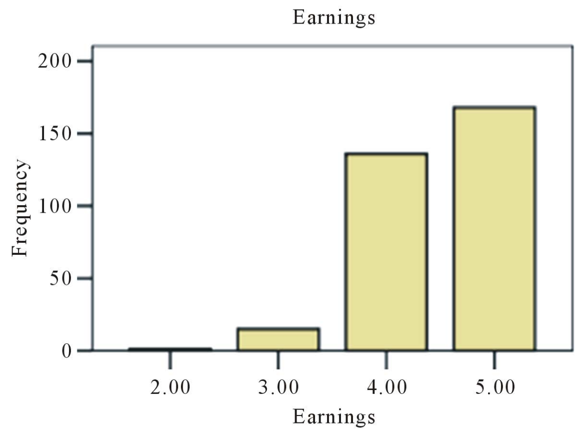

Figure 2. The percentage of PCTPELL criterion.

By excel and spss, we select 320 institutes to invest, they have enough effective date for us to analyze. Than calculate the rate of return and risk rate on investment for these school by Analytic Hierarchy Process.

5.1.2. Calculation the Rate of Return on Investment in Schools

We find the three factors closely related to the rate of return on investment from the college students' score card, which are the average debt, the median of 10 years of monthly debt repayment and the median income of students after 10 years of work. We choose three factors as rate of return on investment, the reasons are as follows:

If the average debt is high for students and the repayment ability of students is weaker, the risk of investment will be higher. So the return on investment would be reduced accordingly.

If the Median to repay debt in 10-year monthly payments is higher, the Ability to repay debt will be higher. The return on investment is higher.

If the median to earn money is higher after 10 years of work, their repayment ability is strong and the risk to their investment is much smaller. So the return on investment is higher.

5.1.3. Calculation of Investment Risk Rate in Schools

We find the three factors closely related to the rate of investment risk from the college students' score card. There are gt_25k_p6 (Share of students earning over $25,000/year (threshold earnings) 6 years after entry), RPY_3YR_RT_SUPP (Repayment rate within 3 years) and C150_4_POOLED_SUPP (150% graduation completion rate). We choose three factors as rate on investment risk, the reasons are as follows:

If the 3-year repayment rate is high in the university, which can prove they have better ability. So the investment risk rate for the school is lower.

If 150% the graduation completion rate is high in the four-year university, the school's average graduation completion rate will be high. So the investment risk rate for the school is lower.

If share of earning over $25,000/year on student is much higher， the school's average income will be much higher. So the investment risk rate for the school is lower.

5.2. Building a Model

Model one. On behalf of ROI indicators were weight analysis. Analytic Hierarchy Process (AHP) is a simple and feasible method to make a decision for complicated problems, and it is applied to the problems of quantitative analysis which are hard to solve. AHP is proposed by Stay, which is widely used in many fields such as business, industry, healthcare, and education [1].

Evaluation of weight, that is based on the establishment of an orderly hierarchical criterion system, and through the comparison of the same level by the relative importance of each indicator to synthesize calculate the weight of criterion coefficients [2].

The principle and steps of AHP:

1) The hierarchical structure establishment

When applying Analytic Hierarchy Process (AHP) to analyze problems and make decisions, firstly, we must to make the objectives methodical and construct a hierarchical structure model [3]. Secondly, in this model, complex problem is decomposed into elemental components. It consists of the goal and decision criteria and a group of options. If necessary, we can further break down the criteria into sub-criteria. The number of the hierarchies depends on the complexity of the problem [4].

The overall goal is the ranking of ROI and the rate of risk. The criterion includes A1, A2, A3. The last hierarchy is the alternatives of the problem. Hierarchical structure of the problem is shown in **Figure 3**.

Three criterion include A1, A2 and A3 (**Table 1**).

2) Judgment Matrix

Analyzing the same level of the relative importance of *n* indicators is completed by a number of experts. By 9 quintile ratio scale (**Table 2**).

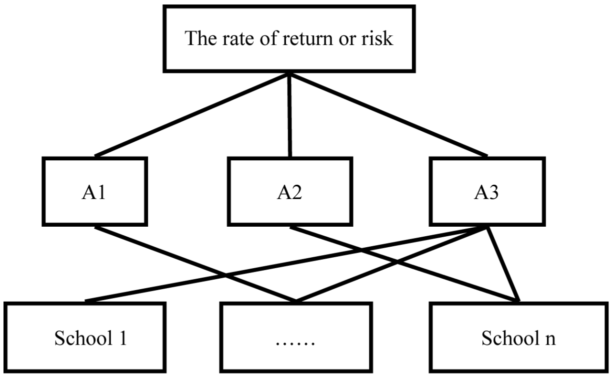

Figure 3. Hierarchy of criteria and sub-criteria.

Table 1. The full name of attributes and simplified symbols.

Symbol	Attribute
A1	GRAD_DEBT_MDN_SUPP
A2	GRAD_DEBT_MDN10YR_SUPP
A3	md_earn_wne_p10

Table 2. The definition of the scale.

Scale a_{ij}	Mean
1	Equally important
3	Moderately important
5	Strongly important
7	Very strongly important
9	Extremely important
$\frac{1}{3}$	Weaker
$\frac{1}{5}$	Weak
$\frac{1}{7}$	Strongly weak
$\frac{1}{9}$	Very strongly weak
8, 6, 4, 2, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$	Intermediate values

1) Each element (a_{ij}) of judgment matrix stands for the value of criterion i row and j column relative importance of pairwise comparisons. In judgment matrix,

$$a_{ij} > 0,\ a_{ij} = 1,\ a_{ij} = \frac{1}{a_{ji}}\left(i, j = 1, 2, \cdots, n\right).$$

Below, a matrix constructed for 3 criterion is shown.

$$\begin{bmatrix} & A1 & A2 & A3 \\ A1 & a_{11} & a_{12} & a_{13} \\ A2 & a_{21} & a_{22} & a_{23} \\ A3 & a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1}$$

2) Calculate the weight and consistency inspection

The literature contains calculations formulas for the subsequent steps leading to the calculation of the value of a priority criterion [5].

a) Calculations of the multiplication of each row element

$$M_i = \prod_{j=1}^{n} a_{ij},\ i = 1, 2, \cdots, n. \tag{2}$$

b) Calculation of the root of M_i

$$\overline{W_i} = \sqrt[n]{M_i}. \tag{3}$$

c) Normalized matrix:

$$w = \left[\overline{W_1}, \overline{W_2}, \cdots, \overline{W_n}\right]^{\mathrm{T}} \quad w_i = \frac{\overline{W_i}}{\sum_{n=1}^{n} \overline{W_i}} \quad w \text{ stands for weight.} \tag{4}$$

d) The maximum characteristic root,

$$\lambda_{\max} = \frac{1}{n}\sum_{i=1}^{n}\frac{(AW)_i}{w_i}. \tag{5}$$

e) value of the consistency criterion:

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n-1}. \tag{6}$$

f) consistency ratio:

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} \tag{7}$$

where the CR should reach a value <10%, ($n > 2$).

Finding the corresponding average random consistency criterion RI.

For $n = 1, 2, \cdots, 9$, Saaty [5] gives the value of RI. As shown in **Table 3**.

Model two. According to the maximum return on investment and the smallest investment risk, we want to seek the optimal investment scheme. We set up the following multi-objective risk investment model [6].

The objective function:

$$\max R = \sum_{i=1}^{n} r_i x_i \tag{8}$$

$$\min Q = \sum_{i=1}^{n} q_i x_i \tag{9}$$

$$s.t \begin{cases} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \end{cases} \tag{10}$$

Table 3. Value of the random criteria (RI).

n	1	2	3	4	5	6	7	8	9
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45

In addition, R—represents the total return on investment;
Q—represents the overall investment risk;
r_i—represents the i school's rate of return on investment for a year;
q_i—represents the i school's rate of investment risk for a year;
x_i—represents the i school's investment for a year, the unit is hundred million;
i—represents the school of investment, i from 1 to n.

5.3. Resolve the Model

5.3.1. Analytic Hierarchy Process

1) The ROI of institution

The comparison matrix for the main criteria

$$\begin{bmatrix} & A1 & A2 & A3 \\ A1 & 1 & 1 & 9 \\ A2 & 1 & 1 & 9 \\ A3 & \frac{1}{9} & \frac{1}{9} & 1 \end{bmatrix} \tag{11}$$

Result (**Table 4**)

$$\lambda_{\max} = 1.421053 \tag{12}$$

$$CI = \frac{\lambda_{\max} - n}{n-1} = -0.78947 \tag{13}$$

$$\begin{gathered} \mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} = \frac{-0.78947}{0.52} = -1.52 < 0.10 \\ n = 3,\ \ \mathrm{RI} = 0.52 \end{gathered} \tag{14}$$

Calculation the rate of return (**Table 5**)
a_{i1}—the value of GRAD_DEBT_MDN_SUPP $\left(i = 1, 2, \cdots, 320,\ j = 1\right)$
a_{i2}—the value of GRAD_DEBT_MDN10YR_SUPP $\left(i = 1, 2, \cdots, 320,\ j = 2\right)$
a_{i3}—the value of md_earn_wne_p10 $\left(i = 1, 2, \cdots, 320,\ j = 3\right)$
ROI—the rate of return of institution

$$a_{i4} = \sum_{i=1}^{320} C2 * a_{ij} + D2 * a_{ij} + E2 * a_{ij},\quad j = 1, 2, 3 \tag{15}$$

$$\mathrm{ROI} = \frac{a_{i4}}{\sum_{i=1}^{320} a_{i4}} \tag{16}$$

2) The rate of risk of institution

The comparison matrix for the main criteria

$$\begin{bmatrix} & B1 & B2 & B3 \\ B1 & 1 & 1 & 9 \\ B2 & 1 & 1 & 9 \\ B3 & \frac{1}{9} & \frac{1}{9} & 1 \end{bmatrix} \tag{17}$$

Table 4. The ROI of institution.

	A1	A2	A3	M_i	$\overline{W_i}$	Weight
A1	1	1	9	9	2.080084	0.473684
A2	1	1	9	9	2.080084	0.473684
A3	1/9	1/9	1	0.01234568	0.23112	0.052632
					4.391288	
		Check consistency				
		λ_{max}	1.421053			
		CI	−0.78947			
		CR	−1.52			

Table 5. The rate of return of part institutions.

		GRAD_DEBT_ MDN_SUPP	GRAD_DEBT_ MDN10YR_SUPP	md_earn_wne_p10		
UNITID	INSTNM	0.473684	0.473684	0.052632		RIO
100706	University of Alabama in Huntsville	24,738	274.6425	46,600	14,300.74	0.003419
100830	Auburn University at Montgomery	21,791	241.9248	34,800	12,268.24	0.002938
102094	University of South Alabama	25,000	277.5513	38,300	13,989.38	0.003344
102368	Troy University	26,000	288.6533	36,600	14,378.85	0.003437
102669	Alaska Pacific University	25,125	278.939	47,400	14,528.2	0.003473
104151	Arizona State University-Tempe	20,375	226.2043	45,200	12,137.43	0.002901
104586	Embry-Riddle Aeronautical University-Prescott	26,000	288.6533	60,900	15,657.8	0.003743
105330	Northern Arizona University	21,450	238.139	38,800	12,315.45	0.002944
105589	Prescott College	25,000	277.5513	35,200	13,826.22	0.003305
105899	Arizona Christian University	22,625	251.1839	31,000	12,467.67	0.00298
106245	University of Arkansas at Little Rock	23,000	255.3472	34,800	12,847.28	0.003071
106458	Arkansas State University-Main Campus	20,959	232.6879	32,700	11,759.23	0.002811
106467	Arkansas Tech University	17,528	194.5967	35,000	10,237.03	0.002447

*B*1—stands for GRAD_DEBT_MDN_SUPP
*B*2—stands for GRAD_DEBT_MDN10YR_SUPP
*B*3—stands for md_earn_wne_p10
Result (**Table 6**)

$$\lambda_{\max} = 1.636434 \tag{18}$$

$$\mathrm{CI} = \frac{\lambda_{\max} - n}{n-1} = -0.68178 \tag{19}$$

$$\mathrm{CR} = \frac{\mathrm{CI}}{\mathrm{RI}} = \frac{-0.68178}{0.52} = -1.31 < 0.10 \qquad (20)$$
$$n = 3, \ \ \mathrm{RI} = 0.52$$

Calculation the rate of risk (Table 7)

b_{i1}—the value of GRAD_DEBT_MDN_SUPP $(i = 1, 2, \cdots, 320, \ j = 1)$

b_{i2}—the value of GRAD_DEBT_MDN10YR_SUPP $(i = 1, 2, \cdots, 320, \ j = 2)$

b_{i3}—the value of md_earn_wne_p10 $(i = 1, 2, \cdots, 320, \ j = 3)$

I—the rate of risk

$$b_{i4} = \sum_{i=1}^{320} C2 * b_{ij} + D2 * b_{ij} + E2 * b_{ij}, \quad j = 1, 2, 3 \qquad (21)$$

Table 6. The rate of risk of institution.

	B1	B2	B3	M_i	$\overline{W_i}$	Weight
B1	1	3	1/9	0.33333333	0.693361	0.221125
B2	1/3	1	9	3	1.44225	0.459958
B3	9	1/9	1	1	1	0.318917
					3.135611	
		Check consistency				
		λ_{max}	1.636434			
		CI	−0.68178			
		CR	−1.52			

Table 7. The rate of risk of part institutions.

		RPY_3YR_ RT_SUPP	C150_4_POOLED _SUPP	gt_25k_p6		
UNITID	INSTNM	0.221125	0.459958	0.318917		risk rate
100706	University of Alabama in Huntsville	0.781998	0.478211	0.660566	0.603542	0.003183
100830	Auburn University at Montgomery	0.628856	0.285309	0.554537	0.447137	0.002358
102094	University of South Alabama	0.701489	0.350632	0.6088	0.510549	0.002692
102368	Troy University	0.543508	0.34558	0.613959	0.474937	0.002505
102669	Alaska Pacific University	0.76824	0.463924	0.678571	0.599671	0.003162
104151	Arizona State University-Tempe	0.802838	0.580479	0.71778	0.673436	0.003551
104586	Embry-Riddle Aeronautical University-Prescott	0.86524	0.571099	0.773429	0.700667	0.003695
105330	Northern Arizona University	0.760114	0.491979	0.65243	0.602441	0.003177
105589	Prescott College	0.815552	0.407981	0.538922	0.539864	0.002847
105899	Arizona Christian University	0.748918	0.491084	0.521739	0.557874	0.002942
106245	University of Arkansas at Little Rock	0.657656	0.226839	0.571885	0.432144	0.002279
106458	Arkansas State University-Main Campus	0.646863	0.382615	0.514133	0.48299	0.002547
106467	Arkansas Tech University	0.656991	0.405856	0.547864	0.506677	0.002672

$$I = \frac{b_{i4}}{\sum_{i=1}^{320} b_{i4}} \tag{22}$$

5.3.2. Multi Objective Programming

Solving multi-objective problem is more difficult, so we need to convert them to more easily problem of a single target. Because of a variety of methods: The main objective method, linear weighted sum method, the weighted sum of squares method, ideal point method, multiplication and division, efficacy coefficient method, stepwise method, constraint method, etc. So, our team decided to use the most simple constraint method to solve it, which convert one of the goals to constraints conditions, while another goal will be goal function. Like this, Solving Multi-objective programming problem with two objectives will be transformed into solving a single [7].

Using the constraint method to following:

The objective function:

$$\begin{aligned} &\max R \\ &\min Q \\ &s.t.\begin{cases} R = \sum_{i=1}^{n} r_i x_i \\ Q = \sum_{i=1}^{n} q_i x_i \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \ge 0 \end{cases} \end{aligned} \tag{23}$$

In addition, R—represents the total return on investment;

Q—represents the overall investment risk;

r_i—represents the i school's rate of return on investment for a year;

q_i—represents the i school's rate of investment risk for a year;

x_i—represents the i school's investment for a year, the unit is hundred million US dollars;

i—represents the school of investment, i from 1 to 320.

According constraint equations listed above, calculated by the Analytic Hierarchy Process risk rate and rate of return, we can get a result. Because amount of data are so big, we use c language to solve. C language implementation process is as follows:

In order to ensure the maximum return and minimum risk, so we consider the recursive iterative algorithm in the end. First, putting the results of the risk rate and rate of return on the txt file and write a program to reads file data. Secondly, putting program into two parts, one is to calculate max and min, another is to calculate the money of investment in every school. According to Iterative Algorithms, until the value from large to small equal value from small to large, the program is end [8].

The following procedure is recursive algorithm processes (**Figure 4**):

In a recursive algorithm, I advance its recursive algorithm, There are three aspects:

First, each call will narrowed in size (usually the size is half);

Second, there is a close link between adjacent repeated twice, once time is prepare for the time after (usually output on a previous post is input);

Third, when a size of the problem is extremely small, you must be given directly to answer rather than a recursive call, so each recursive call is conditional, unconditional recursive calls will become died loop and not have a normal end.

By recursive algorithm, we got the investment for 320 schools.

6. Results and Analysis

On the basis of solving process, we can get the following results:

1) Every school's investment for every year, part of the data listed below (**Table 8**):

2) The annual return on investment is 59.3854 million dollar.

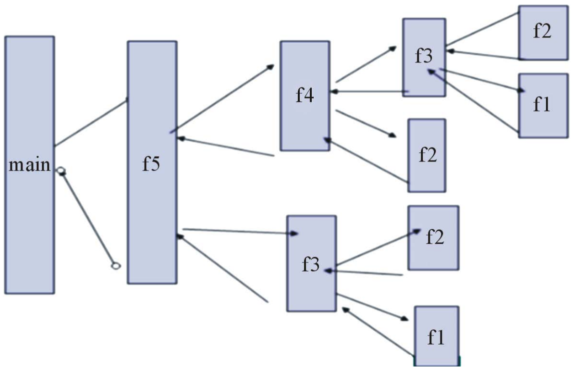

Figure 4. Recursive algorithm.

Table 8. Part of the data for every school's investment (Unit: Hundred Million US Dollars).

UNITID	INSTNM	Investment
152363	Saint Josephs College	0.00471
160065	Our Lady of Holy Cross College	0.004701
196042	Farmingdale State College	0.004698
392840	Watkins College of Art Design & Film	0.004696
228529	Tarleton State University	0.004672
221661	Southern Adventist University	0.004664
127556	Colorado Mesa University	0.004663
204820	Ohio University-Chillicothe Campus	0.004642
152363	Saint Josephs College	0.004623
214713	Pennsylvania State University-Penn State Harrisburg	0.004613
196237	SUNY College at Old Westbury	0.004612
149222	Southern Illinois University-Carbondale	0.004593
169327	Cleary University	0.004591
184612	Felician College	0.004588
214625	Pennsylvania State University-Penn State New Kensington	0.004567

3) The annual risk loss money on investment is 0.3128 million dollar.
4) The annual net return on investment is 59.3854 − 0.3128 = 59.0726 million dollar.
5) Forecasting total return on investment for the five-year is 295.363 billion dollar.

From the results, we can see that Goodgrant foundation invest 100 million on 320 schools. In the end of year, they can obtain 60% of repay money. The risk loss of investment for these schools will reduce year by year. The repay on investment will be stable in the fifth year, the overall return on investment in five years will reach 60% of the total investment. The most important thing is that schools obtain the investment within five years will improve largely comprehensive capabilities and students' ability in these schools will have greatly improved too.

7. Strengths and Weaknesses

7.1. Strengths

1) Our main model's strength is its enormous edibility. For instance, all these factors into a single, robust framework, our model enables.

2) We developed a theoretical line formation model which agrees without rough data. Our computer model agrees with both despite working on different principles, implying it behaves as we want.

3) This main model allows us to make substantive conclusions.

4) The fundamental strengths of our model are its robustness and flexibility. All of the data is fully parameterized, so the model can be applied to solve practical problems.

7.2. Weaknesses

Some special data can't be found, and it makes that we have to do some proper assumption before the solution of our models. A more abundant data resource can guarantee a better result in our models. Current line length is not taken into account by the line formation model. In real life, it's not so ideal.

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A Numerical Method for Nonlinear Singularly Perturbed Multi-Point Boundary Value Problem

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Abstract

We consider a uniform finite difference method for nonlinear singularly perturbed multi-point boundary value problem on Shishkin mesh. The problem is discretized using integral identities, interpolating quadrature rules, exponential basis functions and remainder terms in integral form. We show that this method is the first order convergent in the discrete maximum norm for original problem (independent of the perturbation parameter ε). To illustrate the theoretical results, we solve test problem and we also give the error distributions in the solution in Table 1 and Figures 1-3.

Keywords

Singular Perturbation, Fitted Finite Difference Method, Shishkin Mesh, Nonlocal Boundary Condition, Uniform Convergence

1. Introduction

In this paper we shall consider singularly perturbed multi-point nonlinear problem

$$-\varepsilon^2 u'' + f(x,u) = 0, \quad 0 \le x \le 1, \tag{1}$$

$$u(0) = 0 \tag{2}$$

$$k_0 u(1) = \sum_{i=1}^{m} k_i u(s_i) + k_{m+1} \int_0^1 u(x)\,\mathrm{d}x + d \tag{3}$$

$$f(x,u) \in C^1([0,1]xR), \quad \frac{\partial f}{\partial u}(x,u) \ge \alpha > 0, \quad s_i \in (0,1), \quad i = 1,\cdots,m, \quad k_0 \ge 0$$

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where, $0<\varepsilon\ll 1$ is small perturbation parameter. The solution $u(x)$ has boundary layers at $x=0, x=1$.

Singularly perturbed differential equations arise many applications such as, fluid mechanics, chemical-reactor theory, the Navier-Stokes equations of fluid flow at high Reynolds number, control theory, electrical networks, and other physical models. In recent years, singularly perturbed differential equations were studied by many authors in various fields of applied mathematics and engineering. For examples, Cziegis [1] studied the numerical solution of singularly pertürbed nonlocal problem. Cziegis [2] analyzed the difference schemes for problems with nonlocal conditions. Bakhvalov [3] investigated on optimization of methods for solving boundary-value problems in the presence of a boundary layer. Amiraliyev and Çakır [4] applied the difference method on a Shishkin mesh to the singularly perturbed three-point boundary value problem. Amiraliyev and Çakır [5] researched a uniformily convergent difference scheme for singularly perturbed problem with convective term end zeroth order reduced equation. Amiraliyev and Çakır [6] studied numerical solution of the singularly perturbed problem with nonlocal boundary condition. Amiraliyev and Duru [7] estimated a note on a parameterized singular perturbation problem. Amiraliyev and Erdoğan [8] studied uniform method for singularly perturbed delay differential equations. Amiraliyeva, Erdoğan and Amiraliyev [9] applied a uniform numerical method for dealing with a singularly perturbed delay initial value problem. Adzic and Ovcin [10] studied nonlinear spp with nonlocal boundary conditions and spectral approximation. Amiraliyev, Amiraliyeva and Kudu [11] applied a numerical treatment for singularly perturbed differential equations with integral boundary condition. Herceg [12] studied the numerical solution of a singularly perturbed nonlocal problem. Herceg [13] researched solving a nonlocal singularly perturbed problem by splines in tension. Çakır [14] studied uniform second-order difference method for a singularly perturbed three-point boundary value problem. Geng [15] applied a numerical algorithm for nonlinear multi-point boundary value problems.

In this study we present uniformly convergent difference scheme on an equidistant mesh for the numerical solution of the problem (1)-(3). The difference scheme is constructed by the method integral identities with the use exponential basis functions and interpolating quadrature rules with the weight and remainder terms in integral form [5]-[7]. In Section 2, the asymptotic estimations of the problem (1)-(3) are established. The difference scheme constructed on Shishkin mesh for numerical solution (1)-(3) is presented in Section 3 and in Section 4. We prove that the method is first-order convergent in the discrete maximum norm. In Section 5, a numerical example is considered. The results show that the uniform finite difference method on Shishkin mesh is more powerful method than other methods for nonlinear singularly perturbed multi-point boundary value problem.

2. The Continuous Problem

In this section, we describe some properties of the solution of (1) with Lemma 2.1. we use $\|g\|_\infty$ for the continuous maximum norm on the [0, 1], where $g(x)$ is any continuous function.

Lemma 2.1.

Let $f(x,u)\in C^1([0,1]xR)$ and $\frac{\partial f}{\partial u}(x,u)$ is uniformly bounded in $u(x)$. We assume that

$$\sum_{i=1}^{m} k_i w_0(s_i)+k_{m+1}\int_0^1 u(x)\,\mathrm{d}x<k_0 \tag{4}$$

where $w_0(x)\ge |w(x)|$,

$$-\varepsilon^2 w''+a(x)w(x)=0 \tag{5}$$

$$w(0)=0,\ w(1)=1 \tag{6}$$

solution of this problem.

So, the solution of the Equation (1) satisfies the inequalities

$$\|u(x)\|_{C[0,1]}\le C_0 \tag{7}$$

and

$$|u'(x)|\le C\left\{1+\frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right\},\ 0<x<1 \tag{8}$$

where, C_0 and C are constants independent of ε.

Proof. We rewrite the Equation (1). Hence, we use intermediate value theorem for $f(x,u)$,

$$\frac{f(x,u)-f(x,0)}{u-0}=\frac{\partial f}{\partial u}(x,\vartheta),\ \ \vartheta=\gamma u,\ \ 0<\gamma<1$$

$$f(x,u)=f(x,0)+u(x)\frac{\partial f}{\partial u}(x,\vartheta)=-F(x)+a(x)u(x)$$

where $a(x),F(x)$ are sufficiently smooth on $[0,1],[0,1]xR$ and

$$\frac{\partial f}{\partial u}(x,\vartheta)=a(x)\geq\alpha>0,\ \ \vartheta\text{-intermediate value.}$$

Consequently, we obtain the following linear equation,

$$-\varepsilon^2u''+a(x)u(x)=F(x). \tag{9}$$

Now, let $u(1)=\lambda$ according to the Equation (3).

We can write the solution of the Equation (9) as follows

$$u(x)=v(x)+\lambda w(x) \tag{10}$$

$$-\varepsilon^2v''+a(x)v(x)=F(x) \tag{11}$$

$$v(0)=0,\ \ v(1)=0 \tag{12}$$

where $v(x)$ is solution of the Equations (11), (12).

First, we prove the estimate $v(x)$,

$$|v(x)|\leq|v(0)|+|v(1)|+\alpha^{-1}\|F\|_\infty\leq C_1. \tag{13}$$

Second, we prove the estimate $w(x)$,

$$|w(x)|\leq|w(0)|+|w(1)|+\alpha^{-1}\|0\|_\infty\leq 1. \tag{14}$$

According to the Equation (4), λ is a finite number. Then, from the Equations (13), (14) we have the following inequality

$$|u(x)|\leq|v(x)|+|\lambda||w(x)|\leq C_1+|\lambda|$$

$$|u(x)|\leq C_0 \tag{15}$$

we now prove the estimate the Equation (8).

If $u''(x)$ is pulled from the Equation (9), we obtain

$$u''(x)=\frac{-1}{\varepsilon^2}\left[F(x)-a(x)u(x)\right] \tag{16}$$

and from the Equation (16)

$$|u''(x)|\leq\left|\frac{1}{\varepsilon^2}\left[F(x)-a(x)u(x)\right]\right|\leq\frac{C}{\varepsilon^2}. \tag{17}$$

Now, we take derivative of the Equation (9) and if it called $u'(x)=v_0(x)$, the Equation (9) takes the form with boundary condition

$$-\varepsilon^2v_0''+a(x)v_0(x)=\varnothing(x) \tag{18}$$

$$u'(0)=v_0(0)=0,\ \ u'(1)=v_0(1)=1. \tag{19}$$

Now, we proceed with the estimation of $\varnothing(x),u'(0),u'(1)$, respectively, from the Equation (7)

$$\varnothing(x)=F'(x)-a'(x)u(x)\leq C_1. \tag{20}$$

We use the following relation for $g \in C^2[0,1]$,

$$g'(x) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} - \int_\alpha^\beta \left[\frac{\beta - t}{\beta - \alpha} - T_0(x - t) \right] g''(t)\,\mathrm{d}t, \quad \alpha < x < \beta, \ \alpha < \beta \tag{21}$$

where

$$T_0(x-t) = \begin{cases} 1, & x - t > 0 \\ 0, & x - t < 0 \end{cases}$$

the Equation (21) with the values $g(x) = u(x), \alpha = 0, \beta = \varepsilon, x = 0$ and from the Equations (7)-(17)

$$|u'(0)| \le \left| \frac{u(\varepsilon) - u(0)}{\varepsilon} \right| + \int_0^\varepsilon |u''(t)|\,\mathrm{d}t \le \frac{C}{\varepsilon}. \tag{22}$$

In a similar manner, the Equation (21) with the values $g(x) = u(x),\ \alpha = 1 - \varepsilon,\ \beta = 1,\ x = 1$ and from the Equations (7)-(17)

$$|u'(1)| \le \left| \frac{u(1) - u(1-\varepsilon)}{\varepsilon} \right| + \int_{1-\varepsilon}^1 |u''(t)|\,\mathrm{d}t \le \frac{C}{\varepsilon}. \tag{23}$$

We write the solution of the Equations (18), (19) in the form,

$$v_0(x) = v_1(x) + v_2(x)$$

where $v_1(x), v_2(x)$ are respectively the solution of the following problems,

$$-\varepsilon^2 v_1''(x) + a(x) v_1(x) = \varnothing(x) \tag{24}$$

$$v_1(0) = 0,\ u'(1) = v_1(1) = 0 \tag{25}$$

$$-\varepsilon^2 v_2''(x) + a(x) v_2(x) = 0 \tag{26}$$

$$v_2(0) = |v_0(0)|,\ u'(1) = v_2(1) = |v_0(1)|.$$

According to the maximum principle in the Equations (24), (25), we can the following Barrier function,

$$\psi_1(x) = \mp v_1(x) + \alpha^{-1} \|\varnothing(x)\|_\infty.$$

This Barrier function provides the conditions of the maximum principle and

$$v_1(x) \le C.$$

In a similar manner, according to the maximum principle in the Equation (26), we can write

$$v_2(x) \le \theta(x)$$

where $\theta(x)$ is the solution of the following problem with constant coefficient,

$$-\varepsilon^2 \theta''(x) + \alpha \theta(x) = 0 \tag{27}$$

$$\theta(0) = |v_0(0)| \le \frac{C}{\varepsilon},\ \theta(1) = |v_0(1)| \le \frac{C}{\varepsilon} \tag{28}$$

where $a(x) \ge \alpha > 0$ and the solution of $\theta(x)$ as follows,

$$\theta(x) = \frac{v_0(0)\left[\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} - \mathrm{e}^{\frac{\sqrt{\alpha}(1-x)}{\varepsilon}} \right] + v_0(1)\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}} - \mathrm{e}^{\frac{\sqrt{\alpha}x}{\varepsilon}} \right]}{\left[-\mathrm{e}^{\frac{\sqrt{\alpha}}{\varepsilon}} + \mathrm{e}^{\frac{-\sqrt{\alpha}}{\varepsilon}} \right]} \tag{29}$$

after some arragement, we can obtain,

$$\theta(x) \le \frac{C}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} \right]. \tag{30}$$

Finally, from $u'(x) = v_0(x), |v_1(x)| \le C, \theta(x) \le v_2(x), v_0(x) = v_1(x) + v_2(x)$, we have the following inequality,

$$\begin{aligned} |u'(x)| = |v_0(x)| \le |v_1(x)| + |v_2(x)| \le C + \frac{C}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} \right] \\ \le C\left\{ 1 + \frac{1}{\varepsilon}\left[e^{-\frac{\sqrt{\alpha}x}{\varepsilon}} + e^{-\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} \right] \right\} \end{aligned}$$

which leads to the Equation (8).

3. Discretizaton and Non-Uniform Mesh

Let us consider the following any non-uniform mesh on $[0,1]$,

$$\bar{\omega}_N = \left\{ 0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, i \ge 1, h_i = x_i - x_{i-1}, x_{N_i} = s_i, N_i = \frac{s_i N}{1} \right\}.$$

We present some properties of the mesh function $g(x)$ defined on $\bar{\omega}_N$, which is needed in this section for analysis of the numerical solution.

$$g_i = g(x_i)$$

$$g_{x,i} = \frac{g_{i+1} - g_i}{h_{i+1}}, \quad g_{\bar{x},i} = \frac{g_i - g_{i-1}}{h_i}$$

$$g_{\hat{x},i} = \frac{g_{i+1} - g_i}{\hbar_i}$$

$$g_{\bar{x}\hat{x},i} = \frac{g_{x,i} - g_{\bar{x},i}}{\hbar_i} = \frac{1}{\hbar_i}\left(\frac{g_{i+1} - g_i}{h_{i+1}} - \frac{g_i - g_{i-1}}{h_i} \right)$$

$$h_i = x_i - x_{i-1}, \quad \hbar_i = \frac{h_i + h_{i+1}}{2}$$

$$\|g\|_\infty = \|g\|_{\infty,\bar{\omega}_N} = \max_{0 \le i \le N} |g_i|.$$

Now, We will construct the difference scheme for the Equation (1). First, we integrate the Equation (1) over (x_{i-1}, x_{i+1}),

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} -\varepsilon^2 u''(x) \varphi_i(x) \mathrm{d}x + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x,u) \varphi_i(x) \mathrm{d}x = 0, \; i = \overline{1, N-1} \tag{31}$$

where $\{\varphi_i(x)\}, i = \overline{1, N-1}$ are the linear basis functions and having the form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \dfrac{x - x_{i-1}}{h_i}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \dfrac{x_{i+1} - x}{h_{i+1}}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}) \end{cases}$$

$\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems,

$$-\varepsilon^2 \varphi'' = 0 \tag{32}$$

$$\varphi(x_{i-1}) = 0,\ \ \varphi(x_i) = 1 \tag{33}$$

$$-\varepsilon^2\varphi'' = 0 \tag{34}$$

$$\varphi(x_i) = 1,\ \ \varphi(x_{i+1}) = 0. \tag{35}$$

If we rearrange the Equation (31) it gives

$$-\varepsilon^2\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u''(x)\varphi_i(x)\mathrm{d}x + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,u)\varphi_i(x)\mathrm{d}x = 0,\ \ i = \overline{1, N-1}. \tag{36}$$

After doing some calculation

$$\varepsilon^2\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} u'(x)\varphi_i'(x)\mathrm{d}x + f(x_i,u_i) + R_i = 0,\quad i = \overline{1, N-1} \tag{37}$$

where

$$R_i = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \mathrm{d}x\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}} \frac{\mathrm{d}}{\mathrm{d}x} f(x,u)K_0^*(x,\xi)\mathrm{d}\xi,\quad i = \overline{1, N-1} \tag{38}$$

and

$$K_0^*(x,\xi) = T_0(x-\xi) - T_0\left(\frac{x_{i+1}-x_{i-1}}{2} - \xi\right) + (x_{i+1}-x_{i-1})^{-1}(x_{i+1}-\xi)^0\left(\frac{x_{i+1}-x_{i-1}}{2} - x\right).$$

So, from the Equation (37), the difference scheme is defined by

$$-\varepsilon^2 u_{\bar{x}\hat{x},i} + f(x_i,u_i) + R_i = 0,\ \ i = \overline{1, N-1}. \tag{39}$$

Now, we define an approximation for the second boundary condition of the Equation (1). We accepted that x_{N_i} is the mesh point nearest to s_i.

$$\begin{aligned} k_0 u_N &= \sum_{i=1}^{m} k_i u_{N_i} + k_{m+1}\int_0^1 u(x)\mathrm{d}x + d \\ &= \sum_{i=1}^{m} k_i u_{N_i} + k_{m+1}\left[\sum_{i=1}^{N} h_i u_i + r_i\right] + d \end{aligned} \tag{40}$$

where remainder term

$$r_i = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} (\xi - x_{i-1})\frac{\mathrm{d}}{\mathrm{d}x}u(\xi)\mathrm{d}\xi. \tag{41}$$

By neglecting R_i, r_i in the Equation (39) and the Equation (40), we suggest the following difference scheme for approximating the Equations (1)-(3)

$$-\varepsilon^2 y_{\bar{x}\hat{x},i} + f(x_i,y_i) = 0,\ \ i = \overline{1, N} \tag{42}$$

$$y_0 = 0 \tag{43}$$

$$k_0 y_N = \sum_{i=1}^{m} k_i y_{N_i} + k_{m+1}\sum_{i=1}^{N} h_i y_i + d. \tag{44}$$

We will use the Shishkin mesh to be ε-uniform convergent of the difference scheme the Equations (42)-(44). So the Shishkin mesh divides each of the interval $[0,\sigma]$ and $[\sigma, 1-\sigma]$ into $N/4$ equidistant subintervals and $[1-\sigma, 1]$ into $N/2$ equidistant subinterval, where σ and $1-\sigma$ are transition points which are defined as

$$\sigma = \min\left\{\frac{1}{4}, \left(\sqrt{\alpha}\right)^{-1}\varepsilon\ln N\right\}$$

if $h^{(1)}$, $h^{(2)}$ and $h^{(3)}$, respectively, are the stepsize on $[0,\sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$. We have as

$$h^{(1)} = \frac{4\sigma}{N},\ h^{(2)} = \frac{2(1-2\sigma)}{N},\ h^{(3)} = \frac{4\sigma}{N},$$

$$\frac{\left(h^{(1)}+h^{(3)}\right)}{2} = \frac{2}{N},\ h^{(1)} = h^{(3)} \le N^{-1},\ N^{-1} \le h^{(2)} \le 2N^{-1}$$

$$\overline{\omega}_N = \left\{ x_i = ih^{(1)}, i = \overline{0, \frac{N}{4}}, h^{(1)} = \frac{4\sigma}{N}; x_i = \sigma + \left(i - \frac{N}{4}\right)h^{(2)}, i = \overline{\frac{N}{4}+1, \frac{3N}{4}}, \right.$$

$$\left. h^{(2)} = \frac{2(1-2\sigma)}{N}; x_i = 1-\sigma + \left(i - \frac{3N}{4}\right)h^{(3)}, i = \overline{\frac{3N}{4}+1, N}, h^{(3)} = \frac{4\sigma}{N} \right\}$$

where, N is even number, $\sigma \ll 1$.

4. Error Analysis

Let $z = y - u$, $x \in \overline{w}_N$, which is the error function of the difference scheme the Equations (42), (43) and the solution of the discrete problem

$$lz \equiv -\varepsilon^2 z_{\overline{x}\hat{x},i} + \left[f(x_i, y_i) - f(x_i, u_i) \right] = R_i,\ i = \overline{1, N-1} \tag{45}$$

$$z_0 = 0 \tag{46}$$

$$k_0 z_N = \sum_{i=1}^{m} k_i z_{N_i} + k_{m+1} \sum_{i=1}^{N} h_i z_i = r_i,\ i = \overline{1, N} \tag{47}$$

where R_i, r_i are defined in the Equation (38) and the Equation (41).

Lemma 4.2. Let z_i be the solution of the Equations (45)-(47) and approximation error R_i and r_i. Then there are the following inequalities,

$$\|R\|_{\infty,\overline{w}_N} \le CN^{-1} \ln N \tag{48}$$

$$|r|_{\infty,\overline{w}_N} \le CN^{-1} \ln N \tag{49}$$

Proof. We evaluate the Equation (38) and the Equation (41), respectively

$$R_i = \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \mathrm{d}x\, \varphi_i(x) \int_{x_{i-1}}^{x_{i+1}} \frac{\mathrm{d}}{\mathrm{d}x} f(x,u) K_0^*(x,\xi) \mathrm{d}\xi$$

$$|R_i| \le \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) \mathrm{d}x \int_{x_{i-1}}^{x_{i+1}} \left| \frac{\partial f(\xi, u(\xi))}{\partial \xi} + \frac{\partial f}{\partial u} \frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi} \right| \mathrm{d}\xi \le C \int_{x_{i-1}}^{x_{i+1}} \left(1 + u'(\xi)\right) \mathrm{d}\xi.$$

Consequently,

$$|R_i| \le C \left\{ \int_{x_{i-1}}^{x_{i+1}} \left(1 + \frac{1}{\varepsilon} \left[\mathrm{e}^{\frac{-\sqrt{\alpha} x}{\varepsilon}} + \mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}} \right] \right) \mathrm{d}x \right\}. \tag{50}$$

In the beginning, we consider the case $\sigma = 1 - \sigma = \frac{1}{4}$ and so $x_i \in [0, \sigma]$,

$$\frac{1}{4} < \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N,\ h = h^{(1)} = h^{(2)} = h^{(3)} = \frac{1}{N}$$

it then follows from the Equation (50) that

$$|R_i| \le C \left\{ h + \frac{1}{\varepsilon} \int_{x_{i-1}}^{x_{i+1}} \mathrm{d}x \right\} \le C \left\{ h + \varepsilon^{-1} h \right\} \le CN^{-1} \ln N,\ 1 < i < N. \tag{51}$$

Second, we consider the case $\sigma = \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N$, and so $\frac{1}{4} > \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N$, we estimate R_i on $[0,\sigma]$, $[\sigma, 1-\sigma]$ and $[1-\sigma, 1]$, respectively.

In the seperate $[0,\sigma]$, the Equation (50) reduces to

$$|R_i| \le C\left(1+\varepsilon^{-1}\right) h^{(1)} \le C\left(1+\varepsilon^{-1}\right)\frac{4\sigma}{N} \le CN^{-1}\ln N,\ 1 \le i < \frac{N}{4} - 1. \tag{52}$$

In the seperate $[\sigma, 1-\sigma]$, the Equation (50) reduces to

$$\begin{aligned} |R_i| &\le C\left\{ h^{(2)} + \left(\sqrt{\alpha}\right)^{-1}\left[\left(\mathrm{e}^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}x_{i+1}}{\varepsilon}}\right) + \left(\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x_{i+1})}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}(1-x_{i-1})}{\varepsilon}}\right)\right]\right\} \\ &\le C\left(1+\varepsilon^{-1}\right)\frac{4\sigma}{N} \le CN^{-1}\ln N,\quad 1 \le i < \frac{N}{4} - 1 \end{aligned} \tag{53}$$

where for $x_i = \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N + \left(i - \frac{N}{4}\right) h^{(2)}$,

$$x_{i-1} = \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N + \left(i - 1 - \frac{N}{4}\right) h^{(2)},\ x_{i+1} = \left(\sqrt{\alpha}\right)^{-1} \varepsilon \ln N + \left(i + 1 - \frac{N}{4}\right) h^{(2)}$$

and so

$$\mathrm{e}^{\frac{-\sqrt{\alpha}x_{i-1}}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}x_{i+1}}{\varepsilon}} \le N^{-1}. \tag{54}$$

Analogously for $x_i = 1 - \sigma + \left(i - \frac{3N}{4}\right) h^{(3)}$,

$$x_{i-1} = 1 - \sigma + \left(i - 1 - \frac{3N}{4}\right) h^{(3)},\ x_{i+1} = 1 - \sigma + \left(i + 1 - \frac{3N}{4}\right) h^{(3)}$$

thus

$$\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x_{i+1})}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}(1-x_{i-1})}{\varepsilon}} \le N^{-1} \tag{55}$$

according to the Equation (54) and the Equation (55), we can rewrite the the Equation (53)

$$|R_i| \le C\left\{ h^{(2)} + \left(\sqrt{\alpha}\right)^{-1}\left[N^{-1} + N^{-1}\right]\right\} \le CN^{-1}. \tag{56}$$

In the seperate $[1-\sigma, 1]$, the Equation (50) reduces to

$$|R_i| \le C\left(1+\varepsilon^{-1}\right) h^{(3)} \le C\left(1+\varepsilon^{-1}\right)\frac{4\sigma}{N} \le CN^{-1}\ln N,\ \frac{3N}{4} + 1 \le i \le N. \tag{57}$$

The last estimate is for $x_{N/4}$ and $x_{3N/4}$:

We rewrite the the Equation (50) for $i = \frac{N}{4}$,

$$\left|R_{\frac{N}{4}}\right| \le C\left\{\int_{x_{\frac{N}{4}}}^{x_{\frac{N}{4}+1}} \left(1 + \frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}} + \mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)\mathrm{d}x\right\}. \tag{58}$$

We take integrate in the Equation (58) and so

$$\left|R_{\frac{N}{4}}\right| \le C\left\{\left(\left(x_{\frac{N}{4}+1} - x_{\frac{N}{4}} - \frac{1}{\varepsilon}\frac{\varepsilon}{\sqrt{\alpha}}\left[\left(\mathrm{e}^{\frac{-\sqrt{\alpha}x_{\frac{N}{4}+1}}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}x_{\frac{N}{4}}}{\varepsilon}}\right) + \left(\mathrm{e}^{\frac{-\sqrt{\alpha}\left(1-x_{\frac{N}{4}+1}\right)}{\varepsilon}} - \mathrm{e}^{\frac{-\sqrt{\alpha}\left(1-x_{\frac{N}{4}}\right)}{\varepsilon}}\right)\right]\right)\right)\right\} \tag{59}$$

where

$$\mathrm{e}^{\frac{-\sqrt{\alpha}x_{\frac{N}{4}+1}}{\varepsilon}}-\mathrm{e}^{\frac{-\sqrt{\alpha}x_{\frac{N}{4}}}{\varepsilon}}\le N^{-1} \tag{60}$$

$$\mathrm{e}^{\frac{-\sqrt{\alpha}\left(1-x_{\frac{N}{4}+1}\right)}{\varepsilon}}-\mathrm{e}^{\frac{-\sqrt{\alpha}\left(1-x_{\frac{N}{4}}\right)}{\varepsilon}}\le N^{-1} \tag{61}$$

we rewrite the Equation (59) with the Equation (60) and the Equation (61), thus,

$$\left|R_{\frac{N}{4}}\right|\le C\left\{x_{\frac{N}{4}+1}-x_{\frac{N}{4}}+2N^{-1}\right\}\le C\left\{h^{(2)}+N^{-1}\right\}\le CN^{-1}\ln N \tag{62}$$

where $h^{(2)}=2\left(1-\left(\sqrt{\alpha}\right)^{-1}\varepsilon\ln N\right)$.

We use in a similar way as above for $i=\frac{3N}{4}$, and so

$$\left|R_{\frac{3N}{4}}\right|\le CN^{-1}\ln N. \tag{63}$$

Next, we use estimate for the remainder term r:

From the Equation (41) we can write

$$|r_i|\le\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}\left(x-x_{i-1}\right)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\,\mathrm{d}x\le\sum_{i=1}^{N}h_i\int_{x_{i-1}}^{x_i}u'(x)\,\mathrm{d}x$$

from the Equation (8)

$$|r_i|\le\sum_{i=1}^{N}h_i\int_{x_{i-1}}^{x_i}C\left\{1+\frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right\}\mathrm{d}x,\ 1\le i\le N$$

$$\begin{aligned}|r_i|&\le\sum_{i=1}^{N/4}h^{(1)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)\mathrm{d}x+\sum_{i=N/4+1}^{3N/4}h^{(2)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)\mathrm{d}x\\&\quad+\sum_{i=3N/4+1}^{N}h^{(3)}\int_{x_{i-1}}^{x_i}\left(1+\frac{1}{\varepsilon}\left[\mathrm{e}^{\frac{-\sqrt{\alpha}x}{\varepsilon}}+\mathrm{e}^{\frac{-\sqrt{\alpha}(1-x)}{\varepsilon}}\right]\right)\mathrm{d}x\\&=h^{(1)}\int_0^{\sigma}\left(1+\frac{1}{\varepsilon}\right)\mathrm{d}x+h^{(2)}\int_{\sigma}^{1-\sigma}\left(1+\frac{1}{\varepsilon}\right)\mathrm{d}x+h^{(3)}\int_{1-\sigma}^{1}\left(1+\frac{1}{\varepsilon}\right)\mathrm{d}x\end{aligned}$$

$$|r_i|\le C\left[h^{(1)}+h^{(2)}+h^{(3)}\right]\le C\left[N^{-1}+2N^{-1}+N^{-1}\right]\le CN^{-1} \tag{64}$$

Lemma 4.3. Let z_i be solution of the Equations (45)-(47). Then there is the following inequality,

$$\|z\|_{\infty,\bar{w}_N}\le C\left[\|R\|_{\infty,w_N}+|r|\right]. \tag{65}$$

Proof. Rearranging the Equation (45) gives

$$lz\equiv-\varepsilon^2 z_{\bar{x}\hat{x},i}+a_i z_i=R_i,\quad i=\overline{1,N-1}$$

where

$$a_i=\frac{\partial F}{\partial u}\left(t_i,u_i+\gamma z_i\right),\ 0<\gamma<1$$

according to the proof of Lemma 2.1, we can use the maximum principle, and so it is easy to obtain,

$$
\begin{aligned}
z_{\infty,\bar{w}_N} &\le |z_N| + \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right) \\
&\le \left|\sum_{i=1}^{m} k_i z_{N_i} + k_{m+1}\sum_{i=1}^{N} h_i z_i + d - r_i\right| + \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right) \\
&\le \alpha^{-1}\left(\|R\|_{\infty,w_N} + |r|\right) \le C\left[\|R\|_{\infty,w_N} + |r|\right]
\end{aligned} \tag{66}
$$

Conclusion 4.1. We know that the solution of the Equations (1)-(3) is $u(x)$ and the solution of the Equations (45)-(47) is y_i. Then Lemma (4.2) and Lemma (4.3) give us the following convegence result

$$
\|y-u\|_{\infty,\bar{w}_N} \le CN^{-1}\ln N. \tag{67}
$$

5. Numerical Example

In this section, an example of nonlinear singularly perturbed multi-point boundary value problem is given to illustrate the efficiency of the numerical method described above. The example is computed using maple 10. Results obtained by the method are compared with the exact solution of example and found to be good agreement with each other. We compute approximate errors e_ε^N, e^N and the convergence rates p_ε^N on the Shishkin Mesh $\bar{\omega}_N$ for different values of ε, N.

Example 5.1.

We solve the difference scheme the Equations (42), (44) using the following iteration technique,

$$
\begin{aligned}
&\left[\frac{\varepsilon^2}{\hbar_i h_i}\right] y_{i-1}^{(n)} - \left[\frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\right] y_i^{(n)} + \left[\frac{\varepsilon^2}{\hbar_i h_{i+1}}\right] y_{i+1}^{(n)} \\
&= f(x_i) + f\left(x_i, y_i^{(n-1)}\right) - y_i^{(n-1)}\frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right), \quad i = 1,\cdots,N-1, \quad n = 1,2,\cdots
\end{aligned} \tag{68}
$$

$$
y_0^{(n)} = 0, \quad k_0 y_N^{(n)} = \sum_{i=1}^{m} k_i y_{N_i}^{(n-1)} + k_{m+1}\sum_{i=1}^{N} h_i y_i^{(n-1)} + d \tag{69}
$$

$$
\begin{aligned}
&\left[\frac{\varepsilon^2}{\hbar_i h_i}\right] y_{i-1}^{(n)} - \left[\frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)\right] y_i^{(n)} + \left[\frac{\varepsilon^2}{\hbar_i h_{i+1}}\right] y_{i+1}^{(n)} \\
&= f(x_i) + f\left(x_i, y_i^{(n-1)}\right) - y_i^{(n-1)}\frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right), \quad i = 1,\cdots,N-1,
\end{aligned} \tag{70}
$$

$$
y_{\frac{N}{2}} = \mu_{n-1}, \quad y_N = \mu_{n-1} - 1
$$

where

$$
\mu_n = \frac{\left(\frac{\varepsilon}{h_2}\right)^2 y_{\frac{N}{2}-1,n} + \left(\frac{\varepsilon}{h_2}\right)^2 y_{\frac{N}{2}+1,n} - 1 + \left(y_{\frac{N}{2}+1,n-1}\right)^2}{2\left(\frac{\varepsilon}{h_2}\right)^2 + y_{\frac{N}{2},n-1}}
$$

$$
y_{\frac{N}{2}} = \mu_n, \quad \mu_0 = C_0 \ge 1.
$$

The system of the Equations (68)-(70) is solved by the following procedure,

$$
A_i = \frac{\varepsilon^2}{\hbar_i h_i}, \; B_i = \frac{\varepsilon^2}{\hbar_i h_{i+1}}, \; C_i = \frac{2\varepsilon^2}{h_i h_{i+1}} + \frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right),
$$

$$
F_i = -f\left(x_i, y_i^{(n-1)}\right) + y_i^{(n-1)}\frac{\partial f}{\partial y}\left(x_i, y_i^{(n-1)}\right)
$$

$$\alpha_1 = 0, \beta_1 = 0$$

$$\alpha_{\frac{N}{2}+1} = 0, \beta_{\frac{N}{2}+1} = \mu_{n-1}$$

$$\alpha_{i+1} = \frac{B_i}{C_i - A_i\alpha_i}, \beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - A_i\alpha_i}, i = 1, \cdots, N-1$$

$$y_i^{(n)} = \alpha_{i+1} y_{i+1}^{(n)} + \beta_{i+1}, y_i^{(0)} = 0.5, i = N-1, \cdots, 2, 1.$$

It is easy to verify that

$$A_i > 0, B_i > 0, C_i > A_i + B_i, i = 1, \cdots, N.$$

For this reason, the described procedure above is stable. Also, the Equations (42)-(44) has only one solution.

Now, we consider the following test problem,

$$-\varepsilon^2 u'' + u^2(x) - f(x) = 0, \tag{71}$$

$$u(0) = 0 \tag{72}$$

$$u(1) = u(0.5) + d \tag{73}$$

which has the exact solution,

$$u(x) = \frac{(2x-1)\left(e^{\frac{1-x}{\varepsilon}} - e^{\frac{x}{\varepsilon}}\right)}{e^{\frac{1}{\varepsilon}} - 1} + 1$$

In the computations in this section, we will take $\alpha = 2, d = -1$ the initial guess in the iteration procedure is $y_i^{(0)} = 0.5$. The stopping criterion is taken as

$$\max_i \left| y_i^{(n+1)} - y_i^{(n)} \right| \le 10^{-5}.$$

The error estimates are denoted by

$$e_\varepsilon^N = \|y - u\|_{\infty, \bar{\omega}_N}$$

and

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

The convergence rates are

$$P_\varepsilon^N = \log_2\left(\frac{e^N}{e^{2N}}\right).$$

The numerical results obtained from the problem of the difference scheme by comparison, the error and uniform rates of convergence were found and these are shown in **Table 1**. Consequently, numerical results show that the proposed scheme is working very well.

The results point out that the convergence rate of the established scheme is really in unision with theoretical analysis.

From the graps it is show that the error is maximum near the boundary layer and it is almost zero in outer region in the **Figure 3**. Approximate solution compared with exact solution in **Figure 2**. Approximate solutions are given for different values of ε in **Figure 1**.

Table 1. The computed maximum pointwise errors e^N and e^{2N}, the numerical rate of convergence p^N on the Shishkin mesh $\bar{\omega}_N$ for different values of N and ε.

ε	*N* değerleri							
	8	16	32	64	128	256	512	1024
2^{-10}	0.1350816058 p = 0.826	0.0761888740 p = 0.806	0.00435607918 p = 0.821	0.00246466665 p = 0.884	0.00133500352 p = 1.02	0.0065502193 p = 1.300	0.0026594582 p = 1.72	0.0008027767
2^{-12}	0.1365365902 p = 0.811	0.0778089349 p = 0.777	0.0454015276 p = 0.763	0.0267503827 p = 0.768	0.0157072143 p = 0.797	0.0090342052 p = 0.869	0.0049452671 p = 1.016	0.0024447312
2^{-14}	0.1369028476 p = 0.807	0.0782173980 p = 0.769	0.0458717849 p = 0.748	0.0273010175 p = 0.739	0.0163561085 p = 0.740	0.0097927781 p = 0.753	0.0058067718 p = 0.789	0.0033601955
2^{-16}	0.1369926017 p = 0.806	0.0783189688 p = 0.768	0.0459894145 p = 0.745	0.0274402097 p = 0.731	0.0165223717 p = 0.725	0.0099918880 p = 0.725	0.0060447308 p = 0.731	0.0036407030
2^{-18}	0.1370138364 p = 0.806	0.07834719995 p = 0.767	0.0460165306 p = 0.744	0.0274742487 p = 0.730	0.0165639692 p = 0.721	0.0100425784 p = 0.717	0.0061060714 p = 0.717	0.0037143751
2^{-20}	0.1370841205 p = 0.806	0.0783970437 p = 0.768	0.0460092097 p = 0.743	0.0274772611 p = 0.729	0.0165727285 p = 0.720	0.0100552118 p = 0.715	0.0061220254 p = 0.713	0.0037341617
2^{-22}	0.1369241824 p = 0.802	0.0784855980 p = 0.770	0.0460110587 p = 0.743	0.0274794446 p = 0.727	0.0165923346 p = 0.722	0.0100583725 p = 0.715	0.0061258653 p = 0.712	0.0037388392
2^{-24}	0.1370859052 p = 0.806	0.0783990459 p = 0.767	0.0460605903 p = 743	0.0275085873 p = 0.729	0.0165929886 p = 0.720	0.0100693779 p = 0.715	0.0061330123 p = 0.712	0.0037437729
2^{-26}	0.1370859946 p = 0.806	0.0783991464 p = 0.767	0.0460607056 p = 0.743	0.0275087238 p = 0.729	0.0165931517 p = 0.720	0.0100695762 p = 0.715	0.00613332523 p = 0.712	0.0037440664

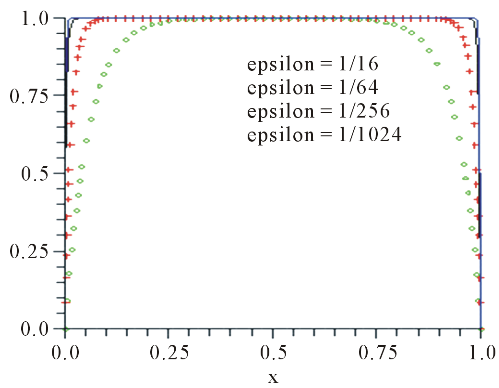

Figure 1. Approximate solution distribution for $\varepsilon = 2^{-4}, 2^{-6}, 2^{-8}, 2^{-10}$ using $N = 256$.

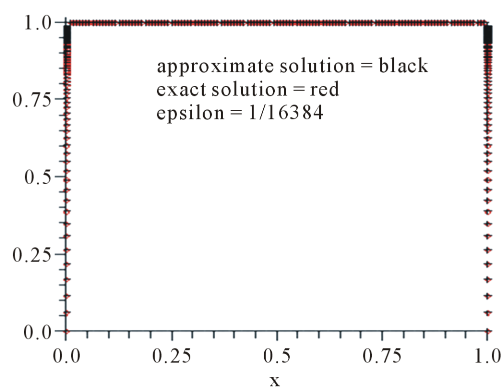

Figure 2. Comparison of approximate solution and exact solution for $\varepsilon = 2^{-14}$.

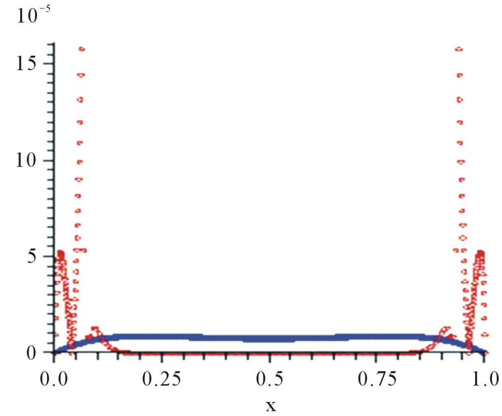

Figure 3. Error distribution for $\varepsilon = 2^{-2}, 2^{-8}, 2^{-10}, 2^{-12}$ using $N = 256$.

6. Conclusion

Consequently, the aim of this paper was to give uniform finite difference method for numerical solution of nonlinear singularly perturbed problem with nonlocal boundary conditions. The numerical method was constructed on Shishkin mesh. The method was pointed out to be convergent, uniformly in the ε-parameter, of first order in the discrete maximum norm. The numerical example illustrated in practice the result of convergence proved theoretically.

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Boundedness for Commutators of Calderón-Zygmund Operator on Herz-Type Hardy Space with Variable Exponent

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Abstract

Our aim in this paper is to prove the boundedness of commutators of Calderón-Zygmund operator with the Lipschitz function or BOM function on Herz-type Hardy space with variable exponent.

Keywords

Commutator, Variable Exponent, Herz-Taype Hardy Spaces, BMO, Calderón-Zygmund Operator

1. Introduction

In 2012, Hongbin Wang and Zongguang Liu [1] discussed boundedness Calderón-Zygmund operator on Herz-type Hardy space with variable exponent. M. Luzki [2] introduced the Herz space with variable exponent and proved the boundedness of some sublinear operator on these spaces. Li'na Ma, Shuhai Li and Huo Tang [3] proved the boundedness of commutators of a class of generalized Calderón-Zygmund operators on Labesgue space with variable exponent by Lipschitz function. Mitsuo Izuki [4] proved the boundedness of commutators on Herz spaces with variable exponent. Lijuan Wang and S. P. Tao [5] proved the boundedness of Littlewood-Paley operators and their commutators on Herz-Morrey space with variable exponent. In this paper we prove the boundedness of commutators of singular integrals with Lipschitz function or BMO function on Herz-type Hardy space with variable exponent.

In this section, we will recall some definitions.

Definition 1.1. Let T be a singular integral operator which is initially defined on the Schwartz space $S(\mathbb{R}^n)$.

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Its values are taken in the space of tempered distributions $S'(\mathbb{R}^n)$ such that for x not in the support of f,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\mathrm{d}y, \tag{1.1}$$

where f is in $L_c^\infty(\mathbb{R}^n)$, the space of compactly bounded function.

Let $0 < \delta, D < \infty$. Here the kernel k is function in $(\mathbb{R}^n)$ away from the diagonal $x = y$ and satisfies the standard estimate

$$|K(x,y)| \le \frac{D}{|x-y|^n}, x \ne y \tag{1.2}$$

and

$$|k(x,y) - k(x',y)| \le \frac{D|x-x'|^\sigma}{|x-y| + |x'-y|^{n+\sigma}}, \tag{1.3}$$

provided that $|x-x'| \le \frac{1}{2}\max\{|x-y|, |x'-y|\}$

$$|k(x,y) - k(x',y)| \le \frac{D|x-x'|^\sigma}{|x-y| + |x'-y|^{n+\sigma}}, \tag{1.4}$$

provided that $|y-y'| \le \frac{1}{2}\max\{|x-y|, |x-y'|\}$ such that is called standard kernel and the class of all kernels that satisfy (1.2), (1.3), (1.4) is denoted by $SK(\sigma, D)$. Let T be as in (1.1) with kernel $SK(\sigma, D)$. If T is bounded from L^p to L^p with $1 < p < \infty$, then we say that T is Calderón-Zygmund operator.

Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$. We first defined Lebesgue spaces with variable exponent.

Definition 1.2. [4] Let $p(\cdot): \Omega \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}. \tag{1.5}$$

The space $L_{Loc}^{p(\cdot)}(\Omega)$ is defined by

$$L_{Loc}^{p(\cdot)}(\Omega) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset \Omega \right\}.$$

The Lebesgue space $L^{p(\cdot)}(\Omega)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \le 1 \right\}. \tag{1.6}$$

We denote

$$p_- = \operatorname{essinf}\{p(x) : x \in \Omega\}, \quad p_+ = \operatorname{esssup}\{p(x) : x \in \Omega\}.$$

Then $\mathcal{P}(\Omega)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathfrak{B}(\Omega)$ to be the set of all function $p(\cdot) \in \mathcal{P}(\Omega)$ satisfying that M is bounded on $L^{p(\cdot)}(\Omega)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}, C_k = B_k \setminus B_{k-1}, \chi_k = \chi_{B_k}, k \in \mathbb{Z}$.

Proposition 1.1. See [1]. If $q(\cdot) \in \mathcal{P}(\Omega)$ satisfies

$$|q(x) - q(y)| \le \frac{-A}{\operatorname{Log}(|x-y|)}, |x-y| \le 1/2, \tag{1.7}$$

$$|q(x) - q(y)| \le \frac{A}{\operatorname{Log}(e + |x|)}, |y| \ge |x|, \tag{1.8}$$

then, we have $q(\cdot)\in\mathfrak{B}(\Omega)$.

Proposition 1.2. [6] Suppose that $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n), 0<\gamma<n/(q_1)_+$, if $\gamma/n=1/q_1(\cdot)-1/q_2(\cdot)$ then

$$\left\|\chi_{B_k}\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\le C2^{-k\gamma}\left\|\chi_B\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}, \tag{1.9}$$

for all balls $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}$ with $k\in\mathbb{Z}$.

Definition 1.3. [7] Let $\alpha\in\mathbb{R}$, $0<p_1\le\infty$ and $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{q(\cdot)}_{loc}(\mathbb{R}^n):\|f\|_{\dot{K}^{\alpha,p}_{q(\cdot)}\mathbb{R}^n}<\infty\right\}, \tag{1.10}$$

where

$$\|f\|_{K^{\alpha,p_2}_{q(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f\chi_k\|^{p_1}_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p_1}}. \tag{1.11}$$

The non-homogeneous Herz space with variable exponent $K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in L^{q(\cdot)}_{loc}(\mathbb{R}^n):\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}<\infty\right\} \tag{1.12}$$

where

$$\|f\|_{K^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p}\|f\tilde{\chi}_k\|^{p}_{L^{q(\cdot)}(\mathbb{R}^n)}\right\}^{\frac{1}{p}}. \tag{1.13}$$

Definition 1.4. [1] Let $\alpha\in\mathbb{R}$, $0<p\le\infty$ and $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $N>n+1$. Suppose that $G_N f(x)$ is maximal function of *f*. Homogeneous variable exponent Herz-tybe Hardy spaces $H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)=\left\{f\in S'(\mathbb{R}^n):G_N f(x)\in\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)\right\}, \tag{1.14}$$

with norm

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}=\left\|G_N f(x)\right\|_{\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}. \tag{1.15}$$

Definition 1.5. [1] Let $n\delta_2\le\alpha<\infty, q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $(0<\delta_2<1)$, and non negative integer $s\ge[\alpha-n\delta_2]$. A function *g* on $\mathbb{R}^n$ is said to be a central $(\alpha,q(\cdot))$, if satisfies

1) $\operatorname{supp} g\subset B(0,r)=\{x\in\mathbb{R}^n:|x|<r\}$;

2) $\|g\|_{q(\cdot)(\mathbb{R}^n)}\le|B(0,r)|^{\frac{-\alpha}{n}}$;

3) $\int_{\mathbb{R}^n}g(x)x^{\ell}\mathrm{d}x=0,|\ell|\le s$.

What's more, when $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$,

$$\|f\|_{H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)}\approx inf\left\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right\}^{\frac{1}{p}}. \tag{1.16}$$

Definition 1.6. [7] $1<\gamma\le 0$ the Lipschiz space is defined by

$$Lip_\gamma(\mathbb{R}^n)=\left\{f:\|f\|_{Lip_\gamma}=\sup_{x,y\in\mathbb{R}^n;x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\gamma}<\infty\right\}. \tag{1.17}$$

Definition 1.7. For $b\in L^1_{loc}(\mathbb{R}^n)$, the bounded mean oscillation space $BMO(\mathbb{R}^n)$ is defined by

$$\|b\|_{BMO(\mathbb{R}^n)}=\sup_{B:balls\in(\mathbb{R}^n)}\int_B\frac{1}{|B|}|b(x)-b_B|\mathrm{d}x.$$

2. Main Result and Proof

In order to prove result, we need recall some lemma.

Lemma 2.1. ([3]) Let $b \in lip_\gamma (0<\gamma<1)$, T be Calderón-Zygmund operator, $q_1(\cdot), q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $\frac{1}{q_1(\cdot)} - \frac{1}{q_2(\cdot)} = \frac{\beta}{n}$ Then,

$$\|[b,T]\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\gamma} \|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{2.1}$$

Lemma 2.2. ([8]) Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$; if $f \in L^{q(\cdot)}(\mathbb{R}^n)$ and $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le r_q \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \tag{2.2}$$

where $r_q = 1 + \frac{1}{q_-} - \frac{1}{q_+}$.

Lemma 2.3. ([2]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then for all ball B in $\mathbb{R}^n$,

$$|B|^{-1} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \le C. \tag{2.3}$$

Lemma 2.4. ([2]) Let $q_1(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then for all measurable subsets $S \subset B$, and all ball B in $\mathbb{R}^n$

$$\frac{\|\chi_S\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \quad \frac{\|\chi_S\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2} \tag{2.4}$$

where δ_1, δ_2 are constants with $0<\delta_1,\delta_2<1$.

Lemma 2.5. ([4]) Let $b \in BMO(\mathbb{R}^n)$, and $i, j \in \mathbb{Z}$ with $i<j$ then

$$C^{-1}\|b\|_{BMO(\mathbb{R}^n)} \le \sup_B \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}} \|(b-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{BMO(\mathbb{R}^n)}$$

$$\left\|(b-b_{B_i})\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C(j-i)\|b\|_{BMO(\mathbb{R}^n)} \left\|\chi_{B_j}\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.6. ([9]) Let $q(\cdot) \in \mathcal{B}(\mathbb{R}^n), b \in BMO$ function and T be a Calderón-Zygmund operator. Then

$$\|[b,T]f\|_{q(\cdot)}(\mathbb{R}^n) \le C\|b\|_{BMO(\mathbb{R}^n)} \|f\|_{q(\cdot)}(\mathbb{R}^n).$$

Theorem 2.1. Let $q_1(\cdot)$, $q_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $b \in Lip_\gamma$, $0<p_1<\infty$, $1/q_1(\cdot) - 1/q_2(\cdot) = \gamma/n$ and $-n\delta_1 < \alpha < n\delta_2$ where δ_1, δ_2 are a constants, then $[b,T]$ are bounded from $H\dot{K}^{\alpha,p}_{q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}^{\alpha,p}_{q_2(\cdot)}(\mathbb{R}^n)$.

Proof: we suffices to prove homogeneous case. Let $f(x) \in H\dot{K}^{\alpha,p}_{q(\cdot)}(\mathbb{R}^n)$, $f = \sum_{j=-\infty}^{\infty} \lambda_j g_j$ in the $S'(\mathbb{R}^n)$ sense, where each g_j is a central $(\alpha, q(\cdot))$-atom with supp $g_j \subset B_j$. Write

$$\|f\|_{H\dot{K}^{\alpha,p_1}_{q_1(\cdot)}(\mathbb{R}^n)} \approx inf\left\{\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}\right\}^{\frac{1}{p_1}}.$$

We have

$$\|([b,T]f)\chi_k\|_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|([b,T]f)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \tag{2.5}$$

$$\begin{aligned}\|([b,T]f)\chi_k\|^{p_1}_{\dot{K}^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &+ C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k-1}^{k+1} |\lambda_j| \|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &+ C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\left(\sum_{j=k+1}^{\infty} |\lambda_j| \|([b,T]g_j)\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1} \\ &= F_1 + F_2 + F_3\end{aligned} \tag{2.6}$$

By virtue of Lemma 2.1, we can easily see that

$$F_2 \le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.$$

First we estimate F_1. For each $j \le k-2$ and we shall get

$$\begin{aligned}
|[b,T]g_j| &\le \int_{\mathbb{R}^n} |K(x,y)(b(x)-b(y))g_j(y)|\mathrm{d}y \\
&\le C\int_{\mathbb{R}^n} \frac{|(b(x)-b(y))g_j(y)|}{|x-y|^n}\mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|x-y|^\gamma |g_j(y)|}{|x-y|^n}\mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|g_j(y)|}{|x-y|^{n-\gamma}}\mathrm{d}y \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^1(\mathbb{R}^n)} \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}
\end{aligned} \tag{2.7}$$

$$\|([b,T]g_j)\chi_k\|_{\dot K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\beta(\mathbb{R}^n)} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

Thus by Lemma 2.3, Lemma 2.4 and Proposition 1.2, we get

$$\begin{aligned}
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{-k\gamma} 2^{-(k-2)(n-\gamma)} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{2(n-\gamma)} 2^{-nk+k\gamma-k\gamma} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \left(|B_k| \|\chi_{B_k}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\right) \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{nk} 2^{-nk} \|g_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \left(\frac{\|\chi_{B_j}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}} \right) \\
&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{(j-k)\delta_2 n - j\alpha}
\end{aligned} \tag{2.8}$$

When $1 < p_1 < \infty$ and $\alpha < \delta_2 n$, by Hölder's inequality and (2.8), we calculations

$$\begin{aligned}
F_1 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|[b,T]g_j\chi_k\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|b\|_{lip_\gamma(\mathbb{R}^n)} 2^{(j-k)\delta_2 n - j\alpha} \right)^{p_1} \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^{p_1} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1}{2}} \right) \times \left(\sum_{j=-\infty}^{k-2} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1'}{2}} \right)^{\frac{p_1}{p_1'}} \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \left(\sum_{k=j+2}^{\infty} 2^{[(j-k)(\delta_2 n-\alpha)]\frac{p_1}{2}} \right) \\
&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}
\end{aligned} \tag{2.9}$$

where $0 < p_1 \le 1$ by $\alpha < \delta_2 n$, we get

$$\begin{aligned}F_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\right)^{p_1}\\&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(j-k)\delta_2 n-j\alpha}\right)^{p_1}\\&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|2^{(j-k)\delta_2 n-j\alpha}\right)^{p_1}\\&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^{p_1}2^{(j-k)(\delta_2 n-\alpha)p_1}\right)\\&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\left(\sum_{k=j+2}^{\infty}2^{(j-k)(\delta_2 n-\alpha)p_1}\right)\\&\le C\|b\|^{p_1}_{lip_\gamma(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}|\lambda_j|^{p_1}\end{aligned}\tag{2.10}$$

Now we estimate F_3. For each $k \ge j+2$, we shall get

$$\begin{aligned}\left|[b,T]g_j\right| &\le \int_{\mathbb{R}^n}\left|K(x,y)\left(b(x)-b(y)\right)g_j(y)\right|\mathrm{d}y\\&\le C\int_{\mathbb{R}^n}\frac{\left|\left(b(x)-b(y)\right)g_j(y)\right|}{|x-y|^n}\mathrm{d}y\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}\int_{R^n}\frac{|x-y|^\gamma\left|g_j(y)\right|}{|x-y|^n}\mathrm{d}y\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{\left|g_j(y)\right|}{|x-y|^{n-\gamma}}\mathrm{d}y\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-(k-2)(n-\gamma)}\left\|g_j\right\|_{L^1(\mathbb{R}^n)}\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-(j-2)(n-\gamma)}\left\|g_j\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\end{aligned}\tag{2.11}$$

$$\left\|\left([b,T]g_j\right)\chi_k\right\|_{\dot K^{\alpha,p_2}_{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{lip_\beta(\mathbb{R}^n)}2^{-(j-2)(n-\gamma)}\left\|g_j\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

Using the Lemma 2.3 and Lemma 2.4 and Proposition 1.2, we obtain

$$\begin{aligned}&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{-j\gamma}2^{-(j-2)(n-\gamma)}\left\|g_j\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{2(n-\gamma)}2^{-nj+j\gamma-j\gamma}\left\|g_j\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_k}\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\left|B_j\right|\left\|\chi_{B_j}\right\|^{-1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{nj}2^{-nj}\left\|g_j\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\left(\frac{\left\|\chi_{B_k}\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_j}\right\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}}\right)\\&\le C\|b\|_{lip_\gamma(\mathbb{R}^n)}2^{(k-j)\delta_1 n-j\alpha}\end{aligned}\tag{2.12}$$

When $1 < p_1 < \infty$ and $\alpha > -\delta_1 n$, by Hölder's inequality and (2.12), we have

$$\begin{aligned}F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p_1}\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}2^{(k-j)\delta_1 n-j\alpha}\right)^{p_1}\\
&\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|^{p_1}2^{\left[(k-j)(\delta_1 n+\alpha)\right]\frac{p_1}{2}}\right)\times\left(\sum_{j=k+2}^{\infty}2^{\left[(k-j)(\delta_1 n+\alpha)\right]\frac{p_1'}{2}}\right)^{\frac{p_1}{p_1'}}\\
&\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^{p_1}\left(\sum_{k=-\infty}^{j-2}2^{\left[(k-j)(\delta_1 n+\alpha)\right]\frac{p_1}{2}}\right),\\
&\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^{p_1}\end{aligned}\tag{2.13}$$

When $1 < p_1 \le 1$ by $\alpha > -n\delta_1$, we have

$$\begin{aligned}F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q_2(\cdot)}\left(\mathbb{R}^n\right)}\right)^{p_1}\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}2^{(k-j)\delta_1 n-j\alpha}\right)^{p_1}\\
&\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}|\lambda_j|^{p_1}2^{\left[(k-j)(\delta_1 n+\alpha)\right]p_1}\right)\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^{p_1}\left(\sum_{k=-\infty}^{j-2}2^{\left[(k-j)(\delta_1 n+\alpha)\right]p_1}\right)\\
&\le C\|b\|_{lip_\gamma\left(\mathbb{R}^n\right)}^{p_1}\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^{p_1}\end{aligned}\tag{2.14}$$

Combining (2.10)-(2.14), we get

$$\left\|[b,T]f\right\|_{\dot{K}_{q_2(\cdot)}^{\alpha,p_2}\left(\mathbb{R}^n\right)}^{p_1}\le C\|f\|_{H\dot{K}_{q_1(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)}^{p_1}.$$

Theorem 2.2. Let $q(\cdot)\in\mathcal{B}\left(\mathbb{R}^n\right)$, $b\in BMO\left(\mathbb{R}^n\right)$, $0<p<\infty$, and $-n\delta_1<\alpha<n\delta_2$ where $\delta_1,\delta_2>0$ are a constants, then $[b,T]$ are bounded from $H\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)$ to $\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)$.

Proof: we suffices to prove homogeneous case. Let $f(x)\in H\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)$, $f=\sum_{j=-\infty}^{\infty}\lambda_j g_j$ in the $S'\left(\mathbb{R}^n\right)$ sense, where each g_j is a central $(\alpha,q(\cdot))$-atom with supp $g_j\subset B_j$. Write

$$\|f\|_{H\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)}\approx inf\left\{\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\right\}^{\frac{1}{p}}.$$

We have

$$\left\|\left([b,T]f\right)\chi_k\right\|_{\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)}=\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left\|\left([b,T]f\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}.$$

By inequality (2.5)we have

$$\begin{aligned}\left\|\left([b,T]f\right)\chi_k\right\|_{\dot{K}_{q(\cdot)}^{\alpha,p}\left(\mathbb{R}^n\right)}^{p}&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^p\\
&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^p\\
&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+1}^{\infty}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^p\\
&=F_1+F_2+F_3\end{aligned}$$

Firstly we estimate F_2 by Lemma 2.6 we can see

$$F_2 \le C\|b\|^p_{BMO(\mathbb{R}^n)} \sum_{j=-\infty}^{\infty} |\lambda_j|^p$$

Now we consider the estimates of F_1. Note that for each $x \in A_k$, $y \in A_j$, and $j \le k-2$, by generalized Hölder's inequality and Lemma 2.2, we have

$$\begin{aligned} \left|[b,T]g_j\right| &\le \int_{A_j} \left|K(x,y)(b(x)-b(y))g_j(y)\right|\mathrm{d}y \le C\int_{A_j} \frac{\left|(b(x)-b(y))g_j(y)\right|}{|x-y|^n}\mathrm{d}y \\ &\le C2^{-nk}\left|b(x)-b_{B_j}\right|\int_{A_j}\left|g_j(y)\right|\mathrm{d}y + \int_{A_j}\left|b_{B_j}-b(y)\right|\left|g_j(y)\right|\mathrm{d}y \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[\left|b(x)-b_{B_j}\right|\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\|\left(b_{B_j}-b\right)\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right] \end{aligned}$$

Thus by Lemma 2.5 we get

$$\begin{aligned} &\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[\left\|\left(b-b_{B_j}\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \left\|\left(b_{B_j}-b\right)\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right] \\ &\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left[(k-j)\|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} + \|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\right] \\ &\le C(k-j)2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)} \times \left\|\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left\|\chi_j\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \end{aligned} \tag{2.16}$$

Thus by Lemma 2.3, Lemma 2.4 and noting that $\left\|\chi_i\right\|_{L^{s(\cdot)}(\mathbb{R}^n)} \le \left\|\chi_{B_i}\right\|_{L^{s(\cdot)}(\mathbb{R}^n)}$ we get

$$\begin{aligned} &\le C(k-j)2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|_{BMO(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}\left(\left|B_k\right|\left\|\chi_{B_k}\right\|^{-1}_{L^{q'(\cdot)}(\mathbb{R}^n)}\right) \\ &\le C\|b\|_{BMO(\mathbb{R}^n)}2^{nk}2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\left(\frac{\left\|\chi_{B_j}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}{\left\|\chi_{B_k}\right\|_{L^{q'(\cdot)}(\mathbb{R}^n)}}\right) \\ &\le C(k-j)\|b\|_{BMO(\mathbb{R}^n)}2^{(j-k)n\delta_2 - j\alpha} \end{aligned} \tag{2.17}$$

When $1 < p < \infty$ and $\alpha < \delta_2 n$, by Hölder's inequality and (2.17), we calculations

$$\begin{aligned} F_1 &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|[b,T]g_j\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\ &\le C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|(k-j)\|b\|_{BMO(\mathbb{R}^n)}2^{(j-k)n\delta_2 - j\alpha}\right)^p \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p 2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right) \times \left(\sum_{j=-\infty}^{k-2}(k-j)^{p'}2^{(k-j)[-n\delta_2+\alpha]\frac{p'}{2}}\right)^{\frac{p_1}{p_1'}} \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p 2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\left(\sum_{k=j+1}^{\infty}2^{(k-j)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\ &\le C\|b\|^p_{BMO(\mathbb{R}^n)}\sum_{k=-\infty}^{\infty}\left|\lambda_j\right|^p \end{aligned} \tag{2.18}$$

when $0<p_1\le 1$ by $\alpha<\delta_2 n$, we get

$$\begin{aligned}
F_1 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right)^p\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|(k-j)\|b\|_{BMO\left(\mathbb{R}^n\right)}2^{(j-k)n\delta_2-j\alpha}\right)^p\\
&\le C\|b\|_{BMO\left(\mathbb{R}^n\right)}^p\sum_{k=-\infty}^{\infty}2^{k\alpha p}\sum_{j=-\infty}^{k-2}\left|\lambda_j\right|^p(k-j)2^{[(j-k)n\delta_2-j\alpha]p}\\
&\le C\|b\|_{BMO\left(\mathbb{R}^n\right)}^p\sum_{k=-\infty}^{\infty}\left|\lambda_j\right|^p\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)[-n\delta_2+\alpha]p}\\
&\le C\|b\|_{BMO\left(\mathbb{R}^n\right)}^p\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p
\end{aligned}\tag{2.19}$$

Finally we consider the estimates of F_3. Note that for each $x\in A_k$, $y\in A_j$, and $k\ge j+2$, by generalized Hölder's inequality and Lemma 2.2. we have

$$\begin{aligned}
\left|[b,T]g_j\right| &\le \int_{A_j}\left|K(x,y)\left(b(x)-b(y)\right)g_j(y)\right|\mathrm{d}y\\
&\le C\int_{A_j}\frac{\left|\left(b(x)-b(y)\right)g_j(y)\right|}{|x-y|^n}\mathrm{d}y\\
&\le C2^{-nj}\left|b(x)-b_{B_k}\right|\int_{A_j}\left|g_j(y)\right|\mathrm{d}y+\int_{A_j}\left|b_{B_k}-b(y)\right|\left|g_j(y)\right|\mathrm{d}y\\
&\le C2^{-nk}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left[\left|b(x)-b_{B_k}\right|\left|\chi_j\right|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}+\left\|\left(b_{B_k}-b\right)\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\right]
\end{aligned}\tag{2.20}$$

Thus by Proposition 1.2, and Lemma 2.5, we get

$$\begin{aligned}
&\left\|\left([b,T]g_j\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\\
&\le C2^{-nj}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left[\left\|\left(b-b_{B_k}\right)\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}+\left\|\left(b_{B_k}-b\right)\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\right]\\
&\le C2^{-nj}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left[\|b\|_{BMO\left(\mathbb{R}^n\right)}\left\|\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}+(j-k)\|b\|_{BMO\left(\mathbb{R}^n\right)}\left\|\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}\right]\\
&\le C(j-k)2^{-nj}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|b\|_{BMO\left(\mathbb{R}^n\right)}\times\left\|\chi_k\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_j\right\|_{L^{q'(\cdot)}\left(\mathbb{R}^n\right)}
\end{aligned}\tag{2.21}$$

Thus by Lemma 2.3, Lemma 2.4 and noting that $\left\|\chi_i\right\|_{L^{s(\cdot)}\left(\mathbb{R}^n\right)}\le\left\|\chi_{B_i}\right\|_{L^{s(\cdot)}\left(\mathbb{R}^n\right)}$ we get

$$\begin{aligned}
&\le C(j-k)2^{-nj}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\|b\|_{BMO\left(\mathbb{R}^n\right)}\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left(\left|B_j\right|\left\|\chi_{B_j}\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}^{-1}\right)\\
&\le C(j-k)\|b\|_{BMO\left(\mathbb{R}^n\right)}2^{nj}2^{-nj}\left\|g_j\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}\left(\frac{\left\|\chi_{B_k}\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}}{\left\|\chi_{B_j}\right\|_{L^{q(\cdot)}\left(\mathbb{R}^n\right)}}\right)\\
&\le C(j-k)\|b\|_{BMO\left(\mathbb{R}^n\right)}2^{(k-j)n\delta_1-j\alpha}
\end{aligned}\tag{2.22}$$

When $1<p<\infty$ and $\alpha>-\delta_1 n$, by Hölder's inequality and (2.22),we calculations

$$\begin{aligned}
F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\left\|\left[b,T\right]g_j\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\left(j-k\right)\|b\|_{BMO(\mathbb{R}^n)}2^{(k-j)n\delta_1-j\alpha}\right)^p \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|^p 2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right)\times\left(\sum_{j=k+2}^{\infty}\left(j-k\right)^{p'}2^{(j-k)[-n\delta_2+\alpha]\frac{p'}{2}}\right)^{\frac{p}{p'}} \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|^p 2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\left(\sum_{k=\infty}^{j-2}2^{(j-k)[-n\delta_2+\alpha]\frac{p}{2}}\right) \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p
\end{aligned}\tag{2.23}$$

when $0<p\le 1$ by $\alpha>-\delta_1 n$, we get

$$\begin{aligned}
F_3 &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\left\|\left(\left[b,T\right]g_j\right)\chi_k\right\|_{L^{q(\cdot)}(\mathbb{R}^n)}\right)^p \\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|\|b\|_{BMO(\mathbb{R}^n)}2^{(k-j)n\delta_1-j\alpha}\right)^p \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+2}^{\infty}\left|\lambda_j\right|2^{(j-k)[-n\delta_1-\alpha]}\right)^p \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p\left(\sum_{k=\infty}^{j-2}2^{(j-k)[-n\delta_1-\alpha]p}\right) \\
&\le C\|b\|_{BMO(\mathbb{R}^n)}^p\sum_{j=-\infty}^{\infty}\left|\lambda_j\right|^p
\end{aligned}\tag{2.24}$$

combining (2.14)-(2.24) the prove is completed.

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The Impact of the Earth's Movement through the Space on Measuring the Velocity of Light

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Abstract

Goal of this experiment is basically measuring the velocity of light. As usual we will measure two-way velocity of light (from A to B and back). In contrast to the similar experiments we will not assume that speeds of light from A to B and from B to A are equal. To achieve this we will take into account Earth's movement through the space, rotation around its axis and apply "least squares method for cosine function", which will be explained in Section 9. Assuming that direction East-West is already known, one clock, a source of light and a mirror, is all equipment we need for this experiment.

Keywords

Speed of Light, One Way Speed of Light, Least Squares Method for Cosine Function

1. Introduction

Observe the planet Earth. The Earth orbits the Sun. For this motion we will join the vector $\mathbf{v}_1$. Sun orbits the center of the Milky Way. For this motion we will join the vector $\mathbf{v}_2$. In relation to the center of the Milky Way, we can join to the Earth movement sum of vectors

$$\mathbf{v}_1 + \mathbf{v}_2.$$

It is also known that our Galaxy is moving relative to other galaxies (or to a point in the space outside the Milky Way Galaxy). Similarly, to this motion we could join the vector $\mathbf{v}_3$.

Denote by $\mathbf{v}$ the sum of all these vectors

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \cdots. \tag{1}$$

At the end of the sum three points are left, because eventually there may be some other movements.

In the period of 24 h vectors $\mathbf{v}_2$, $\mathbf{v}_3$ can be taken as constants, while the vector $\mathbf{v}_1$ by making a certain error

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could also be taken as constant.

Thus for the Earth's motion through the space within 24 h, we can join the constant vector **v**.

The speed and direction Earth orbits the Sun are known, and let v_0 represent its avarage speed.

Suppose that some approximate values for vectors $\mathbf{v}_2$ and $\mathbf{v}_3$ are known as well. On the basis of these values, let suppose that we have inequality

$$|\mathbf{v}| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| \geq \mathrm{v}_0. \tag{2}$$

2. Planning an Experiment

Suppose that an arbitrary point **A** is given. Earth rotation axis will be taken as the **z** coordinate, and as the plane **xy** we will take the plane passing through point **A** and perpendicular to the **z** axis. In this case it is natural to take section of the plane **xy** and **z** axis as the center of the coordinate system. In addition to point **A** let the points **B** and **D** are given. Line **AB** lies in the plane **xy** and parallel to the direction of the Earth's rotation. Distance **AB** will be marked with L. For the **x** axis, at some initial time $\mathbf{t}_0$, we will take the line in the plane **xy**, parallel to **AB**. The projection of the vector **v** in the plane **xy** denote by $\mathbf{v}_{xy}$. Due to the Earth's rotation the direction of **AB** will be changed, so that it will be changed the angle, marked by Φ, between the **x** axis (which remained fixed) and the line **AB**. Let at point **A** we have a clock and some source of light. Suppose that speed of light in the direction **AB** is given by equation

$$\mathbf{c}_{\mathrm{AB}} = \mathbf{c} - |\mathbf{v}_{\mathrm{xy}}| * \cos(\Phi). \tag{1}$$

Point **D** will be chosen so the line **AD** is parralel to direction South-North. Distance **AD** is marked by L_1. Angle between line **AD** and **z** axis we will denote by φ. Angle φ actually represents Latitude of point **A** on the Earth's surface, thus it remains unchanged during the experiment.

The projection of the vector **v** on **z** axis denote by $\mathbf{v}_z$ (actually $\mathbf{v}_2 + \mathbf{v}_3$, because $\mathbf{v}_1$ is perpendicular on **z** axis). Assume that the speed of signal in the direction **AD** is given by equation

$$\mathbf{c}_{\mathrm{AD}} = \mathbf{c} - |\mathbf{v}_{\mathrm{z}}| * \cos(\varphi) \tag{2}$$

where **c** represents "velocity of light in vacuum for a body at rest". Our aim is to find the constant **c**, vectors $\mathbf{v}_{xy}$ and $\mathbf{v}_z$.

3. Conducting an Experiment

In some moment T_0 we will send signal from point **A** to point **B**. The angle between the axis **x** and $\mathbf{v}_{xy}$ is marked by Θ.

Once the signal arrived at point **B** it will be reflected back to point **A**.

Difference between the time when the signal was being sent from point **A**, and the time when the signal reached to the point **A** is denoted by $\mathbf{t}_0$.

At the same time we will send signal from point **A** to **D** and return back to point **A**. Difference between the time when signal was being sent and reached to point **A** we will denote by $\boldsymbol{\tau}_0$.

The same procedure will be within 24 h repeated N (N > 4) times, whereas the time between the two sets of consecutive procedure to be same and equal to 24 h/N.

In that way we will get the series $\{\mathbf{t}_i\}$ and $\{\boldsymbol{\tau}_i\}$

$$\{t_i\}, \{\tau_i\} \quad i \in \{0, 1, \cdots, N-1\}. \tag{1}$$

To the each t_i we can join an angle α_i between **x** axis and line **AB**.

In that way we get the series

$$\{\alpha_i\} \quad \text{where} \quad \alpha_i = i * 2\Pi/N - \Theta, \; i \in \{0, 1, \cdots, N-1\}. \tag{2}$$

By assumption (3.1) the speed of the signal $\mathbf{c}_i$ in the direction **AB** is equal to

$$\mathbf{c}_{\mathrm{i}}(AB) = \mathbf{c} - |\mathbf{v}_{\mathrm{xy}}| * \cos(\mathrm{i} * 2\Pi/\mathrm{N} - \Theta) \tag{3}$$

and in opposite direction **BA**

$$\mathbf{c}_i(BA) = \mathbf{c} + |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta). \tag{4}$$

It follows that

$$t_i = \frac{L}{\mathbf{c} - |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta)} + \frac{L}{\mathbf{c} + |\mathbf{v}_{xy}| * \cos(i * 2\Pi/N - \Theta)} \Rightarrow \tag{5}$$

$$t_i = \frac{2 * L * \mathbf{c}}{\mathbf{c}^2 - |\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta)}. \tag{6}$$

If we swap the roles of the points **A** and **B**, we would get the same formula as in (6). Therefore it is completely irrelevant whether direction of the vector $\mathbf{v}_{xy}$ is equal to direction **AB** or **BA**.

We assume that

$$\mathbf{c}^2 - |\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta) > 0 \Leftrightarrow t_i > 0$$

for $i \in \{0, 1, \cdots, N-1\}, \Theta \in [-\Pi/2, \Pi/2]$.

It would be in principle our experiment.

4. Computing the Values of c, $|\mathbf{v}_{xy}|$ and Θ

In this section we will deal only with the measurements in direction East-West.

Let t_i is given by (3.6) and

$$c_i = \frac{2 * L}{t_i}, \quad i \in \{0, 1, \cdots, N\} \tag{1}$$

denote the average speed c_i (from point **A** to point **B** and back to **A**).

It follows that c_i can be written as

$$c_i = \mathbf{c} - \frac{|\mathbf{v}_{xy}|^2 * \cos^2(i * 2\Pi/N - \Theta)}{\mathbf{c}} + e_i \Rightarrow \tag{2}$$

where e_i represents some experimental error. Replacing

$$\cos^2(i * 2\Pi/N - \Theta) = (\cos(2 * i * 2\Pi/N - 2\Theta) + 1)/2$$

we get

$$c_i = \left(\mathbf{c} - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}}\right) - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}} * \cos(2 * (i * 2\Pi/N - \Theta)) + e_i \tag{3}$$

in short form

$$c_i = \mathbf{B} - \mathbf{A} * \cos(2 * (i * 2\Pi/N - \Theta)) + e_i, \quad i \in \{0, 1, \cdots, N-1\} \tag{4}$$

$$\mathbf{B} = \mathbf{c} - \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}} \tag{5}$$

$$\mathbf{A} = \frac{|\mathbf{v}_{xy}|^2}{2 * \mathbf{c}}, \text{ where } \mathbf{A} \ge \mathbf{0}. \tag{6}$$

The coefficients **A**, **B** and **Θ** will be chosen so the sum of squares

$$S_1(\mathbf{B}, \mathbf{A}, \Theta) = \sum e_i^2 = \sum \left(c_i - \mathbf{B} + \mathbf{A} * \cos(2 * (i * 2\Pi/N - \Theta))\right)^2 \tag{7}$$

has a minimum value.

To acheive our goal we are going to apply **Theorem 1** for k = 2.

For the sake of simplicity we've only considered cases when

$$\sum a_i * \cos(2*\alpha_i) \neq 0 \text{ and } \mathbf{A}_0 \neq 0.$$

Thus we have

$$\mathbf{B}_0 = \mathbf{c}_\text{m} = \left(\sum_{i=0}^{N-1} c_i\right) \Big/ N \tag{8}$$

$$\mathbf{tg}(2*\Theta_\mathbf{0}) = \frac{\sum_{i=0}^{N-1} a_i * \sin(2*\alpha_i)}{\sum_{i=0}^{N-1} a_i * \cos(2*\alpha_i)} \tag{9}$$

$$\mathbf{A_0} = -\frac{2*\sum_{i=0}^{N-1} a_i * \cos(2*\alpha_i - 2*\Theta_0)}{N} \tag{10}$$

$$a_i = c_i - \mathbf{c}_\text{m},\ \alpha_i = i*2*\Pi/N,\ \ i \in \{0,1,\cdots,N-1\}.$$

We'll make a small digression. From **Lemma 1** it follows

$$\begin{aligned}\sum a_i * \cos(k*\alpha_i) &= \sum (c_i - \mathbf{c_m}) * \cos(k*\alpha_i) \\ &= \sum c_i * \cos(k*\alpha_i) - \sum \mathbf{c}_\text{m} * \cos(k*\alpha_i) \\ &= \sum c_i * \cos(k*\alpha_i)\end{aligned}$$

In the similiar way we can get

$$\sum a_i * \sin(k*\alpha_i) = \sum c_i * \sin(k*\alpha_i).$$

Generally we have $\text{tg}(\text{x}) = \text{tg}(\text{x} - \Pi) \Rightarrow \text{tg}(2*\Theta) = \text{tg}(2*\Theta - \Pi)$. From (9) $\Rightarrow$

$$\Theta_1 = \frac{1}{2} * \mathbf{Atan}\left(\frac{\sum a_i * \sin(2*\alpha_i)}{\sum a_i * \cos(2*\alpha_i)}\right) \tag{11}$$

Function Atan () takes values at interval (−Π/2, Π/2).

$$\Theta_2 = \Theta_1 - \Pi/2$$

If we consider $\mathbf{A}_0$ as function of $\Theta \Rightarrow \mathbf{A_0}(\Theta_2) = \mathbf{A_0}(\Theta_1 - \Pi/2) = -\mathbf{A_0}(\Theta_1)$.

From (6) it folows that between the values Θ_1 and Θ_2 we have to choose that one for which $\mathbf{A}_0 > 0$.

From (5) and (6) we can derive values for **c** and $|\mathbf{v}_\text{xy}|$.

$$\mathbf{c} = \mathbf{B}_0 + \mathbf{A}_0 = \mathbf{c}_\text{m} + \mathbf{A}_0 \tag{12}$$

$$|\mathbf{v}_\text{xy}| = \pm\sqrt{2*\mathbf{A}_0 * \mathbf{c}} \tag{13}$$

We don't know exact direction of vector $\mathbf{v}_\text{xy}$, thus positive and negative value are assigned to $|\mathbf{v}_\text{xy}|$.

5. Comparison between Two Methods

In this section we will make comparison between "the least squares method" and "the least squares method for cosine function".

Let consider $\{c_i\}$ given by (4.1) as the series of mutually independent measurements.

Let $\mathbf{c}_\text{m}$ represents the mean value of serial $\{c_i\}$.

$$\mathbf{c}_\text{m} = \left(\sum c_i\right) \big/ N \tag{1}$$

If we apply Least squares method, Variance V_1 is given by

$$V_1 = \sum \left(c_i - \mathbf{c}_\mathrm{m}\right)^2 \tag{2}$$

and standard deviation σ_1 by

$$\sigma_1 = \sqrt{V_1/N}. \tag{3}$$

Suppose that to the each c_i we joined the time when measurement took place, or rather the angle between the direction of **AB** and vector $\mathbf{v}_{xy}$. Expected value $E_2(\alpha_i)$ for "The Least squares method for cosine function" is given by

$$E_2\left(\alpha_i\right) = y_i = \mathbf{B}_0 - \mathbf{A}_0 \cos\left(k * \left(\alpha_i - \Theta_0\right)\right) \tag{4}$$

where

$$\alpha_i = i * 2\Pi/N,\ i \in \{0,1,\cdots,N-1\}. \tag{5}$$

Denote a_i by

$$a_i = c_i - \mathbf{B}_0 = c_i - \mathbf{c}_m.$$

Let us find Variance V_2 for this method

$$\begin{aligned} V_2 &= \sum \left(c_i - y_i\right)^2 = \sum \left(c_i - \mathbf{c}_m + \mathbf{A}_0 * \cos\left(k * \left(\alpha_i - \Theta_0\right)\right)\right)^2 \\ &= \sum \left(a_i + \mathbf{A}_0 * \cos\left(k * \left(\alpha_i - \Theta_0\right)\right)\right)^2 \\ &= \sum a_i^2 + 2 * \mathbf{A}_0 * \sum a_i * \cos\left(k * \left(\alpha_i - \Theta_0\right)\right) + \mathbf{A}_0^2 * \sum \frac{1 + \cos\left(2 * k\left(\alpha_i - \Theta_0\right)\right)}{2} \\ &= \sum a_i^2 + 2 * \mathbf{A}_0 * \sum a_i * \cos\left(k * \left(\alpha_i - \Theta_0\right)\right) + \frac{\mathrm{N} * \mathbf{A}_0^2}{2} \\ &= \left|\text{from (10.5)}\right| = \sum a_i^2 - \frac{\mathrm{N} * \mathbf{A}_0^2}{2} = \sum \left(\mathrm{c_i} - \mathbf{c}_\mathrm{m}\right)^2 - \frac{\mathrm{N} * \mathbf{A}_0^2}{2} \end{aligned} \tag{6}$$

$$V_2 = V_1 - \frac{\mathrm{N} * \mathbf{A}_0^2}{2} \geq 0 \Rightarrow V_1 \geq V_2. \tag{7}$$

Standard deviation σ_2 for this method is given by

$$\sigma_2 = \sqrt{V_2/N}. \tag{8}$$

From (7) $\Rightarrow \sigma_1 \geq \sigma_2$ From (7) $\Rightarrow V_2 \geq 0 \Rightarrow \sum \left(c_i - \mathbf{c}_m\right)^2 \geq \frac{N * \mathbf{A}_0^2}{2} \Rightarrow \sqrt{2} * \sigma_1 \geq \left|\mathbf{A}_0\right|$.

If standard deviation σ_2 is bigger then some expected value it means either our measurement are not accurate enough or our method (curve) doesn't suit to our data.

6. Analysys of South-North Measurements

In this chapter we will deal with the series $\{\tau_i\}$ given by (3.1).

Just to remind that $\boldsymbol{\tau}_i$ represents time it takes for signal to travel from **A** to **D** and back to **A** in direction South-North.

$$\tau_i = \frac{L_1}{\mathbf{c} - \left|\mathbf{v}_z\right| * \cos(\varphi)} + \frac{L_1}{\mathbf{c} + \left|\mathbf{v}_z\right| * \cos(\varphi)} \Rightarrow \tag{1}$$

$$\tau_i = \frac{2 * L_1 * \mathbf{c}}{\mathbf{c}^2 - \left|\mathbf{v}_z\right|^2 * \cos^2(\varphi)} \tag{2}$$

Let

$$\gamma_1 = \frac{2 * L_1}{\tau_i} \quad i \in \{0,1,\cdots,N-1\} \tag{3}$$

denote the average speed γ_i. In that way we get the series $\{\gamma_i\}$

$$\gamma_i = \mathbf{c} - \frac{|\mathbf{v_z}|^2 * \cos^2(\varphi)}{\mathbf{c}} + e_i \tag{4}$$

where e_i represents some experimental error.

Since angle φ kept constant value during the experiment we could apply Least squares method to the series given by (4).

Let denote γ_{m} by

$$\gamma_{\mathrm{m}} = \left(\sum \gamma_i\right) / N \tag{5}$$

mean value of the series $\{\gamma_i\}$.

We can calculate Variance V_1

$$V_1 = \sum \left(\gamma_i - \gamma_{\mathrm{m}}\right)^2 \tag{6}$$

and standard deviation σ_1

$$\sigma_1 = \sqrt{\frac{V_1}{N}}. \tag{7}$$

If standard deviation σ_1 is bigger then some expected value we should declare the experiment failed.

Combining equations (4) and (5) we get

$$|\mathbf{v}_z| = \pm \sqrt{\frac{\mathbf{c}^2 - \mathbf{c} * \gamma_{\mathrm{m}}}{\cos(\varphi)}} \quad \left(\mathbf{c} \ge \gamma_{\mathrm{m}}, \cos\varphi \ne 0\right). \tag{8}$$

We don't know exact direction of vector $\mathbf{v}_z$, thus positive and negative value were assigned to $|\mathbf{v}_z|$.

7. Conclusions

From (5.13) and (7.8) it follows that length of vector $\mathbf{v}$ is given by

$$|\mathbf{v}| = \sqrt{\mathbf{v}_{xy}^2 + \mathbf{v}_z^2} \tag{1}$$

while vector $\mathbf{v}$ is given by

$$\mathbf{v} = \pm \mathbf{v}_{xy} \pm \mathbf{v}_z. \tag{2}$$

Recall (from 2.1) that vector $\mathbf{v}$ can be written also as

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3. \tag{3}$$

Suppose that during one year the same experiments have been repeated 2∗K times. In that way we will get the series

$$\left\{|\mathbf{v}(\mathrm{i})|\right\}_{i=1}^{2K} \tag{4}$$

where $|\mathbf{v}(\mathrm{i})|$ represents length of vector given by Equation (2) or (3) at i-*th* try.

Let $\mathbf{v}_1(\mathrm{i}+\mathrm{K})$ and $\mathbf{v}_1(\mathrm{i})$ denote velocity at which Earth orbits the Sun at $(\mathrm{i}+\mathrm{K})$-th and i-th try.

Suppose also that origins of vectors $\mathbf{v}_\mathrm{i}(\mathrm{i}+\mathrm{K})$ and $\mathbf{v}_\mathrm{i}(\mathrm{i})\ \mathrm{i} \in \{1,2,\cdots,\mathrm{K}\}$ lay on the diameter of Earth orbit around the Sun, so they are parallel but in oposite directions.

Mean value $\mathrm{v_m}$ of the serial (3) is given by

$$\mathrm{v_m} = \left(\sum |\mathbf{v}(\mathrm{i})|\right) / (2K). \tag{5}$$

Depending on $\mathrm{v_m}$ we will consider following cases:

1) $\frac{\mathrm{v_m}}{\mathrm{v_0}} \to 0$

In other words $\mathrm{v_m}$ is significantly less than $\mathrm{v_0}$ what is in contradiction to our hypotesis (2.2).

In this case we have to reject hypothesis given by (3.1) and declare that velocity of light is not effected by Earth's movement through the space.

This results is consistent with some other experiments, for example with Michelson-Morley experiment.

2) $\mathrm{v_m} > \mathrm{v_0}$

During the experiments in period of one year $\mathbf{v}_1$ is changing, while $\mathbf{v}_2 + \mathbf{v}_3$ is keeping the constant value.

Recall that vector $\mathbf{v}_1$ is perpendicular to $\mathbf{z}$ axis.

Denote vector $\mathbf{u}$ by

$$\mathbf{u} = \mathbf{v}_2 + \mathbf{v}_3 \tag{6}$$

$$\left(\text{let } \mathrm{proj}_{xy}(\mathbf{a}) \text{ represents orthogonal projection of vector } \mathbf{a} \text{ on plane } \mathbf{xy}\right) \tag{7}$$

$$\mathbf{v}_{\mathrm{xy}} = proj_{xy}(\mathbf{v}) = proj_{xy}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = proj_{xy}(\mathbf{v}_1) + proj_{xy}(\mathbf{u}) = \mathbf{v}_1 + \mathbf{u}_{xy} \Rightarrow \tag{8}$$

$$\left|\mathbf{v}_{\mathrm{xy}}(i)\right|^2 = \left|\mathbf{v}_1(i)\right|^2 + \left|\mathbf{u}_{\mathrm{xy}}\right|^2 + 2*\mathbf{v}_1(i)*\mathbf{u}_{\mathrm{xy}} \tag{9}$$

$$\left|\mathbf{v}_{\mathrm{xy}}(i+K)\right|^2 = \left|\mathbf{v}_1(i+K)\right|^2 + \left|\mathbf{u}_{\mathrm{xy}}\right|^2 + 2*\mathbf{v}_1(i+K)*\mathbf{u}_{\mathrm{xy}}. \tag{10}$$

If we replace $\left|\mathbf{v}_1(i)\right|$ and $\left|\mathbf{v}_1(i+K)\right|$ by v_0

$$\left|\mathbf{v}_1(i)\right| \approx v_0$$

$$\left|\mathbf{v}_1(i+K)\right| \approx v_0$$

(v_0 represents average speed Earth orbits the Sun).

From (9) and (10) we can get approximate value for $\left|\mathbf{u}_{\mathrm{xy}}(i)\right|$

$$\left|\mathbf{u}_{\mathrm{xy}}(i)\right| \approx \sqrt{\frac{\left|\mathbf{v}_{xy}(i)\right|^2 + \left|\mathbf{v}_{xy}(i+K)\right|^2 - 2*\mathrm{v}_0^2}{2}} \quad i \in \{1, 2, \cdots, K\}. \tag{11}$$

We can form serial

$$\left\{\left|\mathbf{u}_{\mathrm{xy}}(\mathrm{i})\right|\right\}_{i=1}^{K}. \tag{12}$$

Mean value $\mathrm{u_{xy}}$ of the serial (12) is given by

$$\mathrm{u_{xy}} = \left(\sum_{i=1}^{K}\left|\mathbf{u}_{\mathrm{xy}}(\mathrm{i})\right|\right)\Big/K. \tag{13}$$

Let find standard deviation σ_1 for serial (13).

If σ_1 is bigger then some expected value we have to decline our hypothesis (2.1) and declare the experiment failed.

$$\mathbf{v}_{\mathrm{z}} = proj_z(\mathbf{v}) = proj_z(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = proj_z(\mathbf{v}_1) + proj_z(\mathbf{u}) = \mathbf{u}_{\mathrm{z}} \tag{14}$$

$$\left\{\left|\mathbf{u}_{\mathrm{z}}(\mathrm{i})\right|\right\}_{i=1}^{2K} \tag{15}$$

where $\left|\mathbf{u}_{\mathrm{z}}(\mathrm{i})\right| = \left|\mathbf{v}_{\mathrm{z}}(i)\right|$ at i-th try.

For serial (15) mean value $\mathrm{u_z}$ is given by

$$\mathrm{u_z} = \left(\sum_{i=1}^{2K}\left|\mathbf{u}_{\mathrm{z}}(\mathrm{i})\right|\right)\Big/(2K).$$

Let standard deviation for serial (15) is marked by σ_2.

If σ_2 is bigger then some expected value we have to decline our hypothesis (2.1) and declare the experiment failed.

Otherwise hypothesis given by (3.1) holds and we can conclude that velocity of light depends on Earth's movement through space. In other words velocity of light depends on the direction in which has been measured, what would be in contradiction with Michelson-Morley experiment [1].

The speed that Solar system moves in the space in this case is given by equation

$$\mathrm{u} = \sqrt{\mathrm{u}_{\mathrm{xy}}^2 + \mathrm{u}_{\mathrm{z}}^2}. \tag{16}$$

Note that while performing the experiment we committed some mistakes.

It was not taken into account the speed of Earth's rotation. This problem can be solved by conducting an experiment at place closer to the Earth's poles, and thus the speed of Earth's rotation taken as small as we want. On other hand this would be counter-productive to our conditions for South-North measurement. Ideally, E-W experiment should be performed on the North/South Pole and S-N experiment at some place on equator.

In addition, within 24 h the Earth changes its direction and the speed at which it revolves around the Sun. We can't solve this problem but we can assume that this speed is relatively small comparing to total speed at which Earth moves through the space.

8. Lemma 1

If N, k are natural numbers ($1 < N$, $0 < k < N$) and Θ an arbitrary angle then

$$\sum_{j=0}^{N-1} \sin\left(\left(j * \mathrm{k}/N\right) * 2\Pi - \Theta\right) = 0 \tag{1}$$

$$\sum_{j=0}^{N-1} \cos\left(\left(j * \mathrm{k}/N\right) * 2\Pi - \Theta\right) = 0 \tag{2}$$

Proof.

$$\sum_{j=0}^{N-1} \cos\left(\left(j * \mathrm{k}/N\right) * 2\Pi - \Theta\right) + \mathbf{i} * \sin\left(\left(j * \mathrm{k}/N\right) * 2\Pi - \Theta\right) = \mathrm{e}^{-\Theta * \mathbf{i}} * \sum_{j=0}^{N-1} \mathrm{e}^{\left(j * k/N\right) * 2\Pi * \mathbf{i}} = \mathrm{e}^{-\Theta * \mathbf{i}} * \frac{M}{N}$$

where

$$M = \mathrm{e}^{\left(\left(N * k/N\right) * 2\Pi\right) * \mathbf{i}} - 1 = \mathrm{e}^{\left(k * 2\Pi\right) * \mathbf{i}} = 1 - 1 = 0$$

$$N = \mathrm{e}^{\left(\left(k/N\right) * 2\Pi\right) * \mathbf{i}} - 1 \neq 0, \quad 0 < \left(k/N\right) * 2\Pi < 2\Pi$$

Q.E.D.

9. Theorem 1. Least Squares Method for Cosine Function

Suppose we are given the series $\{c_i\}$, $c_i > 0$, $i \in \{0, 1, \cdots, N-1\}$ and there are at least two p, q thus $c_p <> c_q$

Let take arbitrary coefficients **B**, **A**, $\boldsymbol{\Theta}$ and form equations

$$c_i = \mathbf{B} - \mathbf{A} * \cos\left(k * \left(i * 2\Pi/N - \Theta\right)\right) + e_i, \quad 0 < k < N/2 \tag{1}$$

$$\sum e_i^2 = \sum \left(c_i - \mathbf{B} + \mathbf{A} * \cos\left(k * \left(\alpha_i - \Theta\right)\right)\right)^2$$

Define function g(B, A, Θ) by

$$g\left(\mathbf{B}, \mathbf{A}, \Theta\right) = \sum e_i^2 = \sum \left(c_i - \mathbf{B} + \mathbf{A} * \cos\left(k * \left(\alpha_i - \Theta\right)\right)\right)^2 \tag{2}$$

We will prove that in case $\mathbf{A_0} \neq 0$, function g() has a minimum value at point $\left(\mathbf{B}_0, \mathbf{A}_0, \Theta_0\right)$

$$\mathbf{B}_0 = \mathbf{c}_{\mathrm{m}} = \left(\sum_{i=0}^{N-1} c_i\right) \Big/ N \tag{3}$$

$$\mathbf{tg}\left(k*\Theta_{\mathbf{0}}\right)=\frac{\sum_{i=0}^{N-1}a_i*\sin\left(k*\alpha_i\right)}{\sum_{i=0}^{N-1}a_i*\cos\left(k*\alpha_i\right)} \tag{4}$$

$$\mathbf{A_0}=-\frac{2*\sum_{i=0}^{N-1}a_i*\cos\left(k*\alpha_i-k*\Theta_0\right)}{N} \tag{5}$$

where $a_i=c_i-\mathbf{c}_{\mathrm{m}}$, $\alpha_i=i*2\Pi/N$, $i\in\{0,1,\cdots,N-1\}$.

Proof.

Let **B**, **A** and **Θ** have arbitrary values

$$\begin{aligned}
g(\mathbf{B},\mathbf{A},\Theta) &= \sum\left(c_i-\mathbf{B}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2\\
&= \sum\left(\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)+\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)\right)^2\\
&= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\sum\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)\\
&\quad +\sum\left(c_i-\mathbf{c}_{\mathrm{m}}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2\\
&= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\sum\left(c_i-\mathbf{c}_{\mathrm{m}}\right)+2*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)*\mathbf{A}*\sum\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right)\\
&= \mathrm{N}*\left(\mathbf{c}_{\mathrm{m}}-\mathbf{B}\right)^2+\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right)
\end{aligned}$$

thus we get

$$\mathrm{g}(\mathbf{B},\mathbf{A},\Theta)\geq\mathrm{g}\left(\mathbf{c}_{\mathrm{m}},\mathbf{A},\Theta\right) \tag{6}$$

In that way we can reduce function g() from function of three variables to fuction of two variables A and **Θ**, keeping coefficent **B** fixed and equal to $\mathbf{c}_{\mathrm{m}}$.

Now we can write the function g() in the form

$$\begin{aligned}
g(\mathbf{A},\Theta) &= \sum\left(c_i-\mathbf{c_m}+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2=\sum\left(a_i+\mathbf{A}*\cos\left(k*\left(\alpha_i-\Theta\right)\right)\right)^2\\
&= \sum a_i^2+2*\mathbf{A}*\sum a_i*\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\mathbf{A}^2*\sum\cos^2\left(k*\left(\alpha_i-\Theta\right)\right)\\
&\Rightarrow\left|\cos^2\left(k*\left(\alpha_i-\Theta\right)\right)=\left(\cos\left(2k*\left(\alpha_i-\Theta\right)\right)+1\right)/2\right|
\end{aligned}$$

$$\mathrm{g}(\mathbf{A},\Theta)=\frac{\mathrm{N}*\mathbf{A}^2}{2}+2*\mathbf{A}*\sum a_i*\cos\left(k*\left(\alpha_i-\Theta\right)\right)+\sum a_i^2. \tag{7}$$

In order to find minimum for function g(), first we have to find partial derivates with respect to A and Θ and critical point (A_0, Θ_0)

$$\frac{\partial\mathrm{g}\left(\mathbf{A}_0,\Theta_0\right)}{\partial\mathbf{A}}=0,\quad\frac{\partial\mathrm{g}\left(\mathbf{A}_0,\Theta_0\right)}{\partial\Theta}=0. \tag{8}$$

Let us find the first partial derivatives

$$\begin{aligned}
\frac{\partial g}{\partial\Theta} &= 2*\mathrm{k}*\mathbf{A}*\sum a_i*\sin\left(k*\alpha_i-k*\Theta\right)\\
&= 2*\mathrm{k}*\mathbf{A}*\left(\cos\left(\mathrm{k}*\Theta\right)*\sum\mathrm{a_i}*\sin\left(\mathrm{k}*\alpha_i\right)-\sin\left(\mathrm{k}*\Theta\right)*\sum\mathrm{a_i}*\cos\left(\mathrm{k}*\alpha_i\right)\right)
\end{aligned} \tag{9}$$

$$\frac{\partial\mathrm{g}}{\partial\Theta}=0\Rightarrow 2*\mathrm{k}*\mathbf{A}*\sum\mathrm{a_i}*\sin\left(\mathrm{k}*\alpha_i-\mathrm{k}*\Theta\right)=0\Rightarrow. \tag{10}$$

1) $\mathbf{A}=0$

In this case we would have

$$g(\mathbf{B},\mathbf{A},\Theta)=g(\mathbf{B})=\sum e_i^2=\sum(c_i-\mathbf{B})^2$$

It's easy to prove that g() has minimum at

$$\mathbf{B}_0=\mathbf{c}_\mathrm{m}=\left(\sum_{i=0}^{N-1}c_i\right)\Big/N.$$

2) $\mathbf{A}\neq 0\Rightarrow\sum \mathrm{a_i}*\sin(\mathrm{k}*\alpha_i-\mathrm{k}*\Theta)=0$

$$\Rightarrow\cos(\mathrm{k}*\Theta)*\sum \mathrm{a_i}*\sin(\mathrm{k}*\alpha_i)-\sin(\mathrm{k}*\Theta)*\sum \mathrm{a_i}*\cos(\mathrm{k}*\alpha_i)=0$$

$$\begin{aligned}\frac{\partial \mathrm{g}}{\partial\mathbf{A}}&=\mathrm{N}*\mathbf{A}+2*\sum a_i*\cos(\mathrm{k}*\alpha_\mathrm{i}-\mathrm{k}*\Theta)\\&=\mathrm{N}*\mathbf{A}+2*\left(\cos(\mathrm{k}*\Theta)*\sum \mathrm{a_i}*\cos(\mathrm{k}*\alpha_i)+\sin(\mathrm{k}*\Theta)*\sum \mathrm{a_i}*\sin(\mathrm{k}*\alpha_i)\right)\end{aligned}\tag{11}$$

$$\begin{aligned}\frac{\partial \mathrm{g}}{\partial\mathbf{A}}=0\Rightarrow\mathbf{A}&=-\frac{2*\sum a_i*\cos(\mathrm{k}*(\alpha_\mathrm{i}-\Theta))}{\mathrm{N}}\\&=-\frac{2*\left(\cos(\mathrm{k}*\Theta)*\sum a_i*\cos(\mathrm{k}*\alpha_\mathrm{i})+\sin(\mathrm{k}*\Theta)*\sum a_i*\sin(\mathrm{k}*\alpha_\mathrm{i})\right)}{\mathrm{N}}\end{aligned}\tag{12}$$

Let us look at the Equations (10) and (12)

For $\mathrm{A}\neq 0$ we will consider three cases:

1) $\sum \mathrm{a_i}*\cos(\mathrm{k}*\alpha_i)=0,\ \sum \mathrm{a_i}*\sin(\mathrm{k}*\alpha_i)=0$

From (12) it follows A = 0. We will reject this posibility because $\mathrm{A}\neq 0$.

2) $\sum \mathrm{a_i}*\cos(\mathrm{k}*\alpha_i)=0,\ \sum \mathrm{a_i}*\sin(\mathrm{k}*\alpha_i)\neq 0$

From (10) it follows $\cos(\mathrm{k}*\Theta_0)=0\Rightarrow\Theta_0=\pm\Pi/(2*\mathrm{k})$.

3) $\sum \mathrm{a_i}*\cos(\mathrm{k}*\alpha_i)\neq 0$

From (10) $\Rightarrow$

$$\mathbf{tg}(\mathrm{k}*\Theta_0)=\frac{\sin(\mathrm{k}*\Theta_0)}{\cos(\mathrm{k}*\Theta_0)}=\frac{\sum \mathrm{a}_i*\sin(k*\alpha_i)}{\sum \mathrm{a}_i*\cos(k*\alpha_i)}$$

From (12) $\Rightarrow$

$$\mathbf{A}_0=-\frac{2*\sum a_i*\cos(\mathrm{k}*(\alpha_\mathrm{i}-\Theta_0))}{\mathrm{N}}\quad(\text{for both cases})$$

Now we have to find the second order partial derivatives of g() with respect to A and Θ.

$$\frac{\partial^2\mathrm{g}}{\partial^2\mathrm{A}}=\mathrm{N}\Rightarrow\frac{\partial^2\mathrm{g}(\mathrm{A}_0,\Theta_0)}{\partial^2\mathrm{A}}=\mathrm{N}>0\tag{13}$$

$$\frac{\partial^2\mathrm{g}}{\partial^2\Theta}=-2*\mathrm{k}^2*\mathrm{A}*\sum \mathrm{a}_i*\cos(k*(\alpha_i-\Theta))\Rightarrow\tag{14}$$

$$\frac{\partial^2\mathrm{g}(\mathrm{A}_0,\Theta_0)}{\partial^2\Theta}=-2*\mathrm{k}^2*\mathrm{A}_0*\sum \mathrm{a}_i*\cos(k*(\alpha_i-\Theta_0))\Rightarrow$$

$$\left|\text{from (12)}\Rightarrow\sum a_i*\cos(\mathrm{k}*(\alpha_\mathrm{i}-\Theta_0))=-\frac{\mathrm{N}*\mathrm{A}_0}{2}\right|\Rightarrow$$

$$\frac{\partial^2\mathrm{g}(\mathrm{A}_0,\Theta_0)}{\partial^2\Theta}=-2*\mathrm{k}^2*\mathrm{A}_0*\sum \mathrm{a}_i*\cos(k*(\alpha_i-\Theta_0))\Rightarrow$$

$$\frac{\partial^2\mathrm{g}(\mathrm{A}_0,\Theta_0)}{\partial^2\Theta}=\mathrm{N}*\mathrm{k}^2*\mathrm{A}_0^2$$

$$\frac{\partial^2 g(A,\Theta)}{\partial A \partial \Theta} = \frac{\partial^2 g(A,\Theta)}{\partial \Theta \partial A} = 2 * k * \sum a_i * \sin\left(k * \left(\alpha_i - \Theta\right)\right) \Rightarrow \tag{15}$$

$$\frac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} = \frac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} = 2 * k * \sum a_i * \sin\left(k * \left(\alpha_i - \Theta_0\right)\right) \Rightarrow \tag{16}$$

$$\left| \text{from (10)} \Rightarrow \sum a_i * \sin\left(k * \left(\alpha_i - \Theta_0\right)\right) = 0 \right| \Rightarrow$$

$$\frac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} = \frac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} = 0$$

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial^2 A} & \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial A \partial \Theta} \\ \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial \Theta \partial A} & \dfrac{\partial^2 g(A_0,\Theta_0)}{\partial^2 \Theta} \end{vmatrix} = \begin{vmatrix} N & 0 \\ 0 & N * k^2 * A_0^2 \end{vmatrix} = \left(N * k * A_0\right)^2 \Rightarrow \tag{17}$$

$$\Delta = \left(N * k * A_0\right)^2 > 0 \Leftrightarrow A_0 \neq 0 \tag{18}$$

Equations given by (13) and (18) are sufficient conditions for minimum.
Q.E.D.

References

[1] Ditchburn, R.W. (1991) Light. Dover Publications Inc., New York.