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On New Solutions of Classical Yang-Mills Equations with Cylindrical Sources

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Abstract

Strong fields generated by big electric currents are examined within the framework of the Yang-Mills nonlinear generalization of the Maxwell electrodynamics proposed in our earlier papers. First we consider the case of stationary currents and find a new exact solution to the Yang-Mills equations. Then we study a Yang-Mills field inside a thin circular cylinder with nonstationary plasma and find expressions for field strengths in it. Obtained results are applied to interpret several puzzling natural phenomena.

Keywords: Yang-Mills Equations, *Su*(2) Symmetry, Source Currents, Field Strengths, Lightning, Exploding Wires

1. Introduction

As is well known, the Yang-Mills field theory proposed in 1954 is one of the greatest achievements of the XX century, which plays a leading role in modern quantum physics [1-3]. At the same time, the whole area of its applications can concern not only quantum physics but also classical physics [4-7]. To explain this point of view, let us examine powerful fields generated by sources with very big electric charges and currents. Then the following question should be raised. Are the classical Maxwell equations always applicable to such fields?

It is beyond doubt that the Maxwell equations adequately describe a great diversity of electromagnetic fields for which photons are their carriers. At the same time, powerful sources with very big charges and currents may generate not only photons but also Z^0 *and* $\mathrm{W}^{\pm}$ bosons. In such cases, the Maxwell equations may be incorrect since they are applicable to fields for which only photons are the carriers. On the other hand, there are the well-known Yang-Mills equations with *SU*(2) symmetry which are a nonlinear generalization of the linear Maxwell equations playing a leading role in various models of electroweak interactions caused by photons and Z^0 and $W^{\pm}$ bosons. For this reason, in [4-7] the classical Yang-Mills equations with *SU*(2) symmetry are applied in the case of powerful field sources with very big electric charges and currents when Z^0 *and* $\mathrm{W}^{\pm}$ bosons may be generated, along with photons. These equations can be represented in the form [1-3]

$$D_{\mu}F^{k,\mu\nu} \equiv \partial_{\mu}F^{k,\mu\nu} + g\varepsilon_{klm}F^{l,\mu\nu}A^{m}_{\mu} = (4\pi/c)\,j^{k,\nu}, \qquad (1)$$

$$F^{k,\mu\nu} = \partial^{\mu}A^{k,\nu} - \partial^{\nu}A^{k,\mu} - g\varepsilon_{klm}A^{l,\mu}A^{m,\nu}, \qquad (2)$$

where $\mu,\nu = 0, 1, 2, 3;$ $k,l,m = 1, 2, 3,$ $A^{k,\nu}$ and $F^{k,\mu\nu}$ are potentials and strengths of a Yang-Mills field, respectively, $j^{k,\nu}$ are three 4-vectors of source current densities, ε_{klm} is the antisymmetric tensor, $\varepsilon_{123} = 1$, D_{μ} is the Yang-Mills covariant derivative, g is the constant of electroweak interaction, and $\partial_{\mu} \equiv \partial/\partial x^{\mu}$, where x^{μ} are orthogonal space-time coordinates of the Minkowski geometry.

It is worth noting that Equations (1),(2) have the following well-known consequences [1-3]:

$$D_{\nu}D_{\mu}F^{k,\mu\nu} = 0, \qquad (3)$$

$$D_{\nu}j^{k,\nu} \equiv \partial_{\nu}j^{k,\nu} + g\varepsilon_{klm}j^{l,\nu}A^{m}_{\nu} = 0. \qquad (4)$$

Further we will consider the field sources $j^{k,\nu}$ of the form

$$j^{1,\nu} = j^{\nu}, \quad j^{2,\nu} = j^{3,\nu} = 0, \qquad (5)$$

where j^{ν} is a classical 4-vector of current densities.

Then when the potentials $A^{2,\nu} = A^{3,\nu} = 0$, the Yang-Mills Equations (1),(2) become coinciding with the Maxwell equations for the potentials $\mathrm{A}^{1,\nu}$. Moreover, from (4) with $\mathrm{k} = 1$ and from (5) we obtain the differential charge conservation equation

$$\partial_\nu j^{1,\nu} = 0 . \quad (6)$$

That is why Equations (1),(2) with field sources of Formula (5) can be regarded as a reasonable nonlinear generalization of the classical Maxwell equations. This nonlinear generalization was studied in [4-7], where new classes of spherically symmetric and wave solutions to the considered Yang-Mills equations were obtained. These solutions were applied to interpret puzzling properties of atmospheric electricity, the phenomenon of ball lightning, and some other natural phenomena unexplained within the framework of the linear Maxwell theory [4-7].

It should be noted that from (1)-(6) we come to the identity

$$D_\nu[D_\mu F^{k,\mu\nu} - (4\pi / c)\, j^{k,\nu}] \equiv 0 \quad \text{for } k = 1 \quad (7)$$

This identity shows that there is a differential relation for the Yang-Mills Equation (1) with the classical sources of Formula (5).

Consider now the classical Yang-Mills Equations (1), (2) with cylindrical sources $j^{k,\nu}$ of the following form:

$$\begin{aligned} j^{1,0} &= j^0(\tau,\rho,z), \quad j^{1,3} = j(\tau,\rho,z), \\ j^{1,1} &= j^{1,2} = 0, \quad j^{2,\nu} = j^{3,\nu} = 0, \end{aligned} \quad (8)$$

where

$$\tau = x^0, \quad \rho = \sqrt{x^2 + y^2}, \quad x = x^1, \quad y = x^2, \quad z = x^3 . \quad (9)$$

Then let us seek the potentials $A^{k,\nu}$ in the form

$$\begin{aligned} & A^{k,0} = \lambda^k(\tau,\rho,z), \quad A^{k,1} \\ & = x\gamma^k(\tau,\rho,z), \quad A^{k,2} \\ & = y\gamma^k(\tau,\rho,z), \quad A^{k,3} \\ & = \delta^k(\tau,\rho,z) \end{aligned} , \quad (10)$$

where $\lambda^k, \gamma^k, \delta^k$ are some functions.

Substituting expressions (10) into Formula (2) for the strengths $F^{k,\mu\nu}$ and taking into account the antisymmetry of ε_{klm}, we find

$$\begin{aligned} & F^{k,01} = xu^k, \quad F^{k,02} = yu^k, \quad F^{k,03} = p^k, \\ & F^{k,12} = 0, \quad F^{k,13} = xh^k, \quad F^{k,23} = yh^k, \\ & u^k = u^k(\tau,\rho,z), \quad p^k = p^k(\tau,\rho,z), \quad h^k = h^k(\tau,\rho,z), \end{aligned} \quad (11)$$

where the functions u^k, p^k, and h^k are as follows:

$$\begin{aligned} u^k &= \gamma^k_\tau + \lambda^k_\rho / \rho - g\varepsilon_{klm}\lambda^l\gamma^m, \\ p^k &= \delta^k_\tau + \lambda^k_z - g\varepsilon_{klm}\lambda^l\delta^m, \\ h^k &= \gamma^k_z - \delta^k_\rho / \rho + g\varepsilon_{klm}\delta^l\gamma^m. \end{aligned} \quad (12)$$

Here $\gamma^k_\tau \equiv \partial\gamma^k / \partial\tau, \quad \lambda^k_z \equiv \partial\lambda^k / \partial z, \quad \delta^k_\rho \equiv \partial\delta^k / \partial\rho$.

After substituting expressions (8)-(11) for $j^{k,\nu}$, $A^{k,\nu}$, and $F^{k,\mu\nu}$ into the Yang-Mills Equation (1), we obtain

$$\begin{aligned} & \rho u^k_\rho + 2u^k + p^k_z - g\varepsilon_{klm}(\rho^2 u^l\gamma^m + p^l\delta^m) \\ & = -(4\pi / c)\, s^k j^0 \end{aligned} , \quad (13)$$

$$u^k_\tau - h^k_z + g\varepsilon_{klm}(u^l\lambda^m + h^l\delta^m) = 0 , \quad (14)$$

$$p^k_\tau + \rho h^k_\rho + 2h^k + g\varepsilon_{klm}(p^l\lambda^m - \rho^2 h^l\gamma^m) = (4\pi / c) s^k j , \quad (15)$$

where

$$s^1 = 1, \quad s^2 = s^3 = 0 . \quad (16)$$

In the considered case (8), from the three Equations (4) we obtain

$$j^0_\tau + j_z = 0 , \quad (17)$$

$$j^0\lambda^2 - j\delta^2 = 0, \quad j^0\lambda^3 - j\delta^3 = 0 . \quad (18)$$

From (8) we have $j^{2,\nu} = j^{3,\nu} = 0$. That is why we can choose the following gauge by some rotation about the first axis in the gauge space:

$$j^0\lambda^1 - j\delta^1 = 0 . \quad (19)$$

Then using (18), we obtain

$$j^0\lambda^k = j\delta^k, \quad k = 1,\ 2,\ 3 \quad (20)$$

In the second section we study Yang-Mills fields generated by stationary currents flowing in the direction of the axis z and find a particular exact solution to the Yang-Mills Equations (12)-(16). In the third section we examine these equations in the case of no stationary plasma flowing through a thin circular cylinder and study the Yang-Mills field inside the cylinder. In the fourth section we discuss obtained results and their applications to some puzzling phenomena appearing in lightning and exploding wires.

2. A Particular Exact Solution to the Yang-Mills Equations in the Case of a Cylindrical Source with Stationary Current

Let us turn to the considered Yang-Mills Equations (12)-(16) in the following stationary case:

$$j = j(\rho), \quad j^0 = \chi_0 j(\rho), \quad \chi_0 = \text{const} , \quad (21)$$

where $j(\rho)$ is some function of ρ .

Let us seek the functions λ^k, δ^k, and γ^k in (10) in the form

$$\lambda^k = \delta^k = 0, \quad \gamma^k = (z - \chi_0\tau)\phi^k(\rho) + \psi^k(\rho) , \quad (22)$$

where ϕ^k and ψ^k are some functions of ρ .

Then from Formulas (12) and (22) we find

$$u^k = -\chi_0 \phi^k(\rho), \quad h^k = \phi^k(\rho), \quad p^k = 0\,. \tag{23}$$

Substituting Formulas (21)-(23) into Equations (13-15) and using the antisymmetry of ε_{klm} and hence the identity $\varepsilon_{klm}\phi^l\phi^m \equiv 0$, we come to the following system of equations:

$$\rho(\phi^k)' + 2\phi^k - g\rho^2\varepsilon_{klm}\phi^l\psi^m = (4\pi/c)s^k j, \quad k = 1,\ 2,\ 3\,. \tag{24}$$

Therefore, we have got three equations for the six functions $\phi^k(\rho)$ and $\psi^k(\rho)$.

Taking into account (16), from Equation (24) we find

$$\rho(\phi^1)' + 2\phi^1 + g\rho^2(\phi^3\psi^2 - \phi^2\psi^3) = (4\pi/c)\,j, \tag{25}$$

$$\begin{aligned}\psi^2 &= \frac{\rho(\phi^3)' + 2\phi^3 + g\rho^2\phi^2\psi^1}{g\rho^2\phi^1},\\ \psi^3 &= -\frac{\rho(\phi^2)' + 2\phi^2 - g\rho^2\phi^3\psi^1}{g\rho^2\phi^1}\,.\end{aligned} \tag{26}$$

From (26) we derive

$$\begin{aligned}&g\rho^2(\phi^3\psi^2 - \phi^2\psi^3)\\ &= (\phi^1)^{-1}\{\rho[\phi^2(\phi^2)' + \phi^3(\phi^3)'\,] + 2[(\phi^2)^2 + (\phi^3)^2]\}.\end{aligned} \tag{27}$$

Substituting (27) into Equation (25), we readily obtain

$$\phi(\rho\phi' + 2\phi) = (4\pi/c)\,j\phi^1\,, \tag{28}$$

where

$$(\phi)^2 = (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2\,. \tag{29}$$

Since in the case (8) under consideration the axes with $k = 2,\ 3$ in the gauge space are equivalent, let us choose the relativistic-invariant gauge condition

$$F^{2,\mu\nu}F^2_{\mu\nu} = F^{3,\mu\nu}F^3_{\mu\nu}\,. \tag{30}$$

Then we can take the following form for the components ϕ^k:

$$\phi^1 = \phi\cos\xi, \quad \phi^2 = \phi^3 = 2^{-1/2}\phi\sin\xi\,, \tag{31}$$

which satisfies (29) and (30), where $\phi = \phi(\rho)$ and $\xi = \xi(\rho)$.

From (28) and (31) we find

$$\rho\phi' + 2\phi = (4\pi/c)\,j\cos\xi\,. \tag{32}$$

Equation (32) is the only equation for the two unknown functions $\phi(\rho)$ and $\xi(\rho)$. Therefore, in the case under consideration the Yang-Mills equations cannot allow us to uniquely determine the field strengths $F^{k,\mu\nu}$. To interpret this, let us turn to identity (7). It shows that the considered Yang-Mills Equation (1) with the classical sources of Formula (5) are not independent and there is a differential relation for them.

Therefore, in order to uniquely determine the field strengths $F^{k,\mu\nu}$, we should find an additional equation. For this purpose, let us represent the Yang-Mills Equation (1) in the form

$$\partial_\mu F^{k,\mu\nu} = (4\pi/c)J^{k,\nu}\,, \tag{33}$$

where

$$J^{k,\nu} = j^{k,\nu} - (cg/4\pi)\varepsilon_{klm}F^{l,\mu\nu}A^m_\mu\,. \tag{34}$$

Taking into account (33) and the evident identity $\partial_\nu\partial_\mu F^{k,\mu\nu} \equiv 0$, we find that the components $J^{k,\nu}$ satisfy the three differential equations of charge conservation

$$\partial_\nu J^{k,\nu} = 0\,. \tag{35}$$

In these equations the values $J^{k,\nu}$ can be interpreted as components of full current densities. As is seen from (34), they are the sum of the source components $j^{k,\nu}$ and the second addendum which can correspond to charged field quanta.

Using the components $J^{k,\nu}$ of full current densities and the source current densities $j^{k,\nu}$, the following additional relativistic-invariant equation was proposed in [4-6] to uniquely determine the field strengths $F^{k,\mu\nu}$:

$$\sum_{k=1}^{3} J^{k,\nu}J^k_{\ \nu} = \sum_{k=1}^{3} j^{k,\nu}j^k_{\ \nu}\,. \tag{36}$$

The expressions on the left and right of this equation are proportional to the interaction energy of the full currents and source currents, respectively, in a small part of a field source. That is why Equation (36) implies the conservation of this energy when charged field quanta are created inside the source [5,6].

Using (33), we can represent Equation (36) in the form

$$\sum_{k=1}^{3}\partial_\alpha F^{k,\alpha\nu}\partial_\beta F^{k,\beta}{}_{\nu} = (4\pi/c)^2\sum_{k=1}^{3} j^{k,\nu}j^k_{\ \nu}\,. \tag{37}$$

Substituting expressions (8) and (11) into Equation (37), we find

$$\sum_{k=1}^{3}\begin{aligned}&[(\rho u^k_\rho + 2u^k + p^k_z)^2 - \rho^2(u^k_\tau - h^k_z)^2\\ &-(\rho h^k_\rho + 2h^k + p^k_\tau)^2] = (4\pi/c)^2[(j^0)^2 - (j)^2]\end{aligned}\,. \tag{38}$$

Using (21) and (23), from Equation (38) we obtain

$$\sum_{k=1}^{3}[\rho(\phi^k)' + 2\phi^k]^2 = (4\pi/c)^2 j^2\,. \tag{39}$$

Taking into account Formulas (31), Equation (39) can be represented as

$$(\rho\phi' + 2\phi)^2 + (\rho\phi\xi'\,)^2 = (4\pi/c)^2 j^2\,. \tag{40}$$

From Equations (32) and (40) we obtain

$$\rho\phi\xi' = \pm(4\pi / c)\, j \sin\xi \; . \tag{41}$$

Equations (32) and (41) give

$$\rho\phi' + 2\phi = \pm\rho\phi\xi' \cot\xi \; . \tag{42}$$

Dividing this equation by $\rho\phi$ and then integrating it, we find

$$\int\left(\frac{\phi'}{\phi} + \frac{2}{\rho}\right) d\rho = \pm\int \cot\xi \; d\xi \; . \tag{43}$$

Equation (43) gives

$$\ln\left|\rho^2\phi\right| = \pm\ln\left|\sin\xi\right| + \text{const} \; . \tag{44}$$

In order to have the function $\phi(\rho)$ nonsingular, let us choose the sign '+' in Equation (41) and hence in Equation (44). Then from Equation (44) we find

$$\phi = D_0 \sin\xi / \rho^2, \quad D_0 = \text{const} \; . \tag{45}$$

Since we have chosen the sign '+' in Equation. (41), from it and Formula (45) we obtain

$$\xi' = (4\pi / c)\, j\rho / D_0 \; . \tag{46}$$

From (45) and (46) we have the following nonsingular solution:

$$\phi = D_0 \frac{\sin\xi}{\rho^2}, \quad \xi = \frac{4\pi}{cD_0}\int_0^{\rho} j\rho \; d\rho, \quad j = j(\rho) \; . \tag{47}$$

Formulas (31) and (47) give

$$\begin{aligned} \phi^1 &= \frac{2D}{c}\sin\left(\frac{I}{D}\right)\frac{1}{\rho^2}, \quad D = \frac{cD_0}{4}, \\ \phi^2 &= \phi^3 = \frac{\sqrt{2}D}{c}\left[1 - \cos\left(\frac{I}{D}\right)\right]\frac{1}{\rho^2}, \end{aligned} \tag{48}$$

$$I = 2\pi\int_0^{\rho} j\rho \; d\rho \; , \tag{49}$$

where $I = I(\rho)$ is the source current in the cylindrical region of radius ρ .

From Equations (11), (23), and (48) we find the nonzero strength components

$$F^{k,13}, \; F^{k,23}$$

and

$$F^{k,01} = -\chi_0 F^{k,13}, \quad F^{k,02} = -\chi_0 F^{k,23} \, .$$

For the components $F^{k,13}$ and $F^{k,23}$ we have

$$\begin{aligned} F^{1,13} &= \frac{2I_{\text{eff}}}{c}\frac{x}{\rho^2}, \quad F^{1,23} = \frac{2I_{\text{eff}}}{c}\frac{y}{\rho^2}, \\ I_{\text{eff}} &= D\sin\left(\frac{I}{D}\right), \quad I = I(\rho), \\ F^{2,13} &= F^{3,13} = \frac{\sqrt{2}D}{c}\left[1 - \cos\left(\frac{I}{D}\right)\right]\frac{x}{\rho^2}, \\ F^{2,23} &= F^{3,23} = \frac{\sqrt{2}D}{c}\left[1 - \cos\left(\frac{I}{D}\right)\right]\frac{y}{\rho^2}, \end{aligned} \tag{50}$$

where D is some constant.

Below we use the terms 'actual' and 'effective' for the currents I and $I_{\text{eff}} = D\sin(I / D)$, respectively.

It should be noted that when $|I| << |D|$, the effective current I_{eff} practically coincides with the actual current I and we have the Maxwell field expressions for the strength components $F^{1,13}$ and $F^{1,23}$. The value $|D|$ should be a sufficiently large constant. Then Formula (50) can be regarded as a nonlinear generalization of the corresponding Maxwell field expressions for the strengths components $F^{1,13}$ and $F^{1,23}$ when the actual current I is sufficiently large.

Formula (50) describe a nonlinear effect of field saturation. Namely, let the absolute value of the actual current I be increasing from zero. Then when it reaches the value $\pi|D|$, the strengths components $F^{1,13}$ and $F^{1,23}$ become equal to zero and after that they change theirs signs.

This property could be applied to give a new interpretation for the unusual phenomenon of bipolar lightning that actually changes its polarity (positive becoming negative or vice versa) [8].

It is also interesting to note that puzzling data for lightning were recently obtained by the Fermi Gamma-ray Space Telescope which could be explained by Formula (50). Namely, some of lightning storms had the surprising sign of positrons, and the conclusion was made that the normal orientation for an electromagnetic field associated with a lightning storm somehow reversed [9].

To explain these data, let us note that as follows from (50), the sign of the effective current I_{eff} can differ from the sign of the actual current I when the latter is sufficiently large.

3. Yang-Mills Fields inside Thin Circular Cylinders with Nonstationary Plasma

Consider now a nonstationary thin cylindrical source of Formula (8) and let us assume that the matter inside it is in the plasma state.

Besides, let the functions j^0 and j in (8) have the following form inside the thin source:

$$j^0 = j^0(\tau, z), \quad j = j(\tau, z), \quad 0 \le \rho < \rho_0 \tag{51}$$

where ρ_0 is its radius.

Our objective is to describe the Yang-Mills field inside the source and for this purpose let us apply Equations (12)-(16). In the considered case we seek the functions λ^k, γ^k, and δ^k, describing the field potentials $A^{k,\nu}$, in the following special form:

$$\gamma^k = 0,\ \lambda^k = \lambda^k(\tau, z),\ \delta^k = \delta^k(\tau, z),\ 0 \le \rho < \rho_0 \quad (52)$$

Then substituting (52) into Formula (12) and using (20) and the identity $\varepsilon_{klm}\lambda^l\lambda^m \equiv 0$, since ε_{klm} are anti-symmetric, we find

$$u^k = 0, \quad h^k = 0\,, \quad (53)$$

$$p^k = \delta^k_\tau + \lambda^k_z\,, \quad j^0\lambda^k = j\delta^k \quad (54)$$

It should be noted that in the examined case, as follows from (11) and (53), the values $F^{k,03} = -F^{k,30}$ are the only nonzero field strengths. That is why the chosen Formula (52) for the field potentials (10) provides the absence of currents in the directions orthogonal to the axis z of the considered cylindrical source in the plasma state, in accordance with the used Formula (8) for the source.

Substituting now expressions (52) and (53) into Eqs. (13)-(15), we obtain

$$p^k_z - g\varepsilon_{klm}p^l\delta^m = -(4\pi / c)s^k j^0\,, \quad (55)$$

$$p^k_\tau + g\varepsilon_{klm}p^l\lambda^m = (4\pi / c)s^k j \quad (56)$$

where, as indicated in (16), $s^1 = 1, \quad s^2 = s^3 = 0$.

Let us multiply Equations (55) and (56) by j and j^0, respectively, and add the products. Then using (20): $j^0\lambda^m - j\delta^m = 0$, we obtain

$$j^0 p^k_\tau + jp^k_z = 0\,. \quad (57)$$

To find solutions $p^k(\tau, z)$ to Equation (57), let us introduce the function

$$q = \int_0^\tau j(\tau, z)d\tau - \int_0^z j^0(0, z)dz\,. \quad (58)$$

Consider its partial derivatives. Using expression (58) and Equation (17): $j_z = -j^0_\tau$, we obtain

$$q_\tau = j(\tau, z)\,, \quad (59)$$

$$q_z = \int_0^\tau j_z(\tau, z)d\tau - j^0(0, z)\,.$$

$$= -\int_0^\tau j^0_\tau(\tau, z)d\tau - j^0(0, z) = -j^0(\tau, z) \quad (60)$$

From Formulas (59) and (60) we find

$$j^0 q_\tau + jq_z = 0\,. \quad (61)$$

Taking this into account, we come to the following solutions to the partial differential Equation (57) of the first order for $p^k(\tau, z)$:

$$p^k = p^k(q)\,, \quad (62)$$

where $p^k(q)$ are arbitrary differentiable functions.

Indeed, substituting (62) into Equation (57) and taking into account equality (61), we find

$$j^0 p^k_\tau + jp^k_z = (dp^k / dq)(j^0 q_\tau + jq_z) = 0\,, \quad (63)$$

and hence, Equation (57) are satisfied.

Thus, as follows from (11), (62), and the first term in (58), the nonzero field strengths $F^{k,03}$ inside the cylindrical source under consideration depend on all charge passing through unit area of a cross section of the cylindrical source from beginning of the current flow.

Let us turn to Equation (54) and seek the functions δ^k and λ^k, satisfying them, in the form

$$\begin{aligned} \delta^k &= j^0[b^k(q) + \vartheta(\tau, z)p^k(q)], \\ \lambda^k &= j[b^k(q) + \vartheta(\tau, z)p^k(q)], \end{aligned} \quad (64)$$

where $b^k(q)$ are arbitrary differentiable function and $\vartheta(\tau, z)$ is some differentiable function.

Then substituting expressions (64) into Equation (54) and taking into account equality (17): $j^0_\tau + j_z = 0$, we come to the equations

$$\begin{aligned} p^k &= [db^k / dq + \vartheta dp^k / dq](j^0 q_\tau \\ &+ jq_z) + (j^0\vartheta_\tau + j\vartheta_z)p^k \end{aligned}. \quad (65)$$

Using Formula (61), from (65) we find

$$j^0\vartheta_\tau + j\vartheta_z = 1 \quad (66)$$

When $j^0 = 0$, from (17) and (66) we have $j = j(\tau)$ and $\vartheta = z / j(\tau) + \vartheta_0(\tau)$, where $\vartheta_0(\tau)$ is an arbitrary function.

Consider the case $j^0 \ne 0$. Then in order to solve Eq. (66), it is convenient to choose the variable q instead of the variable z and put

$$\vartheta = \varphi(\tau, q), \quad q = q(\tau, z)\,. \quad (67)$$

Indeed, using (67) and Formulas (59) and (60), we find

$$\vartheta_\tau = \varphi_\tau + j\varphi_q, \quad \vartheta_z = -j^0\varphi_q\,, \quad (68)$$

and substituting (68) into Equation (66), we derive

$$j^0\varphi_\tau = 1\,. \quad (69)$$

Therefore, we obtain

$$\varphi(\tau, q) = \int d\tau / j^0(\tau, q)\,, \quad (70)$$

where j^0 is represented as a function of τ and q.

Let us now substitute Formulas (62) and (64) into Equations (55) and (56). Then using Formulas (59) and (60) and the evident identity, we find that Equations (55) and (56) give the same equations of the following form:

$$dp^k/dq + g\varepsilon_{klm}p^l b^m = (4\pi/c)s^k, \quad k=1,\ 2,\ 3,$$
$$p^k = p^k(q), \quad b^k = b^k(q), \quad s^1 = 1, \quad s^2 = s^3 = 0. \tag{71}$$

Multiplying Equation (71) by $2p^k$ and summing the products over, taking into account the antisymmetry of, we obtain

$$\frac{d}{dq}\sum_{k=1}^{3}(p^k)^2 = (8\pi/c)p^1. \tag{72}$$

Besides this equation, from the second and third equations in (71) we also find relations of the functions $b^2(q)$ and $b^3(q)$ to the functions $b^1(q)$ and $p^k(q)$.

Let us put

$$p^1 = (4\pi/c)\beta(q)\cos\zeta(q),$$
$$p^2 = p^3 = (4\pi/c)2^{-1/2}\beta(q)\sin\zeta(q), \tag{73}$$

where condition (30) is taken into account, and $\beta(q)$ and $\zeta(q)$ are some functions.

Then substituting expressions (73) into Equation (72), we find

$$\beta' = \cos\zeta. \tag{74}$$

Let us now turn to Equation (38). From it and (53) we have

$$\sum_{k=1}^{3}[(p_z^k)^2 - (p_\tau^k)^2] = (4\pi/c)^2[(j^0)^2 - (j)^2]. \tag{75}$$

Using Formulas (59), (60), and (62), from Equtaion (75) we find

$$\sum_{k=1}^{3}(dp^k/dq)^2 = (4\pi/c)^2. \tag{76}$$

Substituting Formula (73) into Equation (76), we obtain

$$(\beta')^2 + (\beta\zeta')^2 = 1, \tag{77}$$

where $\beta = \beta(q)$ and $\zeta = \zeta(q)$.

Substituting now expression (74) for β' into Equation (77), we find

$$\beta\zeta' = \pm\sin\zeta. \tag{78}$$

Equations (74) and (78) give

$$\beta'/\beta = \pm\cot\zeta\ \zeta'. \tag{79}$$

Let us integrate Equation (79) and choose the sign '+' in it and hence in (78), in order to have its nonsingular solution. Then we obtain

$$\beta = B_0\sin\zeta, \tag{80}$$

where B_0 is some constant.

Substituting expression (80) into Equation (78) and taking into account that the sign '+' has been chosen in it, we find

$$\zeta' = 1/B_0, \quad \zeta = q/B_0 + \zeta_0, \tag{81}$$

where ζ_0 is some constant.

From Formulas (80) and (81) we obtain

$$\beta = B_0\sin(q/B_0 + \zeta_0), \tag{82}$$

where q is defined by Formula (58).

Using Formulas (11), (58), (73), and (82), we find

$$F^{1,03} = (2\pi/c)B_0\sin(2q/B_0 + \eta_0),$$
$$F^{2,03} = F^{3,03} = (\sqrt{2}\pi/c)B_0[1 - \cos(2q/B_0 + \eta_0)], \tag{83}$$
$$\eta_0 = 2\zeta_0, \quad q = \int_0^\tau j(\tau,z)d\tau - \int_0^z j^0(0,z)dz.$$

As follows from (83), when the value $2q = -B_0(\eta_0 + \pi n)$, where n is an integer, the strength component $F^{1,03}$ is zero and when n is an even integer, $F^{k,03} = 0$, $k = 1,\ 2,\ 3$.

Let us apply obtained results to the puzzling phenomenon of current pause which takes place in exploding wires [10]. The phenomenon proceeds in three stages. At the instant of closure of the circuit, sufficiently large current flows through the wire and causes its explosion. Then in some time the current flow ceases and the period of current pause begins. After a certain period of time the current pause can end and the current flow can continue.

The origin of the current pause is not well understood within the framework of the Maxwell electrodynamics [10,11]. That is why let us apply its nonlinear generalization based on the Yang-Mills equations which we have studied. For this purpose, let us turn to Formula (83) and apply them to an exploding wire. As follows from Formula (83), after some period of time the strength component $F^{1,03}$ becomes zero. At this moment the current in the wire should cease. Therefore, Formula (83) allow one to interpret the origin of current pause in exploding wires. The pause could end and the current flow could continue after some redistribution of charges in exploding wires.

4. Conclusions

We have studied classical Yang-Mills fields with $SU(2)$ symmetry generated by charged circular cylinders with currents. Our objective was to find solutions to the nonlinear Yang-Mills equations that could generalize the corresponding solutions to the linear Maxwell equations for sufficiently powerful sources.

We considered two cases. In the first of them we studied a Yang-Mills field generated by a stationary current flowing through a circular cylinder. In this case we found a particular exact solution to the Yang-Mills equations. In the obtained solution the strength components $F^{1,13}$ and $F^{1,23}$ have the form $F^{1,13} = (2I_{\text{eff}}/c)x/\rho^2$,

$F^{1,23} = (2I_{\text{eff}}/c)y/\rho^2$, $I_{\text{eff}} = D\sin(I/D)$, where $I = I(\rho)$ and $I_{\text{eff}} = I_{\text{eff}}(\rho)$ are the actual and effective currents in the cylindrical region of radius ρ, respectively, and D is a sufficiently large constant. When the actual current I is not large and $|I/D| << 1$, the effective current I_{eff} is very close to the actual current I and the found expressions for $F^{1,13}$ and $F^{1,23}$ are practically coinciding with the corresponding Maxwell field expressions. At the same time, when the actual current I is sufficiently large, the effective current I_{eff} can substantially differ from the actual current I and, moreover, the values I and I_{eff} can have different signs. Using this result, we gave a new interpretation for the phenomenon of bipolar lightning and explained the puzzling inversion of the normal orientation for electromagnetic fields associated with some lightning storms which was recently detected by the Fermi Gamma-ray Space Telescope.

In the second case we considered a Yang-Mills field inside a thin circular cylinder with nonstationary plasma. We sought field potentials in Formula (52) and came to the partial differential Equations (54-56). Solving these equations, we found expressions for the field strengths inside the cylindrical source under consideration. It was shown that the strengths could depend on all charge passing through unit area of a cross section of the cylindrical source from beginning of the current flow. The obtained Formula (83) shows that the field strengths inside the cylindrical source can become zero after some period of time. This property of the found solution was above used to explain the puzzling phenomenon of current pause in exploding wires.

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A Modified Limited SQP Method For Constrained Optimization*

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Abstract

In this paper, a modified variation of the Limited SQP method is presented for constrained optimization. This method possesses not only the information of gradient but also the information of function value. Moreover, the proposed method requires no more function or derivative evaluations and hardly more storage or arithmetic operations. Under suitable conditions, the global convergence is established.

Keywords: Constrained Optimization, Limited Method, SQP Method, Global Convergence

1. Introduction

Consider the constrained optimization problem

$$\begin{aligned} &\min \quad f(x) \\ &s.t. \quad h_i(x)=0, \quad i\in E \\ &\qquad\ \ g_j(x)\le 0, \quad j\in I \end{aligned} \tag{1}$$

where $f, h_i, g_j : R^n \to R$ are twice continuously differentiable, $E=\{1,2,\cdots,m'\}$, $I=\{m'+1,m'+2,\cdots,m'+l\}, l>0$ is an integer. Let the Lagrangian function be defined by

$$L(x,\mu,\lambda)=f(x)+\mu^T g(x)+\lambda^T h(x) \tag{2}$$

where μ and λ are multipliers. Obviously, the Lagrangian function L is a twice continuously differentiable function. Let S be the feasible point set of the problem (1). We define I^* to be the set of all the subscripts of those inequality constraints which are active at x^*, *i.e.*, $I^*=\{i\,|\,i\in I \text{ and } g_i(x)=0\}$.

It is well known that the SQP methods for solving twice continuously differentiable nonlinear programming problems, are essentially Newton-type methods for finding Kuhn-Tucher points of nonlinear programming problems. These years, the SQP methods have been in vogue [1-8]: Powell [5] gave the BFGS-Newton-SQP method for the nonlinearly constrained optimization. He gave some sufficient conditions, under which SQP method would yield 2-step Q-superlinear convergence rate (assuming convergence) but did not show that his modified BFGS method satisfied these conditions. Coleman and Conn [2] gave a new local convergence quasi-Newton-SQP method for the equality constrained nonlinear programming problems. The local 2-step Q-superlinear convergence was established. Sun [6] proposed quasi-Newton-SQP method for general LC^1 constrained problems. He presented the locally convergent sufficient conditions and superlinear convergent sufficient conditions. But he did not prove whether the modified BFGS-quasi-Newton-SQP method satisfies the sufficient conditions or not. We know that, the BFGS update exploits only the gradient information, while the information of function values of the Lagrangian function (2) available is neglected.

If $x\in R^n$ holds, then the problem (1) is called unconstrained optimization problem (UNP). There are many methods [9-13] for the UNP, where the BFGS method is one of the most effective quasi-Newton method. The normal BFGS update exploits only the gradient information, while the information of function values available is neglected for UNP too. These years, lots of modified BFGS methods [14-19] have been proposed for UNP. Especially, many efficient attempts have been made to modify the usual quasi-Newton methods using both the gradient and function values information (e.g. [19,20]). Lately, in order to get a higher order accuracy in approximating the second curvature of the objective function, Wei, Yu, Yuan, and Lian [18] proposed a new BFGS-type method for UNP, and the reported numerical results show that the average performance is better than that of the standard BFGS method. The superlinear convergence of this modified has been established for uniformly convex function. Its global convergence is estab-

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lished by Wei, Li, and Qi [20]. Motivated by their ideas, Yuan and Wei [21] presented a modified BFGS method which can ensure that the update matrix are positive definite for the general convex functions. Moreover, the global convergence is proved for the general convex functions.

The limited memory BFGS (L-BFGS) method [22] is an adaptation of the BFGS method for large- scale problems. The implementation is almost identical to that of the standard BFGS method, the only difference is that the inverse Hessian approximation is not formed explicitly, but defined by a small number of BFGS updates. It is often provided a fast rate of linear convergence, and requires minimal storage.

Inspired by the modified method of [21], we combine this technique and the limited memory technique, and give a limited SQP method for constrained optimization. The global convergence of the proposed method will be established for generally convex function. The major contribution of this paper is an extension of, based on the basic of the method in [21], the method for the UNP to constrained optimization problems. Unlike the standard SQP method, a distinguishing feature of our proposed method is that a triple $\{s_i, y_i, A_i^*\}$ being stored, where $s_i = x_{i+1} - x_i$, $y_i = \nabla_x L(z_{i+1}) - \nabla_x L(z_i) + A_i^* s_i$, $z_{i+1} = (x_{i+1}, \mu_{i+1}, \lambda_{i+1})$, $z_i = (x_i, \mu_i, \lambda_i)$, μ_i and λ_i are the multipliers which are according to the Lagrangian objective function at x_i, while μ_{i+1} and λ_{i+1} are the multipliers which are according to the Lagrangian objective function at x_{i+1}, and A_i^* is a scalar related to Lagrangian function value. Moreover, a limited memory SQP method is proposed. Compared with the standard SQP method, the presented method requires no more function or derivative evaluations, and hardly more storage or arithmetic operations.

This paper is organized as follows. In the next section, we briefly review some modified method and the L-BFGS method for UNP. In Section 3, we describe the modified limited memory SQP algorithm for (2). The global convergence will be established in Section 4. In the last section, we give a conclusion. Throughout this paper, $\|\cdot\|$ denotes the Euclidean norm of vectors or matrix.

2. Modified BFGS Update and the L-BFGS Update for UNP

We will state the modified BFGS update and the L-BFGS update for UNP in the following subsections, respectively.

2.1. Modified BFGS Update

Quasi-Newton methods for solving UNP often need to update the iterate matrix B_k. In tradition, $\{B_k\}$ satisfies the following quasi-Newton equation:

$$B_{k+1} S_k = \delta_k \tag{3}$$

where $S_k = x_{k+1} - x_k$, $\delta_k = \nabla f(x_{k+1}) - \nabla f(x_k)$ .The very famous update B_k is the BFGS formula

$$B_{k+1} = B_k - \frac{B_k S_k S_k^T B_k}{S_k^T B_k S_k} + \frac{\delta_k \delta_k^T}{S_k^T \delta_k} \tag{4}$$

Let H_k be the inverse of B_k, then the inverse update formula of (4) method is represented as

$$\begin{aligned} H_{k+1} &= H_k - \frac{\delta_k^T (S_k - H_k \delta_k) S_k S_k^T}{(\delta_k^T S_k)^2} + \\ &\quad \frac{(S_k - H_k \delta_k) S_k^T + S_k (S_k - H_k \delta_k)^T}{(\delta_k^T S_k)^2} \\ &= (I - \frac{S_k \delta_k^T}{\delta_k^T S_k}) H_k (I - \frac{\delta_k S_k^T}{\delta_k^T S_k}) + \frac{S_k S_k^T}{\delta_k^T S_k}, \end{aligned} \tag{5}$$

which is the dual form of the *DFP* update formula in the sense that $H_k \leftrightarrow B_k$, $H_{k+1} \leftrightarrow B_{k+1}$, and $s_k \leftrightarrow y_k$. It has been shown that the BFGS method is the most effective in quasi-Newton methods from computation point of view. The authors have studied the convergence of f and its characterizations for convex minimization [23-27]. Our pioneers made great efforts in order to find a quasi-Newton method which not only possess global convergence but also is superior than the BFGS method from the computation point of view [15-17,20,28-31]. For general functions, it is now known that the BFGS method may fail for non-convex functions with inexact line search [32], Mascarenhas [33] showed that the nonconvergence of the standard BFGS method even with exact line search. In order to obtain a global convergence of BFGS method without convexity assumption on the objective function, Li and Fukushima [15,16] made a slight modification to the standard BFGS method. Now we state their work [15] simply. Li and Fukushima (see [15]) advised a new quasi-Newton equation with the following form$B_{k+1} S_k = \delta_k^{1*}$, where $\delta_k^{1*} = \delta_k + t_k \|g_k\| S_k$, $t_k > 0$ is determined by $t_k = 1 + \max\{-\frac{S_k^T \delta_k}{\| S_k \|^2}, 0\}$. Under appropriate conditions, these two methods [15,16] are globally and superlinearly convergent for nonconvex minimization problems.

In order to get a better approximation of the objective function Hessian matrix, Wei, Yu, Yuan, and Lian (see [18]) also proposed a new quasi-Newton equation: $B_{k+1}(2) S_k = \delta_k^{2*} = \delta_k + A_k(3) S_k$, where

$$A_k(3) = \frac{2[f(x_k) - f(x_k + \alpha_k d_k)] + [\nabla f(x_k + \alpha_k d_k) + \nabla f(x_k)]^T S_k}{\| S_k \|^2}.$$

Then the new BFGS update formula is

$$B_{k+1}(2) = B_k(2) - \frac{B_k(2)S_k S_k^T B_k(2)}{S_k^T B_k(2) S_k} + \frac{\delta_k^{2*}\delta_k^{2*T}}{S_k^T \delta_k^{2*}}. \quad (6)$$

Note that this quasi-Newton Formula (6) contains both gradient and function value information at the current and the previous step. This modified BFGS update formula differs from the standard BFGS update, and a higher order approximation of $\nabla^2 f(x)$ can be obtained [18, 20].

It is well known that the matrix B_k are very important for convergence if they are positive definite [24,25]. It is not difficult to see that the condition $S_k^T \delta_k^{2*} > 0$ can ensure that the update matrix $B_{k+1}(2)$ from (6) inherits the positive definiteness of $B_k(2)$. However this condition can be obtained only under the objective function is uniformly convex. If f is a general convex function, then $S_k^T \delta_k^{2*}$ and $S_k^T \delta_k$ may equal to 0. In this case, the positive definiteness of the update matrix B_k can not be sure. Then we conclude that, for the general convex functions, the positive definiteness of the update matrix B_k generated by (4) and (6) can not be satisfied.

In order to get the positive definiteness of $B_k(2)$ based on the definition of δ_k^{2*} and δ_k for the general convex functions, Yuan and Wei [21] give a modified BFGS update, *i. e.*, the modified update formula is defined by

$$B_{k+1}(3) = B_k(3) - \frac{B_k(3)S_k S_k^T B_k(3)}{S_k^T B_k(3) S_k} + \frac{\delta_k^{3*}\delta_k^{3*T}}{\delta_k^{3*T} S_k}, \quad (7)$$

where $\delta_k^{3*} = \delta_k + A_k S_k$, $A_k = \max\{A_k(3), 0\}$. Then the corresponding quasi-Newton equation is

$$B_{k+1}(3) S_k = \delta_k^{3*} \quad (8)$$

which can ensure that the condition $S_k^T \delta_k^{3*} > 0$ holds for the general convex function f (see [21] in detail). Therefore, the update matrix B_{k+1} from (8) inherits the positive definiteness of B_k for the general convex function.

2.2. Limited Memory BFGS-Type Method

The limited memory BFGS (L-BFGS) method (see [22]) is an adaptation of the BFGS method for large-scale problems. In the L-BFGS method, matrix H_k is obtained by updating the basic matrix $H_0 (\tilde{m} > 0)$ times using BFGS formula with the previous $\tilde{m}$ iterations. The standard BFGS correction (5) has the following form

$$H_{k+1} = V_k^T H_k V_k + \rho_k S_k S_k^T \quad (9)$$

where $\rho_k = \frac{1}{S_k^T \delta_k}$, $V_k = I - \rho_k \delta_k S_k^T$, I is the unit matrix. Thus, H_{k+1} in the L-BFGS method has the following form:

$$\begin{aligned} H_{k+1} &= V_k^T H_k V_k + \rho_k S_k S_k^T \\ &= V_k^T [V_{k-1}^T H_{k-1} V_{k-1} + \rho_{k-1} S_{k-1} S_{k-1}^T] V_k + \rho_k S_k S_k^T \\ &= \cdots \\ &= [V_k^T \cdots V_{k-\tilde{m}+1}^T] H_{k-\tilde{m}+1} [V_{k-\tilde{m}+1} \cdots V_k] \\ &\quad + \rho_{k-\tilde{m}+1} [V_{k-1}^T \cdots V_{k-\tilde{m}+2}^T] S_{k-\tilde{m}+1} S_{k-\tilde{m}+2}^T [V_{k-\tilde{m}+2} \cdots V_{k-1}] \\ &\quad + \cdots \\ &\quad + \rho_k S_k S_k^T. \end{aligned} \quad (10)$$

3. Modified SQP Method

In this section, we will state the normal SQP method and the modified limited memory SQP method, respectively.

3.1. Normal SQP Method

The first-order Kuhn-Tucker condition of (2) is

$$\begin{cases} \nabla f(x^*) + \mu^T \nabla g(x^*) + \lambda^T \nabla h(x^*) = 0, \\ g(x^*) \le 0,\ \mu_j \ge 0,\ \mu_j g_j(x^*) = 0, \quad for \quad j \in I, \\ h(x^*) = 0. \end{cases} \quad (11)$$

The system (11) can be represented by the following system:

$$H(z) = 0, \quad (12)$$

where $z = (z, \mu, \lambda) \in S$ and $H : R^{n+m'+l'} \to R^{n+m'+l'}$ is defined by

$$H(z) = \begin{pmatrix} \nabla f(x) + \mu^T \nabla g(x) + \lambda^T \nabla h(x) \\ \min\{-g(x), \mu\} \\ h(x) \end{pmatrix}. \quad (13)$$

Since $\nabla f, \nabla g,$ and ∇h are continuously differentiable functions, it is obviously that $H(z)$ is continuously differentiable function. Then, for all $d \in R^{n+m'+l'}$, the directional derivative $H'(z:d)$ of the function $H(z)$ exists. Denote the index sets by

$$\alpha(z) = \{i \mid \mu_i > -g_i(x)\} \quad (14)$$

and $\beta(z) = \{i \mid \mu_i \le -g_i(x)\}$. (15)

Under the complementary condition, it is clearly that $\alpha(z)$ is an index set of strongly active inequality constraints, and $\beta(z)$ is an index set of weakly active and inactive inequality constraints. In terms of these sets, the directional derivative along the direction $d=(d_x, d_\mu, d_\lambda)$ is given as follows

$$H'(z:d)=\begin{pmatrix} Gd \\ (-\nabla g_i^T d_x)_{i\in\alpha(z)} \\ \min\{d_{\mu_i}, (-\nabla g_i^T d_x)\}_{i\in\beta(z)} \\ \nabla h(x)^T d_x \end{pmatrix}, \tag{16}$$

where G is a matrix which elements are the partial derivatives of $\nabla_x L(z)$ to d_x, d_μ, d_λ, respectively. If $\min\{d_{\mu_i}, (-\nabla g_i^T d_x)\}_{i\in\beta(z)}=d_{\mu_i}$ holds, then the set

$$W(z)=\begin{pmatrix} V & \nabla g_\alpha(x) & \nabla g_\beta(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 & 0 \\ 0 & 0 & I_\beta & 0 \\ \nabla h(x)^T & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

By (33) in [6], we know than the system

$$W_k d_k = -H(z_k), \tag{18}$$

where $d_k=(d_{x_k}, d_{\mu_k}, d_{\lambda_k})$ and $W_k=W(z_k)$, define the Kuhn-Tucker condition of problem (2), which also defines the Kuhn-Tucker condition of the following quadratic programming $QP(z_k, V_k)$:

$$\begin{aligned} \min \quad & \nabla f(x_k)^T s + \frac{1}{2}s^T V_k s, \\ s.t. \quad & g_\alpha(x_k)+\nabla g_\alpha(x_k)^T s = 0, \\ & g_\beta(x_k)+\nabla g_\beta(x_k)^T s \le 0, \\ & h(x_k)+\nabla h(x_k)^T s = 0, \end{aligned} \tag{19}$$

where $s=x-x_k, V_k=\nabla_{xx}^2 L(z_k)$.

Generally, suppose that $B_k(1)$ is an estimate of V_k and $B_k(1)$ can be updated by BFGS method of quasi-Newton formula

$$B_{k+1}(1)=B_k(1)-\frac{B_k(1)s_k s_k^T B_k(1)}{s_k^T B_k(1)s_k}+\frac{y_k y_k^T}{y_k^T s_k}, \tag{20}$$

where $s_k=x_{k+1}-x_k$, $y_k=\nabla_x L(z_{k+1})-\nabla_x L(z_k)$, $z_{k+1}=(x_{k+1}, \mu_{k+1}, \lambda_{k+1})$, $z_k=(x_k, \mu_k, \lambda_k)$, μ_k and λ_k are the multipliers which are according to the Lagrangian objective function at x_k, while μ_{k+1} and λ_{k+1} are the multipliers which are according to the Lagrangian objective function at x_{k+1}. Particularly, when we use the update Formula (20) to (19), the above quadratic programming problem can be written as $QP(z_k, B_k)$:

$$\begin{aligned} \min \quad & \nabla f(x_k)^T s + \frac{1}{2}s^T B_k(1)s, \\ s.t. \quad & g_\alpha(x_k)+\nabla g_\alpha(x_k)^T s = 0, \\ & g_\beta(x_k)+\nabla g_\beta(x_k)^T s \le 0, \\ & h(x_k)+\nabla h(x_k)^T s = 0. \end{aligned} \tag{21}$$

Suppose that (s, μ, λ) is a Kuhn-Tucker triple of the sub problem $QP(z_k, B_k)$, therefore, it is obviously that $s=0$ if (x_k, μ_k, λ) is a Kuhn-Tucker triple of (2).

3.2. Modified Limited Memory SQP Method

The normal limited memory BFGS formula of quasi-Newton-SQP method with H_k for constrained optimization (2) is defined by

$$\begin{aligned} H_{k+1} &= V_k^T H_k V_k + \rho_k s_k s_k^T \\ &= V_k^T[V_{k-1}^T H_{k-1} V_{k-1} + \rho_{k-1} s_{k-1} s_{k-1}^T]V_k + \rho_k s_k s_k^T \\ &= \cdots \\ &= [V_k^T \cdots V_{k-\tilde{m}+1}^T] H_{k-\tilde{m}+1} [V_{k-\tilde{m}+1} \cdots V_k] \\ &\quad + \rho_{k-\tilde{m}+1}[V_{k-1}^T \cdots V_{k-\tilde{m}+2}^T] s_{k-\tilde{m}+1} s_{k-\tilde{m}+2}^T [V_{k-\tilde{m}+2} \cdots V_{k-1}] \\ &\quad + \cdots \end{aligned} \tag{22}$$

where $\rho_k=\frac{1}{s_k^T y_k}$, $V_k=I-\rho_k y_k s_k^T$, I is the unit matrix. To maintain the positive definiteness of the limited memory BFGS matrix, some researchers suggested to discard correction $\{s_k, y_k\}$ if $s_k^T y_k>0$ does not hold (e.g. [34]). Another technique was proposed by Powell [35] in which y_k is defined by

$$y=\begin{cases} y_k, & \text{if } s_k^T y_k \ge 0.2 s_k^T B_k s_k, \\ \theta_k y_k + (1-\theta_k)B_k s_k, & \text{otherwise}, \end{cases}$$

where $\theta_k=\frac{0.8 s_k^T B_k s_k}{s_k^T B_k s_k - s_k^T y_k}$, $B_k^{-1}=H_k$ of (22). However, if the Lagrangian objective function $L(x,\mu,\lambda)$ is a general convex function, then $s_k^T y_k$ may equal to 0. In this case, the positive definiteness of the update matrix H_k of (22) can not be sure.

Whether there exists a limited memory SQP method which can ensure that the update matrix are positive definite for general convex Lagrangian objective function $L(x,\mu,\lambda)$. This paper gives a positive answer. Let $\tilde{A}_k=\frac{2[L(z_k)-L(z_{k+1})]+[\nabla_x L(z_{k+1})+\nabla_x L(z_k)]^T s_k}{\|s_k\|^2}$. Considering the discussion of the above section, we discuss $\tilde{A}_k$ for general convex Lagrangian objective function $L(x,\mu,\lambda)$ in the following cases to state our motivation.

Case 1: If $\tilde{A}_k > 0$, we have

$$s_k^T(y_k + \tilde{A}_k s_k) = s_k^T y_k + \tilde{A}_k \| s_k \|^2 > s_k^T y_k \geq 0. \quad (23)$$

Case 2: If $\tilde{A}_k < 0$, we get

$$\begin{aligned} 0 > \tilde{A}_k &= \frac{2[L(z_k) - L(z_{k+1})] + [\nabla_x L(z_{k+1}) + \nabla_x L(z_k)]^T s_k}{\| s_k \|^2} \\ &\geq \frac{-2\nabla_x L(z_{k+1}) s_k + [\nabla_x L(z_{k+1}) + \nabla_x L(z_k)]^T s_k}{\| s_k \|^2} \\ &= \frac{-s_k^T y_k}{\| s_k \|^2}, \end{aligned} \quad (24)$$

which means that $s_k^T y_k > 0$ holds. Then we present our modified limited memory SQP formula

$$\begin{aligned} H_{k+1}^* &= V_k^{*T} H_k^* V_k^* + \rho_k^* s_k s_k^T \\ &= V_k^{*T}[V_{k-1}^{*T} H_{k-1}^* V_{k-1}^* + \rho_{k-1}^* s_{k-1} s_{k-1}^T] V_k^* + \rho_k^* s_k s_k^T \\ &= \cdots \\ &= [V_k^{*T} \cdots V_{k-\tilde{m}+1}^{*T}] H_{k-\tilde{m}+1}^* [V_{k-\tilde{m}+1}^* \cdots V_k^*] \\ &\quad + \rho_{k-\tilde{m}+1}^* [V_{k-1}^{*T} \cdots V_{k-\tilde{m}+2}^{*T}] s_{k-\tilde{m}+1} s_{k-\tilde{m}+1}^T [V_{k-\tilde{m}+2}^* \cdots V_{k-1}^*] \\ &\quad + \cdots \\ &\quad + \rho_k^* s_k s_k^T \end{aligned} \quad (25)$$

where $\rho_k^* = \frac{1}{s_k^T y_k^*}$, $V_k^* = I - \rho_k^* y_k^* s_k^T$, and $y_k^* = y_k + \max\{\tilde{A}_k, 0\} s_k$. It is not difficult to see that the modified limited memory SQP Formula (25) contains both the gradient and function value information of Lagrangian function at the current and the previous step if $\tilde{A}_k > 0$ holds.

Let B_k^* be the inverse of H_k^*. More generally, suppose that B_k^* is an estimate of V_k. Then the above quadratic programming problem (19) can be written as $QP(z_k, B_k^*)$:

$$\begin{aligned} \min \quad & \nabla f(x_k)^T s + \frac{1}{2} s^T B_k^* s, \\ s.t. \quad & g_\alpha(x_k) + \nabla g_\alpha(x_k)^T s = 0, \\ & g_\beta(x_k) + \nabla g_\beta(x_k)^T s \leq 0, \\ & h(x_k) + \nabla h(x_k)^T s = 0. \end{aligned} \quad (26)$$

Suppose that (s, μ, λ) is a Kuhn-Tucker triple of the subproblem $QP(z_k, B_k^*)$, therefore, it is obviously that $s = 0$ if (x_k, μ_k, λ) is a Kuhn-Tucker triple of (2).

Now we state our algorithm as follows.

Modified limited memory SQP algorithm 1 for (2) (M-L-SQP-A1)

Step 0: Star with an initial point $z_0 = (x_0, \mu_0, \lambda_0)$ and an estimate H_0^* of $V_0 = \nabla_{xx}^2 L(z_0)$, H_0^* is a symmetric and positive definite matrix, positive constants $0 < \delta < \sigma < 1, m_0 > 0$ is a positive constant. Set $k = 0$;

Step 1: For given z_k and H_k^*, solve the subproblem

$$\begin{aligned} \min \quad & \nabla f(x_k)^T s + \frac{1}{2} s^T H_k^{*-1} s, \\ s.t. \quad & g_\alpha(x_k) + \nabla g_\alpha(x_k)^T s = 0, \\ & g_\beta(x_k) + \nabla g_\beta(x_k)^T s \leq 0, \\ & h(x_k) + \nabla h(x_k)^T s = 0, \end{aligned} \quad (27)$$

and obtain the unique optimal solution d_k;

Step 2: α_k is chosen by the modified weak Wolfe-Powell (MWWP) step-size rule

$$L(z_k + \alpha_k d_k) \leq L(z_k) + \delta \alpha_k \nabla_x L(z_k)^T d_k, \quad (28)$$

and

$$\nabla_x L(z_k + \alpha_k d_k)^T d_k \geq \sigma \nabla_x L(z_k)^T d_k, \quad (29)$$

then let $x_{k+1} = x_k + \alpha_k d_k$.

Step 3: If z_{k+1} satisfies a prescribed termination criterion (18), stop. Otherwise, go to step 4;

Step 4: Let $\tilde{m} = \min\{k+1, m_0\}$. Update H_0^* for $\tilde{m}$ times to get H_{k+1}^* by Formula (25).

Step 5: Set $k = k+1$ and go to step 1.

Clearly, we note that the above algorithm is as simple as the limited memory SQP method, form storage and cost point of a view at each iteration.

In the following, we assume that the algorithm updates B_k^* -the inverse of H_k^*. The M-L-SQP-A1 with Hessian approximation B_k^* can be stated as follows.

Modified limited memory SQP algorithm 2 for (2) (M-L-SQP-A2)

Step 0: Star with an initial point $z_0 = (x_0, \mu_0, \lambda_0)$ and an estimate B_0^* of $V_0 = \nabla_{xx}^2 L(z_0)$, B_0^* is a symmetric and positive definite matrix, positive constants $0 < \delta < \sigma < 1$, $m_0 > 0$ is a positive constant. Set $k = 0$;

Step 1: For given z_k and B_k^*, solve the subproblem $QP(z_k, B_k^*)$ and obtain the unique optimal solution d_k;

Step 2: Let $\tilde{m} = \min\{k+1, m_0\}$. Update B_k^* with the triples $\{s_i, y_i, A_i^*\}_{i=k-\tilde{m}+1}^k$, *i.e.*, for $l = k - \tilde{m} + 1, \cdots, k$, compute

$$B_k^{l+1^*} = B_k^{l^*} - \frac{B_k^{l^*} s_l s_l^T B_k^{l^*}}{s_l^T B_k^{l^*} s_k} + \frac{y_l^* y_l^{*T}}{y_l^{*T} s_l}, \quad (30)$$

where $s_l = x_{l+1} - x_l$, $y_l^* = y_l + A_l^* s_l$ and $B_k^{k-\tilde{m}+1^*}$ for all k.

Note that M-L-SQP-A1 and M-L-SQP-A2 are mathematically equivalent. In the next section, we will establish the global convergence of M-L-SQP-A2.

4. Convergence analysis of M-L-SQP-A2

Let x^* be a local optimal solution and $z^* = (x^*, \mu^*, \lambda^*)$ be the corresponding Kuhn-Tucker triple of problem (1). In order to get the global convergence of M-L-SQP-A2, the following assumptions are needed.

Assumption A. 1) f, h_i and g_i are twice continuously differentiable functions for all $x \in S$ and S is bounded.

2) $\{\nabla h_i(x^*), i \in E\} \bigcup \{\nabla g_i(x^*), j \in I^*\}$ are positive linear independence.

3) (Strict complementarity) For $j \in I^*, \mu_j^* > 0$.

4) $s^T V s > 0$ for all $s \neq 0$ with $\nabla h_i(x^*)^T s = 0, i \in E$ and $\nabla g_i(x^*)^T s = 0, j \in I^*$, where $V = \nabla_{xx}^2 L(z^*)$.

5) $\{z_k\}$ converges to z^* where $\nabla_x L(z^*) = 0$.

6) The Lagrangian function $L(z)$ is convex for all $z \in S$.

Assumption A(6) implies that there exists a constant $H > 0$ such that

$$\| V \| \leq H, \quad \forall z \in S. \tag{31}$$

Due to the strict complementary Assumption A(3), at a neighborhood of z^*, the method (26) is equivalent to the following equality constrained quadratic programming:

$$\begin{aligned} \min \quad & \nabla f(x_k)^T s + \frac{1}{2} s^T B_k^* s, \\ s.t. \quad & g_\alpha(x_k) + \nabla g_\alpha(x_k)^T s = 0, \\ & h(x_k) + \nabla h(x_k)^T s = 0. \end{aligned} \tag{32}$$

Without loss of generality for the locally convergent analysis, we may discuss that there are only active constraints in (2). Then (18) becomes the following system with B_k^* instead of V_k:

$$\begin{pmatrix} B^* & \nabla g_\alpha(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 \\ \nabla h(x)^T & 0 & 0 \end{pmatrix} \begin{pmatrix} d_{x_k} \\ d_{\mu_k} \\ d_{\lambda_k} \end{pmatrix} = - \begin{pmatrix} \nabla_x L(z_k) \\ g_\alpha(x_k) \\ h(x_k) \end{pmatrix} = -H(z_k) \tag{33}$$

In the case of only considering active constraints, we can suppose that

$$W_k = \begin{pmatrix} V_k & \nabla g_\alpha(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 \\ \nabla h(x)^T & 0 & 0 \end{pmatrix} \tag{34}$$

and

$$D_{H,K} = \begin{pmatrix} B_k^* & \nabla g_\alpha(x) & \nabla h(x) \\ \nabla g_\alpha(x)^T & 0 & 0 \\ \nabla h(x)^T & 0 & 0 \end{pmatrix}, \tag{35}$$

when B_k^* is close to V_k, $D_{H,K}$ is close to W_k.

Lemma 4.1 Let Assumption A hold. Then there exists a positive number M_1 such that

$$\frac{\| y_k^* \|^2}{s_k^T y_k^*} \leq M_1, \quad k = 0, 1, 2, \cdots.$$

Proof. By Assumption A, then there exists a positive number M_0 such that (see [36])

$$\frac{\| y_k \|^2}{s_k^T y_k} \leq M_0, \quad k \geq 0. \tag{36}$$

Since the function $L(x)$ is convex, then we have $L(z_{k+1}) - L(z_k) \geq \nabla_x L(z_k)^T s_k$ and $L(z_k) - L(z_{k+1}) \geq -\nabla_x L(z_{k+1})^T s_k$, the above two inequalities together with the definition of $\tilde{A}_k$ imply that

$$|\tilde{A}_k| \leq \frac{|s_k^T y_k|}{\| s_k \|^2}. \tag{37}$$

Using the definition of y_k^*, we get

$$s_k^T y_k^* = s_k^T y_k + \max\{\tilde{A}_k, 0\} \geq s_k^T y_k \tag{38}$$

and

$$\| y_k^* \| \leq \| y_k \| + \| \max\{\tilde{A}_k, 0\} s_k \| \leq \| y_k \| + \| y_k \| \leq 2 \| y_k \|, \tag{39}$$

where the second inequality of (39) follows (37). Combining (38), (39), and (36), we obtain:

$$\frac{\| y_k^* \|^2}{s_k^T y_k^*} \leq \frac{4 \| y_k \|^2}{s_k^T y_k} \leq 4M_0.$$

Let $M_1 = 4M_0$, we get the conclusion of this lemma. The proof is complete.

Lemma 4.2 Let B_k is generated by (30). Then we have

$$\det(B_{k+1}^*) = \det(B_k^{k-\tilde{m}+1^*}) \prod_{l=k-\tilde{m}+1}^{k} \frac{s_l^T y_l^*}{s_l^T B_l^* s_l}, \tag{40}$$

where $\det(B_k^*)$ denotes the determinant of B_k^*.

Proof. To begin with, we take the determinant in both sides of (20)

$$\begin{aligned} Det(B_{k+1}(1)) &= Det(B_k(1)(I - \frac{s_k s_k^T B_k(1)}{s_k^T B_k(1) s_k} + \frac{B_k(1)^{-1} y_k y_k^T}{s_k^T y_k})) \\ &= Det(B_k(1)) Det(I - \frac{s_k s_k^T B_k(1)}{s_k^T B_k(1) s_k} + \frac{B_k(1)^{-1} y_k y_k^T}{s_k^T y_k}) \\ &= Det(B_k(1))[(1 - s_k^T \frac{B_k(1) s_k}{s_k^T B_k(1) s_k})(1 + (B_k(1)^{-1} y_k)^T \frac{y_k}{y_k^T s_k}) \\ &\quad - (-s_k^T \frac{y_k}{y_k^T s_k})(\frac{(B_k(1) s_k)^T}{s_k^T B_k(1) s_k} B_k(1)^{-1} y_k)] \end{aligned}$$

$$= Det(B_k(1))\frac{y_k^T s_k}{s_k^T B_k(1)s_k},$$

where the third equality follows from the formula (see, e.g., [37] **Lemma 7.6**)

$$\det(I + u_1u_2^T + u_3u_4^T) = (1+u_1^Tu_2)(1+u_3^Tu_4) - (u_1^Tu_4)(u_2^Tu_3).$$

Therefore, there is also a simple expression for the determinant of (30)

$$\det(B_{k+1}^*) = \det(B_k^{k-\tilde{m}+1^*}) \prod_{l=k-\tilde{m}+1}^{k} \frac{s_l^T y_l^*}{s_l^T B_l^* s_l}.$$

Then we complete the proof.

Lemma 4.3 Let Assumption A hold. Then there exists a positive constant δ_1 such that

$$\| s_k \| \ge \delta_1 \eta_k, \quad \text{where} \quad \eta_k = \frac{-\nabla_x L(z_k)^T d_k}{\| d_k \|}.$$

Proof. By Assumption A, we have

$$(\nabla_x L(z_{k+1}) - \nabla_x L(z_k))^T d_k =$$
$$\alpha_k d_k^T \int_0^1 V(z_k + t\alpha_k d_k) d_k dt \le \alpha_k \| d_k \|^2 (H+1).$$

On the other hand, using (29), we get

$$(\nabla_x L(z_{k+1}) - \nabla_x L(z_k))^T d_k \ge -(1-\sigma)\nabla_x L(z_k)^T d_k.$$

Therefore, $\| s_k \| \ge \dfrac{1-\sigma}{H+1}\eta_k$, let $\delta_1 = \dfrac{1-\sigma}{H+1}$. The proof is complete.

Using Assumption A, it is not difficult to get the following lemma.

Lemma 4.4 Let Assumption A hold. Then the sequence $\{L(z_k)\}$ monotonically decreases, and $z_k \in S$ for all $k \ge 0$. Moreover,

$$\sum_{k=0}^{\infty} (-\alpha_k \nabla_x L(z_k)^T d_k) < +\infty.$$

Similar to Lemma 2.6 in [38], it is not difficult to get the following lemma. Here we also give the proof process.

Lemma 4.5 If the sequence of nonnegative numbers m_k $(k = 0,1,\cdots)$ satisfy

$$\prod_{j=0}^{k} m_j \ge c_1^k, c_1 > 0, k = 1,2,\cdots, \tag{41}$$

then $\limsup_k m_k > 0$.

Proof. We will get this result by contradiction. Assume that $\limsup_k m_k = 0$, then, for $0 < \varepsilon_1 < c_1$, there exists $k_1 > 0$, such that $m_k < \varepsilon_1$ for all $k \ge k_1$. Hence, for all $k > k_1$,

$$c_1^k \le \prod_{j=0}^{k_1-1} m_j \prod_{j=k_1}^{k} \varepsilon_1$$

$$+\infty = \limsup_k \left(\frac{c_1}{\varepsilon_1}\right)^k \le \left(\prod_{j=0}^{k_1-1} m_j\right) \varepsilon_1^{1-k_1} < +\infty,$$

which is a contradiction, thus, $\limsup_k m_k > 0$.

Lemma 4.6 Let $\{x_k\}$ be generated by M-L-SQP-A2 and Assumption A hold. If $\liminf_{k\to\infty} \| \nabla_x L(z_k) \| > 0$, then, there exists a constant $\varepsilon_0 > 0$ such that

$$\prod_{j=0}^{k} \eta_j \ge (\varepsilon_0)^{k+1}, \quad \textit{for all } k \ge 0.$$

Proof. Assume that $\liminf_{k\to\infty} \| \nabla_x L(z_k) \| > 0$, *i.e.*, there exists a constant $c_2 > 0$ such that

$$\| \nabla_x L(z_k) \| \ge c_2, \quad k = 0,1,2,\cdots. \tag{42}$$

Now we prove that the update matrix B_{k+1}^* will always be generated by the update Formula (30), *i.e.*, B_{k+1}^* inherits the positive definiteness of B_k^* or $s_k^T y_k^* > 0$ always holds. For $k = 0$, this conclusion holds at hand. For all $k \ge 1$, assume that B_k^* is positive definite. We will deduce that $s_k^T y_k^* > 0$ always holds from the following three cases.

Case 1. If $\tilde{A}_k > 0$. By the definition of y_k^* and Assumption A, we have

$s_k^T y_k^* = s_k^T y_k + \max\{\tilde{A}_k, 0\} > s_k^T y_k \ge 0$.

Case 2. If $\tilde{A}_k < 0$. By the definition of y_k^*, (24), and Assumption A, we get $s_k^T y_k^* = s_k^T y_k > 0$.

Case 3. If $\tilde{A}_k = 0$. By the definition of y_k^*, (29), Assumption A, $d_k = -B_k^{*-1}\nabla_x L(z_k)$, and the positive definiteness of B_k^*, we obtain

$$s_k^T y_k^* = s_k^T y_k \ge -(1-\sigma)\alpha_k d_k^T \nabla_x L(z_k) = (1-\sigma)\alpha_k d_k^T B_k^* d_k > 0,$$

So, we have $s_k^T y_k^* > 0$, and B_{k+1}^* will be generated by the update Formula (30). Thus, the update matrix B_{k+1}^* will always be generated by the update Formula (30).

Taking the trace operation in both sides of (30), we get

$$Tr(B_{k+1}^*) =$$
$$Tr(B_k^{k-\tilde{m}+1^*}) - \sum_{l=k-\tilde{m}+1}^{k} \frac{\| B_l^* s_l \|^2}{s_l^T B_l^* s_l} + \sum_{l=k-\tilde{m}+1}^{k} \frac{\| y_l^* \|^2}{s_l^T y_l^*}, \tag{43}$$

where $Tr(B_k^*)$ denotes the trace of B_k^*. Repeating this trace operation, we have

$$Tr(B_{k+1}^*) = Tr(B_k^{k-\tilde{m}+1^*}) - \sum_{l=k-\tilde{m}+1}^{k} \frac{\| B_l^* s_l \|^2}{s_l^T B_l^* s_l} + \sum_{l=k-\tilde{m}+1}^{k} \frac{\| y_l^* \|^2}{s_l^T y_l^*}$$
$$= \cdots$$
$$= Tr(B_0^*) - \sum_{l=0}^{k} \frac{\| B_l^* s_l \|^2}{s_l^T B_l^* s_l} + \sum_{l=0}^{k} \frac{\| y_l^* \|^2}{s_l^T y_l^*}. \tag{44}$$

Combining (42), (44), $d_k = -B_k^{*-1}\nabla_x L(z_k)$, and Lemma 4.1, we obtain

$$Tr(B_{k+1}^*) \le Tr(B_0^*) - \sum_{l=0}^{k} \frac{c_2^2}{\nabla_x L(z_j)^T H_j^* \nabla_x L(z_j)} + (k+1)M_1. \tag{45}$$

Using B_{k+1}^* is positive definite, we have $Tr(B_{k+1}^*) > 0$. By (45), we obtain

$$\sum_{l=0}^{k} \frac{c_2^2}{\nabla_x L(z_j)^T H_j^* \nabla_x L(z_j)} \le \frac{Tr(B_0^*) + (k+1)M_1}{c_2^2} \tag{46}$$

and

$$Tr(B_{k+1}^*) \le Tr(B_0^*) + (k+1)M_1. \tag{47}$$

By the geometric-arithmetic mean value formula we get

$$\prod_{j=0}^{k} \nabla_x L(z_j)^T H_j^* \nabla_x L(z_j) \ge \left[\frac{(k+1)c_2^2}{Tr(B_0^*) + (k+1)M_1} \right]^{k+1}. \tag{48}$$

Using **Lemma 4.2**, (30), and (38), we have

$$\begin{aligned} \det(B_{k+1}^*) &= \det(B_k^{k-\tilde{m}+1^*}) \prod_{l=k-\tilde{m}+1}^{k} \frac{s_l^T y_l^*}{s_l^T B_l^* s_l} \\ &\ge \det(B_k^{k-\tilde{m}+1^*}) \prod_{l=k-\tilde{m}+1}^{k} \frac{s_l^T y_l}{s_l^T B_l^* s_l} \\ &\ge \det(B_k^{k-\tilde{m}+1^*}) \prod_{l=k-\tilde{m}+1}^{k} \frac{1-\sigma}{\alpha_l} \\ &\ge \cdots \\ &\ge \det(B_0^*) \prod_{j=0}^{k} \frac{1-\sigma}{\alpha_j}. \end{aligned}$$

This implies

$$\frac{\det(B_0^*)}{\det(B_{k+1}^*)} \le \prod_{j=0}^{k} \frac{\alpha_j}{1-\sigma}. \tag{49}$$

By using the geometric-arithmetic mean value formula again, we get

$$\det(B_{k+1}^*) \le \left[\frac{Tr(B_{k+1}^*)}{n} \right]^n. \tag{50}$$

Using (47), (49) and (50), we obtain

$$\begin{aligned} \prod_{j=0}^{k} \frac{\alpha_j}{1-\sigma} &\ge \frac{\det(B_0^*) n^n}{[Tr(B_0^*) + (k+1)M_1]^n} \\ &\ge \frac{1}{k+1} \frac{\det(B_0^*) n^n}{[Tr(B_0^*) + M_1]^n} \\ &\ge \left(\frac{1}{\exp(n)} \right)^{k+1} \min\{ \frac{\det(B_0^*) n^n}{[Tr(B_0^*) + M_1]^n}, 1 \} \\ &\ge \min\left\{ \frac{\det(B_0^*) n^n}{[Tr(B_0^*) + M_1]^n}, 1 \right\} \\ &\ge C_3^{k+1} \end{aligned} \tag{51}$$

where $c_3 \ge \left(\frac{1}{\exp(n)} \right) \min\{ \frac{\det(B_0^*) n^n}{[Tr(B_0^*) + M_1]^n}, 1\}$. Let $\cos\theta_j = \frac{-\nabla_x L(z_j)^T d_j}{\| \nabla_x L(z_j) \| \| d_j \|}$.

Multiplying (48) with (51), for all $k \ge 0$, we get

$$\begin{aligned} \prod_{j=0}^{k} \| s_k \| \| \nabla_x L(z_j) \| \cos\theta_j &\ge c_3^{k+1} [\frac{(k+1)c_2^2}{Tr(B_0^*) + (k+1)M_1}]^{k+1} \\ &\ge [\frac{c_3 c_2^2}{Tr(B_0^*) + M_1}]^{k+1}. \end{aligned} \tag{52}$$

According to **Lemma 4.4** and Assumption A we know that there exists a constant $M_2' > 0$ such that

$$\| s_k \| = \| x_{k+1} - x_k \| \le \| x_{k+1} \| + \| x_k \| \le 2M_2'. \tag{53}$$

Combining the definition of η_k and (53), and noting that $\| \nabla_x L(z_j) \| \cos\theta_j = \eta_j$, we get for all $k \ge 0$,

$$\prod_{j=0}^{k} \eta_j \ge [\frac{c_3 c_2^2}{(Tr(B_0^*) + M_1) 2M_2'}]^{k+1} = \varepsilon_0^{k+1}.$$

The proof is complete.

Now we establish the global convergence theorem for M-L-SQP-A2.

Theorem 4.1 Let Assumption (i) hold and let the sequence $\{z_k\}$ be generated by M-L-SQP-A2. Then we have

$$\liminf_{k\to\infty} \| \nabla_x L(z_k) \| = 0. \tag{54}$$

Proof. By **Lemma 4.3** and (28), we get

$$\begin{aligned} L(z_{k+1}) &\le L(z_k) - \delta \| s_k \| \eta_k \\ &\le L(z_k) - \delta\delta_1 \eta_k^2. \end{aligned} \tag{55}$$

By (55), we have $\sum_{k=0}^{\infty} \eta_k^2 < +\infty$, this implies that

$$\lim_{k\to\infty} \eta_k = 0. \tag{56}$$

Therefore, relation (54) can be obtained from (56) and **Lemma 4.6** directly.

5. Conclusions

For further research, we should study the properties of the modified limited memory SQP method under weak conditions. Moreover, numerical experiments for practically constrained problems should be done in the future.

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Fourier-Bessel Expansions with Arbitrary Radial Boundaries

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Abstract

Series expansion of single variable functions is represented in Fourier-Bessel form with unknown coefficients. The proposed series expansions are derived for arbitrary radial boundaries in problems of circular domain. Zeros of the generated transcendental equation and the relationship of orthogonality are employed to find the unknown coefficients. Several numerical and graphical examples are explained and discussed.

Keywords: Fourier-Bessel Analysis, Boundary Value Problems, Orthogonality of Bessel Functions

1. Introduction

Several boundary value problems in the applied sciences are frequently solved by expansions in cylindrical harmonics with infinite terms. Problems of circular domain with rounded surfaces often generate infinite series of Bessel functions of the first and second types with unknown coefficients. In this case, the intention is to find the series coefficients which should satisfy the boundary conditions.

The subject of Fourier-Bessel series expansions was investigated and examined in many texts [1-10]. Nearly all of them has derived cylindrical harmonics expansions in $J_0(r)$ for the interval $[0, a]$ only, where $J_0(r)$ is the Bessel function of the first kind with order zero and argument r [8]. The existence of the origin point excludes $Y_0(r)$, Bessel function of the second kind with order zero and argument r, because it goes to negative infinity as r approaches zero [9]. Both $J_0(r)$ and $Y_0(r)$ are shown plotted in **Figure 1.**

In many other problems in the applied sciences, the interval of expansion is found to be $[a, b]$ such that $a, b \in \mathbf{R}$. An example of this could be a hollow cylinder in heat conduction problems or a circular band in vibrations analysis solved in the cylindrical coordinate system. In this case, cylindrical harmonics expansions in both $J_0(r)$ and $Y_0(r)$ are necessary.

In this paper, the derivation of cylindrical harmonics expansion of a single variable function in $[a, b]$ in both $J_0(r)$ and $Y_0(r)$ is solved. In accordance with the boundaries at $r = a$ and $r = b$, zeros of the obtained transcendental equation are first calculated. As shown in **Figure 2**, the solution region is for $a \le r \le b$ where the desired series expansions are forced to be zero at $r = a$ and $r = b$ respectively. Unknown coefficients are then found and the complete series expansion can be achieved.

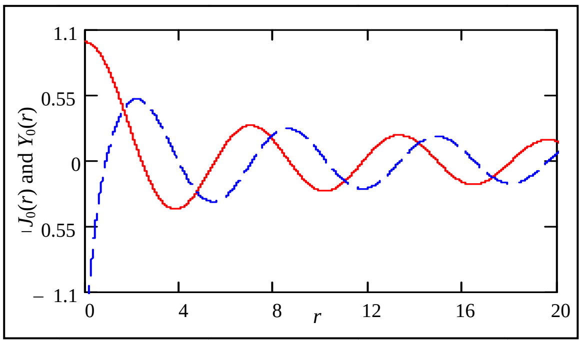

Figure 1. Equation (6), —— $J_0(r)$, – – $Y_0(r)$.

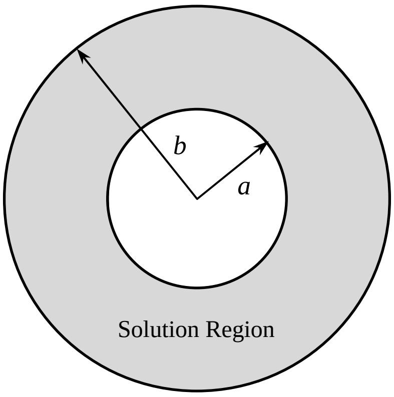

Figure 2. The solution region in radial boundaries.

2. Formulation and Solution

The Bessel differential equation of order zero is well known as [1, 4]:

$$r\frac{d^2}{dr^2}f(r)+\frac{d}{dr}f(r)+\alpha^2 rf(r)=0 \quad (1)$$

$\forall\ \alpha$ and $r \in \mathbf{R}$ and $a \le r \le b$.

The general solution to Equation (1) for real values of α is known to be [2, 3]:

$$f(r)=\sum_{n=0}^{\infty} A_n J_0(\alpha r)+B_n Y_0(\alpha r) \quad (2)$$

As in Equation (1), the assumed boundary conditions at $r = a$ and $r = b$ are of Dirichlet type as $f(a) = 0$ and $f(b) = 0$ respectively. Both A_n and B_n are then related as:

$$A_n=-\frac{Y_0(\alpha a)}{J_0(\alpha a)}B_n \quad (3)$$

$$A_n=-\frac{Y_0(\alpha b)}{J_0(\alpha b)}B_n \quad (4)$$

Going after the elimination method, the transcendental equation can be obtained as:

$$J_0(\alpha a)Y_0(\alpha b)-J_0(\alpha b)Y_0(\alpha a)=0 \quad (5)$$

In order for Equation (5) to be satisfied, there exist many zeros or values of α to be calculated. Thus, in all former and coming equations α can be replaced by α_n which are the zeros obtained from the transcendental equation $\forall\ n \in \mathbf{I}$. That is:

$$J_0(\alpha_n a)Y_0(\alpha_n b)-J_0(\alpha_n b)Y_0(\alpha_n a)=0 \quad (6)$$

The orthogonality feature of Bessel functions can be applied to Equation (2) by multiplying both sides by $r\left[A_m J_0(\alpha_m r)+B_m Y_0(\alpha_m r)\right]$ and integrating it over all possible values of r from a to b as:

$$\sum_{n=0}^{\infty}\int_a^b rC_n(r)C_m(r)dr=\int_a^b rC_m(r)f(r)dr \quad (7)$$

where,

$$C_m(r)=A_m J_0(\alpha_m r)+B_m Y_0(\alpha_m r) \quad (8)$$

$$C_n(r)=A_n J_0(\alpha_n r)+B_n Y_0(\alpha_n r) \quad (9)$$

The terms under the summation in the left side of Equation (7) are zeros for all values of $m \ne n$ [5, 6, 7]. Hence, Equation (7) can be simplified to:

$$\int_a^b rC_n(r)\{C_n(r)-f(r)\}dr=0 \quad (10)$$

Either Equation (3) or (4) can help. Using Equation (3) we can obtain the B_n coefficients as:

$$B_n=\frac{\int_a^b rS_0(\alpha_n r)f(r)dr}{\int_a^b r\left[S_0(\alpha_n r)\right]^2 dr} \quad (11)$$

where, $S_0(\alpha_n r)$ is given by:

$$S_0(\alpha_n r)=Y_0(\alpha_n r)-\frac{Y_0(\alpha_n a)}{J_0(\alpha_n a)}J_0(\alpha_n r) \quad (12)$$

By Equation (3) or (4), the A_n coefficients can also be found. Once the coefficients A_n and B_n are calculated, the function $f(r)$ can be expanded as in Equation (2).

3. Numerical Examples

The transcendental expression in Equation (6) shows a gradual decay as α increases which mean small magnitudes between high zeros. This leads to the convergence of the series in Equation (2) above as n increases. As a consequence, a finite number of terms in Equation (2) can be sufficient for numerical approximations.

The zeros are first evaluated using the transcendental cross product Bessel functions equation for the interval $[a, b]$. A graph of Equation (6) is shown in **Figure 3** for the solution regions [0.65, 2.5] and [0.65, 5]. **Table 1** shows the first 50 zeros of Equation (6) for $a = 0.65$ and $b = 2.5$. Zeros obtained from the transcendental equation changes according to the values of a and b assumed for the solution region. The data presented in **Table 1** indicates that the calculated zeros are not periodic and should be calculated using a proper numerical technique.

Let's assume that the function $f(r)$ to be expanded as in Equation (2) is $\sin(r)$ with a radial solution region in [0.65, 2.5]. The coefficients B_n can be evaluated from Equation (11) and the A_n coefficients are then obtained by Equation (3). Both coefficients are shown in **Tables 2** and **3** respectively for $n = 0$ to 49.

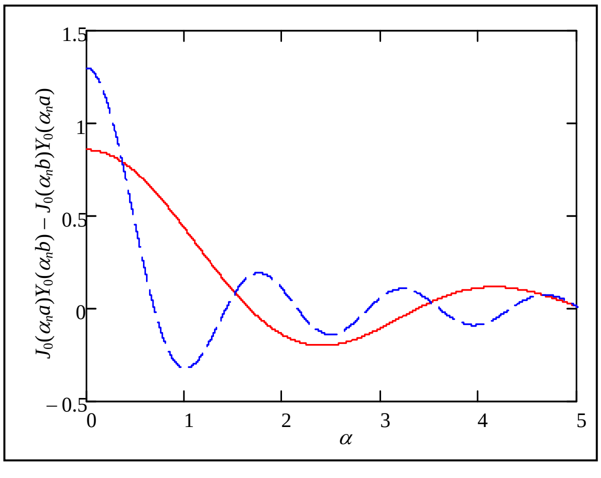

Figure 3. Equation (6), [0.65, 2.5], [0.65, 5].

Many variations can be noticed for the numerical values of A_n and B_n with a general absolute scale of < 1 except for $B_0 = 2.328$. Some coefficients are in the order of $\times 10^{-3}$ meaning that their associated terms are very small such as B_4 and A_{31} in **Tables 2** and **3** respectively.

The function sin(r) and its approximate expansions are plotted in **Figure 4**. Summation over the first 10 terms produced an acceptable estimation in the interval [0.65, 2.5] with some apparent oscillations around the exact function. An improved approximate expansion is also plotted for n = 0 to 49 with less fluctuations in the same radial domain.

In addition, $f(r)$ = cos(r) is expanded as in Equation (2) and the first fifty coefficients are listed in **Tables 4** and **5** for the B_n and A_n respectively. Similar to the sin(r), the cos(r) coefficients go through several variations with a general absolute scale of < 1 except $A_1 = -1.550$. Also, only four coefficients are in the order of $\times 10^{-3}$ implying that their related terms in the series are extremely small such as B_4 and A_{41} in **Tables 4** and **5** respectively.

Table 1. First fifty zeros of Equation (6) in [0.65, 2.5].

n	α_n	n	α_n	n	α_n	n	α_n	n	α_n
0	1.663	10	18.676	20	35.659	30	52.642	40	69.624
1	3.376	11	20.374	21	37.358	31	54.34	41	71.322
2	5.08	12	22.073	22	39.056	32	56.038	42	73.02
3	6.782	13	23.771	23	40.754	33	57.736	43	74.718
4	8.4815	14	25.47	24	42.452	34	59.434	44	76.416
5	10.182	15	27.168	25	44.151	35	61.133	45	78.115
6	11.881	16	28.866	26	45.849	36	62.831	46	79.813
7	13.579	17	30.564	27	47.547	37	64.529	47	81.511
8	15.279	18	32.263	28	49.245	38	66.227	48	83.209
9	16.977	19	33.961	29	50.943	39	67.925	49	84.907

Table 2. First fifty B_n for $f(r)$ = sin(r) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	2.328	10	0.154	20	–0.300	30	–0.114	40	0.206
1	–0.101	11	–0.138	21	7.1E-3	31	0.084	41	1.3E-3
2	–0.703	12	0.228	22	0.267	32	–0.123	42	–0.205
3	0.234	13	0.064	23	–0.082	33	–0.047	43	0.057
4	–4.8E-3	14	–0.385	24	–0.030	34	0.250	44	0.042
5	–0.181	15	0.048	25	0.087	35	–0.025	45	–0.068
6	0.455	16	0.231	26	–0.212	36	–0.173	46	0.148
7	0.030	17	–0.110	27	–0.024	37	0.074	47	0.024
8	–0.478	18	0.082	28	0.272	38	–0.036	48	–0.212
9	0.105	19	0.081	29	–0.054	39	–0.061	49	0.037

Table 3. First fifty A_n for $f(r)$ = sin(r) in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–0.475	10	0.424	20	0.126	30	–0.242	40	–0.109
1	0.462	11	–0.016	21	–0.102	31	2.1E-3	41	0.074
2	–0.547	12	–0.346	22	0.159	32	0.229	42	–0.099
3	–0.114	13	0.111	23	0.052	33	–0.067	43	–0.044
4	0.675	14	0.021	24	–0.297	34	–0.036	44	0.218
5	–0.092	15	–0.110	25	0.034	35	0.075	45	–0.019
6	–0.338	16	0.279	26	0.193	36	–0.174	46	–0.160
7	0.170	17	0.025	27	–0.087	37	–0.024	47	0.064
8	–0.143	18	–0.333	28	0.054	38	0.236	48	–0.023
9	–0.111	19	0.070	29	0.069	39	–0.044	49	–0.057

Table 4. First fifty B_n for $f(r)$ = cos(r) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	–0.129	10	–0.067	20	0.131	30	0.050	40	–0.090
1	0.338	11	0.571	21	–0.030	31	–0.351	41	–5.5E-3
2	0.286	12	–0.099	22	–0.116	32	0.053	42	0.089
3	–0.919	13	–0.264	23	0.342	33	0.195	43	–0.237
4	2.0E-3	14	0.167	24	0.013	34	–0.109	44	–0.018
5	0.732	15	–0.199	25	–0.364	35	0.105	45	0.281
6	–0.196	16	–0.100	26	0.092	36	0.075	46	–0.064
7	–0.122	17	0.457	27	0.100	37	–0.306	47	–0.100
8	0.207	18	–0.036	28	–0.118	38	0.016	48	0.092
9	–0.433	19	–0.338	29	0.223	39	0.256	49	–0.153

Table 5. First fifty A_n for $f(r) = \cos(r)$ in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	0.026	10	–0.184	20	–0.055	30	0.105	40	0.047
1	–1.550	11	0.068	21	0.422	31	–8.9E-3	41	–0.306
2	0.222	12	0.150	22	–0.069	32	–0.100	42	0.043
3	0.448	13	–0.461	23	–0.218	33	0.279	43	0.182
4	–0.287	14	–9.1E-3	24	0.129	34	0.016	44	–0.095
5	0.371	15	0.455	25	–0.139	35	–0.314	45	0.080
6	0.145	16	–0.121	26	–0.084	36	0.076	46	0.070
7	–0.696	17	–0.105	27	0.362	37	0.100	47	–0.268
8	0.062	18	0.145	28	–0.023	38	–0.103	48	0.010
9	0.458	19	–0.289	29	–0.286	39	0.182	49	0.235

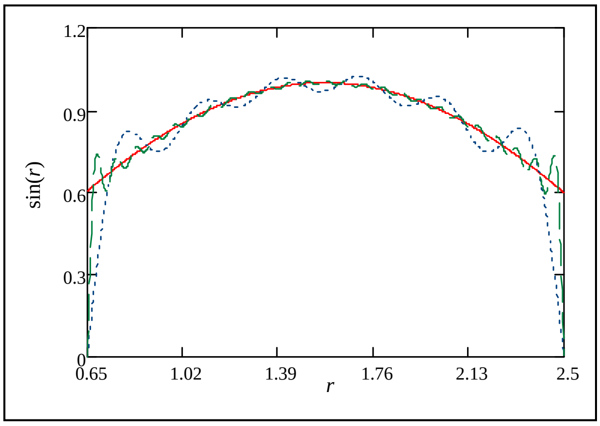

Figure 4. —— sin(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

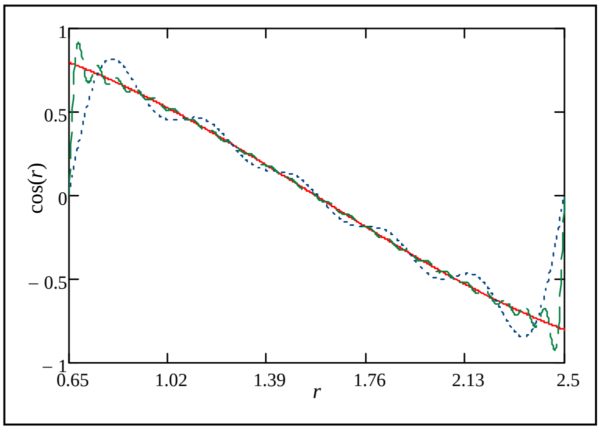

Figure 5. —— cos(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

The function cos(r) and its estimated expansions are shown plotted in **Figure 5**. Finite summation over the first 10 terms generated a satisfactory estimation in the interval [0.65, 2.5] with several obvious oscillations close to the exact function. A better approximate expansion is also plotted for n = 0 to 49 with less fluctuations in the same solution region.

The calculated coefficients for the function e^r are also shown in **Tables 6** and **7** for B_n and A_n respectively. Apparently, the coefficients swing around the exact values with an absolute level of > 1 or < 1.

The greatest values in **Tables 6** and **7** are found as B_0 = 13.852 and A_1 = 11.499. In addition, no coefficients are calculated in the order of $\times 10^{-3}$ implying that all coefficients are to be included in the series expansion.

The function exp(r) and its estimated expansions are shown plotted in **Figure 6** in [0.65, 2.5]. A satisfactory estimation of a finite summation over the first 10 terms are generated with several oscillations close to the exact function. A good approximated expansion is also plotted for n = 0 to 49 with fewer variations in the same solution region.

The last numerical example to be discussed is the square function expressed as:

$$f(r) = \begin{cases} 1 & 1.26 \le r \le 1.88 \\ -1 & otherwise \end{cases} \qquad (13)$$

The calculated B_n and A_n coefficients for this function are shown in **Tables 8** and **9** respectively. Similar to former expansions, both coefficients vary about the exact values of Equation (13). The B_n coefficients have a general absolute level of < 1 except B_2, B_8, B_{14}, B_{20} and B_{26} that have an absolute scale of > 1. Furthermore, the A_n coefficients show an absolute level of < 1 except the absolute values of A_2, A_{32}, A_{38} and A_{44} that are > 1. Some B_n and A_n coefficients are calculated in the order of $\times 10^{-3}$ like A_0 or

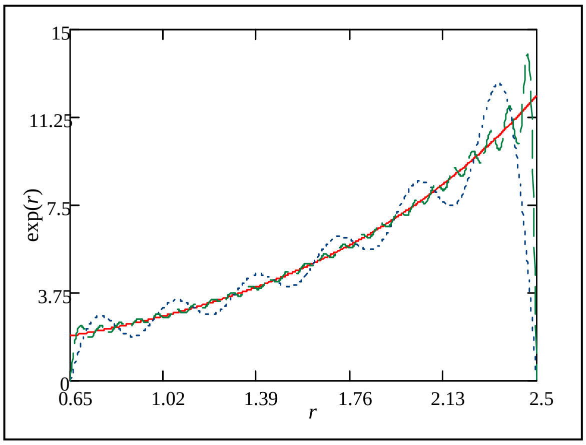

Figure 6. —— exp(r), ••• Equation (2) with n = 0 to 10, — — Equation (2) with n = 0 to 49.

Table 6. First fifty B_n for $f(r)$ = exp(r) in [0.65, 2.5].

n	B_n	N	B_n	n	B_n	n	B_n	n	B_n
0	13.852	10	2.217	20	–4.350	30	–1.660	40	2.987
1	–2.506	11	–5.266	21	0.274	31	3.254	41	0.051
2	–9.361	12	3.298	22	3.873	32	–1.780	42	–2.971
3	8.069	13	2.443	23	–3.169	33	–1.813	43	2.202
4	–0.068	14	–5.566	24	–0.433	34	3.618	44	0.602
5	–6.632	15	1.841	25	3.372	35	–0.971	45	–2.611
6	6.506	16	3.343	26	–3.078	36	–2.505	46	2.140
7	1.113	17	–4.227	27	–0.931	37	2.841	47	0.932
8	–6.867	18	1.182	28	3.937	38	–0.525	48	–3.072
9	3.985	19	3.127	29	–2.069	39	–2.371	49	1.419

Table 7. First fifty A_n for $f(r)$ = exp(r) in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–2.828	10	6.108	20	1.821	30	–3.511	40	–1.575
1	11.499	11	–0.625	21	–3.915	31	0.083	41	2.841
2	–7.286	12	–4.992	22	2.304	32	3.317	42	–1.431
3	–3.934	13	4.263	23	2.019	33	–2.586	43	–1.691
4	9.496	14	0.304	24	–4.301	34	–0.526	44	3.168
5	–3.360	15	–4.213	25	1.292	35	2.912	45	–0.747
6	–4.824	16	4.033	26	2.799	36	–2.523	46	–2.318
7	6.376	17	0.969	27	–3.353	37	–0.925	47	2.490
8	–2.059	18	–4.814	28	0.782	38	3.423	48	–0.336
9	–4.216	19	2.675	29	2.650	39	–1.690	49	–2.185

Table 8. First fifty B_n for Equation (13) in [0.65, 2.5].

n	B_n	n	B_n	n	B_n	n	B_n	n	B_n
0	0.026	10	1.64E-3	20	1.527	30	–0.021	40	–0.051
1	0.1	11	0.081	21	–5E-3	31	–0.404	41	–3E-4
2	3.515	12	0.031	22	–0.025	32	0.614	42	–0.037
3	–0.516	13	–0.205	23	0.042	33	–9E-3	43	–0.329
4	–3E-4	14	1.968	24	–5E-3	34	–0.05	44	–0.201
5	0.105	15	–0.053	25	–0.37	35	8E-3	45	–0.07
6	0.048	16	–9.1E-3	26	1.072	36	–0.033	46	–0.048
7	–0.076	17	0.061	27	6E-3	37	–0.388	47	–3E-3
8	2.447	18	0.013	28	–0.037	38	0.179	48	–0.037
9	–0.17	19	–0.305	29	0.025	39	–0.039	49	–0.228

Table 9. First fifty A_n for Equation (13) in [0.65, 2.5].

n	A_n	n	A_n	n	A_n	n	A_n	n	A_n
0	–5E-3	10	4.5E-3	20	–0.639	30	–0.045	40	0.027
1	–0.457	11	9.6E-3	21	0.069	31	–0.01	41	–0.016
2	2.735	12	–0.047	22	–0.015	32	–1.144	42	–0.018
3	0.252	13	–0.375	23	–0.027	33	–0.013	43	0.253
4	0.039	14	–0.108	24	–0.052	34	7.2E-3	44	–1.056
5	0.053	15	0.121	25	–0.142	35	–0.024	45	–0.02
6	–0.035	16	–0.011	26	–0.975	36	–0.033	46	0.052
7	–0.433	17	–0.014	27	0.02	37	0.126	47	–6.8E-3
8	0.734	18	–0.053	28	–7.3E-3	38	–1.168	48	–4E-3
9	0.18	19	–0.261	29	–0.032	39	–0.027	49	0.352

in the order of $\times 10^{-4}$ such as B_{41} indicating that their associated terms in the series are very small.

The function expressed by Equation (13) and its approximate expansions are plotted in **Figure 7**. Summation over the first 10 terms produced an acceptable estimation in the interval [0.65, 2.5] with some noticeable oscillations around the exact function. A better approximate expansion is also plotted for n = 0 to 49 with less fluctuations in the same radial domain.

In all graphical plots previously shown, the curves return to zero at the assumed boundaries a = 0.65 and b = 2.5. In addition, accuracy of the expanded curves may appear better as n increases due to larger number of terms involved in the series and less fluctuations seen around the exact values.

4. Conclusions

Functions were expanded as a Fourier-Bessel series summation in both $J_0(r)$ and $Y_0(r)$. A finite series expan-

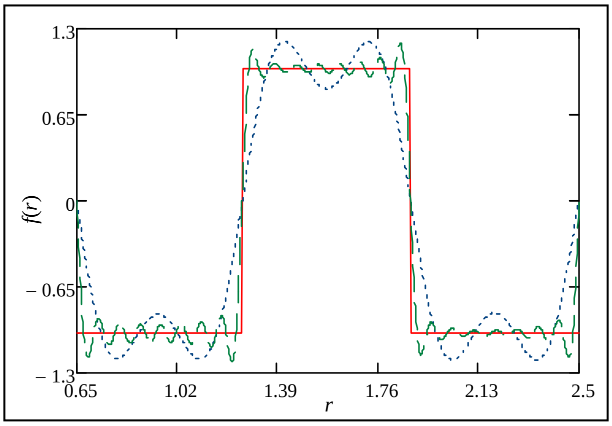

Figure 7. —— Equation (13), ••• Equation (2) with *n* = 0 to 10, — — Equation (2) with *n* = 0 to 49.

sion was obtained for arbitrary radial boundaries in [a, b]. Coefficients were found by calculating the zeros of the transcendental equation and by employing the relationship of orthogonality. A number of examples were numerically and graphically discussed.

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Stationary Distribution of Random Motion with Delay in Reflecting Boundaries*

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Abstract

In this paper we study a continuous time random walk in the line with two boundaries $[a,b]$, $a < b$. The particle can move in any of two directions with different velocities v_1 and v_2. We consider a special type of boundary which can trap the particle for a random time. We found closed-form expressions for the stationary distribution of the position of the particle not only for the alternating Markov process but also for a broad class of semi-Markov processes.

Keywords: Random Motion, Reflecting Boundaries, Semi-Markov, Random Walk

1. Introduction

In this paper we study the stationary distribution of a one-dimensional random motion performed with two velocities, where the random times separating consecutive velocity changes perform an alternating Markov process. The sojourn times of this process are exponentially distributed random variables. There are many papers on random motion devoted to analysis of models in which motions are driven by a homogeneous Poisson process [1-4], however we have not found any paper investigating the stationary distribution of these processes.

We assume that the particle moves on the line $\mathbb{R}$ in the following manner: At each instant it moves according to one of two velocities, namely $v_1 > 0$ or $v_2 < 0$ Starting at the position $x_0 \in \mathbb{R}$ the particle continues its motion with velocity $v_1 > 0$ during random time τ_1, where τ_1 is an exponential random variable with parameter λ_1,then the particle moves with velocity $v_2 < 0$ during random time τ_2, where τ_2 is an exponential distributed random variable with parameter λ_2. Furthermore, the particle moves with velocity $v_1 > 0$ and so on. When the particle reaches boundary a or b it will stay at that boundary a random time given by the time the particle remains in the same direction up to the time such a particle changes direction. Similar partly reflecting (or trapping) boundaries have been considered in [5], and they may be found in optical photon propagation in turbid medium or chemical processes with sticky layers or boundaries.

We also consider a generalization of these results for semi-Markov processes, *i.e.*, when the random variables τ_1 and τ_2 are different from exponential. This paper is divided in two main parts, namely the Markov case and the generalization to the semi-Markov modeling. Our main result, in the first part of this paper, consists on finding the stationary distribution of the well-known telegrapher process on the line with delays in reflecting boundaries. In the second part, we find the stationary distribution of a more general continuous time random walk when the sojourn times are generally distributed.

2. Markov Case

2.1. Mathematical Modeling

Let us set the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. On the phase space $\mathbf{E} = \{1,2\}$ consider an alternating Markov process $\{\xi(t);\ t \geq 0\}$ having the sojourn time τ_i corresponding to the state $i \in \mathbf{E}$, and transition probability matrix of the embedded

Markov chain

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

Denote by $\{x(t);\ t \geq 0\}$ the position of the particle at time t. Consider the function $C(x)$ on the space $\mathbf{E}$ which is defined as

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$$C(x)=\begin{cases} v_1 & \text{if} \quad x=1 \\ v_2 & \text{if} \quad x=2. \end{cases} \tag{2}$$

The position of the particle at any time t can be expressed as

$$x(t)=x_0+\int_0^t C(\xi(s))\,ds, \tag{3}$$

where the starting point $x_0 \in [a,b]$.

Equation (3) determines the random evolution of the particle in the alternating Markov medium $\{\xi(t);\ t \geq 0\}$ [6,7]. So, $x(t)$ is the well-known one-dimensional telegraph process [1,2]. We assume that $a < b$ are two delaying or adhesive boundaries on the line such that if a particle reaches boundary a or b then it is delayed until the instant that the process changes velocity.

Now, consider the two-component stochastic process $\{\zeta(t)=(x(t),\xi(t))\}$ on the phase space $\mathbb{Z}=[a,b]\times\{1,2\}$. The process $\zeta(t)$ is a homogeneous Markov process with the following generating operator [6,7]:

$$A\phi(x,i)=C(x,i)\frac{d}{dx}\phi(x,i)+\lambda_i\left[P\phi(x,i)-\phi(x,i)\right], i=1,2, \tag{4}$$

where $P\phi(x,1)=\phi(x,2)$ and $P\phi(x,2)=\phi(x,1)$.

2.2. Stationary Distribution

Denote by $\pi(\cdot)$ the stationary distribution of $\zeta(t)$. The analysis of the properties of the process $\zeta(t)$ leads up to the conclusion that the stationary distribution π has atoms at points $(a, 2)$ and $(b, 1)$, and we denote them as $\pi[a,2]$ and $\pi[b,1]$ respectively. The continuous part of π is denoted as $\pi(x,i), i\in\mathbf{E}$.

Since π is the stationary distribution of $\zeta(t)$ then for any function $\phi(\cdot)$ from the domain of the operator A we have

$$\int_{\mathbb{Z}} A\phi(z)\,d\pi(z)=0 \tag{5}$$

Now, let A^* be the conjugate or adjoint operator of A. Then by changing the order of integration in (5) (integrating by parts), we can obtain the following expressions for the continuous part of $A^*\pi=0$

$$\begin{cases} v_1\dfrac{d}{dx}\pi(x,1)+\lambda_1\pi(x,1)-\lambda_2\pi(x,2)=0 \\ v_2\dfrac{d}{dx}\pi(x,2)+\lambda_2\pi(x,2)-\lambda_1\pi(x,1)=0. \end{cases} \tag{6}$$

Similarly, from (5) we obtain

$$\begin{cases} \lambda_1\pi(b,1)-v_1\pi(b^-,1)=0 \\ \lambda_1\pi(b,1)+v_2\pi(b^-,2)=0 \\ \lambda_2\pi(a,2)+v_2\pi(a^+,2)=0 \\ \lambda_2\pi(a,2)-v_1\pi(a^+,1)=0. \end{cases} \tag{7}$$

where $\pi(b^-,i):=\lim_{x\uparrow b}\pi(x,i)$ and $\pi(a^+,i):=\lim_{x\downarrow a}\pi(x,i)$ for i=1,2.

It follows from the set (6) that

$$v_1\frac{d}{dx}\pi(x,1)+v_2\frac{d}{dx}\pi(x,2)=0,$$

or equivalently $v_1\pi(x,1)+v_2\pi(x,2)=k=\text{constant}$.

By using (7), we get $v_1\pi(b^-,1)+v_2\pi(b^-,2)=0$, consequently $k=0$ and

$$v_1\pi(x,1)+v_2\pi(x,2)=0 \tag{8}$$

for all $x\in[a,b]$.

By obtaining $\pi(x,2)$ from (8) and substituting such a result into the first equation in the set (6) we have

$$v_1\frac{d}{dx}\pi(x,1)-\lambda_1\pi(x,1)-\lambda_2\frac{v_1}{v_2}\pi(x,1)=0. \tag{9}$$

Solving (9) we obtain for the continuous part of π

$$\pi(x,1)=Ce^{-\mu x} \tag{10}$$

And

$$\pi(x,2)=-C\frac{v_1}{v_2}e^{-\mu x}, \tag{11}$$

where $\mu=\dfrac{\lambda_1}{v_1}+\dfrac{\lambda_2}{v_2}$.

Now, from (7) we obtain for atoms

$$\pi[a,2]=\frac{v_1}{\lambda_2}Ce^{-\mu a} \tag{12}$$

and

$$\pi[b,1]=\frac{v_1}{\lambda_1}Ce^{-\mu b}. \tag{13}$$

The factor C can be calculated from the normalization equation

$$\int_{\mathbb{Z}}\pi(z)\,dz=1, \tag{14}$$

or equivalently

$$\int_a^b\pi(x,1)\,dx+\int_a^b\pi(x,2)\,dx+\pi[a,2]+\pi[b,1]=1. \tag{15}$$

It follows from (15) that

$$C=\left[\left(\frac{v_1}{\lambda_1}-\frac{1}{\mu}\frac{v_2-v_1}{v_2}\right)e^{-\mu b}+\left(\frac{v_1}{\lambda_2}-\frac{1}{\mu}\frac{v_1-v_2}{v_2}\right)e^{-\mu a}\right]^{-1}. \tag{16}$$

We should notice that the stationary distribution $\bar{\pi}(x)$ of the process $x(t)$ over the interval (a,b) is $\bar{\pi}(x) = \pi(x,1)+\pi(x,2)$.

2.3. Balanced and One-Boundary Cases

2.3.1. Balanced Case

Let us call the balanced case when $\mu = \frac{\lambda_1}{v_1}+\frac{\lambda_2}{v_2} = 0$. In this case we can observe that $\pi(x,1)$ and $\pi(x,2)$ do not depend on x. Hence, the continuous part of the stationary distribution of the process $x(t)$ is uniform over the open interval (a,b). Now, the factor C, say C_B, reduces to

$$C_B = \frac{v_2}{(v_2-v_1)(b-a+\delta)}, \tag{17}$$

where $\delta = \frac{v_1}{\lambda_1} = -\frac{v_2}{\lambda_2}$.

Therefore, the stationary distribution can be expressed as

$$\pi(x,1) = C_B \text{and} \pi(x,2) = -\frac{v_1}{v_2}C_B. \tag{18}$$

Thus,

$$\bar{\pi}(x) = \pi(x,1)+\pi(x,2) = \frac{1}{b-a+\delta}, \tag{19}$$

and the atoms are given by

$$\pi[a,2] = \delta\frac{v_1}{v_2}C_B \text{ and } \pi[b,1] = \delta C_B. \tag{20}$$

2.3.2. One-Boundary Case

Now, suppose that there is just the left boundary a, and the starting position of the process $x(t)$ is $x_0 \in [a,+\infty)$. Then for $\mu > 0$ we have the factor C, say C_O, given by

$$C_O = \frac{v_2 e^{\mu a}}{\frac{v_1 v_2}{\lambda_2} - \frac{(v_1-v_2)}{\mu}}. \tag{21}$$

Hence,

$$\pi(x,1) = C_O e^{-\mu x} \text{ and } \pi(x,2) = -\frac{v_1}{v_2}C_O e^{-\mu x}, \tag{22}$$

with the atom

$$\pi[a,2] = \frac{v_1 v_2}{v_1 v_2 - \frac{\lambda_2(v_1-v_2)}{\mu}}. \tag{23}$$

3. Semi-Markov Case

3.1. Mathematical Model

The particle movement is given by the equation

$$x(t) = x_0 + \int_0^t C(\psi(s))ds. \tag{24}$$

where $x_0 \in [a,b]$ is the particle starting point inside the two reflecting boundaries $a < b$, and $\psi(s)$ is an alternating semi-Markov process with phase space $\mathbf{E} = \{1,2\}$ and embedded transition probability matrix P given in (1). The sojourn time at state i is a random variable with a common cumulative distribution function (cdf) $G_i(t), i \in \mathbf{E}$. We assume that $G_1(t)$ and $G_2(t)$ are not degenerated, and that their probability density function (pdf) and first moment, say

$g_i(t) = \frac{dG_i(t)}{dt}$ and $m_i = \int_0^\infty t g_i(t)dt$ respectively, exist.

Now, the hazard rates are given by $r_i(t) = \frac{g_i(t)}{1-G_i(t)}$, and assume $C(1) = v_1 > 0$ and $C(2) = v_2 < 0$.

Define $\tau(t) := t - \sup\{0 \le u \le t : \psi(u) \ne \psi(t)\}$ and consider the three-component process $\chi(t) = (\tau(t), x(t), \psi(t))$ on the phase space $\mathbf{W} = [0,\infty)\times[a,b]\times\{1,2\}$. It is well-known that $\chi(t)$ is a Markov process with the following infinitesimal operator [8,9]

$$\begin{aligned} A\varphi(\tau,x,i) &= \frac{\partial}{\partial\tau}\varphi(\tau,x,i) + \\ & r_i(\tau)\left[P\varphi(0,x,i)-\varphi(\tau,b,i)\right] + C(x,i)\frac{\partial}{\partial x}\varphi(\tau,x,i) \end{aligned} \tag{25}$$

with boundary conditions say $\frac{\partial\varphi(\tau,b,2)}{\partial\tau} = \frac{\partial\varphi(\tau,a,1)}{\partial\tau} = 0$ and $(\tau,x,i) \in \mathbf{W}$. The function $\varphi(\tau,x,i)$ is continuously differentiable on τ and x. We also have that $P\varphi(0,x,1) = \varphi(0,x,2)$ and $P\varphi(0,x,2) = \varphi(0,x,1)$.

3.2. Stationary Distribution

Denote by $\rho(\cdot)$ the stationary distribution of the stochastic process $x(t)$. This stationary distribution has atoms at points $(\tau,a,2)$ and $(\tau,b,1)$, and we denote them as $\rho[\tau,a,2]$ and $\rho[\tau,b,1]$, respectively. The continuous part of ρ is denoted as $\rho(\tau,x,i)$, $i \in \mathbf{E}$.

For any function $\varphi(\cdot)$ belonging to the domain of the operator A we have

$$\int_{\mathrm{W}} A\varphi(z)\rho(dz)=0. \tag{26}$$

By changing the order of integration (integration by parts), we obtain expressions for $A^*\rho$, where A^* is the adjoint operator of A, namely

$$\frac{\partial}{\partial\tau}\rho(\tau,x,i)+r_i\rho(\tau,x,i)+v_i\frac{\partial}{\partial x}\rho(\tau,x,i)=0, i=1,2, \tag{27}$$

and

$$\int_0^\infty r_i(\tau)\rho(\tau,x,i)d\tau=\rho(0,x,j); i\neq j, i,j=1,2, \tag{28}$$

with the limiting behavior $\rho(+\infty,x,i)=0,$ for all $x\in[a,b]$.

For the atoms we have

$$\frac{\partial}{\partial\tau}\rho[\tau,a,2]+r_2(\tau)\rho[\tau,a,2]+v_2\rho(\tau,a^+,2)=0 \tag{29}$$

$$\frac{\partial}{\partial\tau}\rho[\tau,b,1]+r_1(\tau)\rho[\tau,b,1]-v_1\rho(\tau,b^-,1)=0 \tag{30}$$

where

$$\rho(\tau,b^-,i):=\lim_{x\uparrow b}\rho(\tau,x,i) \quad \text{and}$$

$\rho(\tau,a^+,i):=\lim_{x\downarrow a}\rho(\tau,x,i)$, for $i=1,2.$ We also have $\rho[+\infty,a,2]=\rho[0,a,2]=\rho[+\infty,b,1]=\rho[0,b,1]=0$

Now, by taking into account boundary conditions we have

$$\int_0^\infty r_1(\tau)\rho[\tau,b,1]d\tau=-v_2\int_0^\infty \rho(\tau,b^-,2)d\tau, \tag{31}$$

and

$$\int_0^\infty r_2(\tau)\rho[\tau,a,2]d\tau=v_1\int_0^\infty \rho(\tau,a^+,1)d\tau. \tag{32}$$

By solving (27) we obtain

$$\rho(\tau,x,i)= f_i(x-v_i\tau)\exp\left(-\int_0^\tau r_i(t)dt\right), i=1,2, \tag{33}$$

where $f_i\in\mathcal{C}^1$.

By substituting (33) into (28) and by noting that

$$\exp\left(-\int_0^\tau r_i(t)dt\right)=1-G_i(\tau)$$

we obtain

$$\int_0^\infty f_i(x-v_i\tau)g_i(\tau)d\tau= f_j(x), i\neq j, i,j=1,2. \tag{34}$$

It follows from (34) that

$$\int_0^\infty\int_0^\infty f_i(x-v_1\tau-v_2t)g_1(\tau)g_2(t)d\tau dt= f_i(x). \tag{35}$$

From (34) and (35) we can assume that the functions $f_i(x)$ are of the form

$$f_i(x)=c_ie^{\lambda x}, i=1,2. \tag{36}$$

Now, by substituting (36) into (35), we obtain

$$\hat{g}_1(\lambda v_1)\hat{g}_2(\lambda v_2)=1 \tag{37}$$

where $\hat{g}_i(s)=\int_0^\infty g_i(t)e^{-st}dt$ is the Laplace transform of $g_i(t), i=1,2.$ The set of pdf's for which (37) exists is similar to the set of functions which satisfies the Cramér condition.

Lemma 3.2.1 If $v_1m_1+v_2m_2\neq 0,$ where $m_1=\int_0^\infty tg_i(t)dt,$ and there exist $\ell_1<\ell_2, p_1<p_2, \sigma_1>0$ and $\sigma_2>0$ such that $g_1(t)\geq\sigma_1, t\in[\ell_1,\ell_2],$ $g_2(t) \geq\sigma_2,$ $t\in[p_1,p_2]$ and $0<v_1\ell_1+v_2p_2, 0<-(v_1\ell_1+v_2p_1)$.

Then, there exists $\lambda_0\neq 0$ which satisfies (37).

Proof Let us define $p(\lambda)=\hat{g}_1(\lambda v_1)\hat{g}_2(\lambda v_2),$ so $p'(0)=-(v_1m_1+v_2m_2)\neq 0.$

Now, suppose $p'(0)=-(v_1m_1+v_2m_2)<0.$ then

$$p(\lambda)\geq\int_{\ell_1}^{\ell_2}e^{-v_1\lambda t}g_1(t)dt\int_{p_1}^{p_2}e^{-v_2\lambda t}g_2(t)dt,$$

hence

$$p(\lambda)\geq\frac{\sigma_1\sigma_2}{v_1v_2}\left(e^{-v_1\lambda\ell_1}-e^{-v_1\lambda\ell_2}\right)\left(e^{-v_2\lambda p_1}-e^{-v_2\lambda p_2}\right)$$
$$\to+\infty \text{as} \lambda\to+\infty.$$

The case $p'(0)=-(v_1m_1+v_2m_2)>0.$ can be reduced to the previous one by assuming $s=-t$ and using $0<v_1\ell_2+v_2p_2$.

Theorem 3.2.1

A)If $v_1m_1+v_2m_2\neq 0$ and $\lambda_0\neq 0$ is the solution for (37), and $\int_0^\infty e^{-\lambda_0v_1\tau}(1-G_1(\tau))d\tau<+\infty$, then there exists a stationary distribution of $x(t)$ with the following continuous part:

$$\rho(\tau,x,1)=c_1e^{\lambda_0(x-v_1\tau)}(1-G_1(\tau)), \tag{38}$$

$$\rho(\tau,x,2)=c_1\hat{g}_1(\lambda_0v_1)e^{\lambda_0(x-v_2\tau)}(1-G_2(\tau)) \tag{39}$$

and atoms

$$\rho[\tau,b,1]=c_1v_1e^{\lambda_0b}(1-G_1(\tau))\frac{1-e^{-\lambda_0v_1\tau}}{\lambda_0v_1}, \tag{40}$$

$$\rho(\tau,x,2)=c_1v_2\hat{g}_1(\lambda_0v_1)e^{\lambda_0a}(G_2(\tau)-1)\frac{1-e^{-\lambda_0v_2\tau}}{\lambda_0v_2}. \tag{41}$$

The normalization factor c_1 can be calculated from $\int_{\mathrm{W}}\rho(dz)=1.$

B)If $v_1m_1+v_2m_2=0$ and there exists the second moment $\int_0^\infty t^2g_i(t)dt$, $i\in\mathbf{E}$, then the stationary measure of $x(t)$ is as follows

$$\rho(\tau,x,1)=c_2(1-G_1(\tau)),\ \rho(\tau,x,2)=c_2(1-G_2(\tau)) \quad (42)$$

with atoms

$$\rho[\tau,b,1]=c_2v_1\tau(1-G_1(\tau)),\ \rho[\tau,a,2]=c_2v_2\tau(G_2(\tau)-1) \quad (43)$$

where

$$c_2=\left[(m_1+m_2)(b-a)+v_1\frac{m_1^{(2)}}{2}-v_2\frac{m_2^{(2)}}{2}\right]^{-1}.$$

Proof It is easy to see that $f_1(x)=c_1e^{\lambda_0x}$ and $f_2(x)=c_1\hat{g}_1(\lambda_0v_1)e^{\lambda_0x}$ satisfy (34). Substituting these functions f_i into (33) we obtain (38) and (39). Therefore we substitute (38) and (39) into (29) and (30), then by solving these equations we obtain (40) and (41).

It can be easily verified that if $v_1m_1+v_2m_2\neq0$ then the value $\lambda_0\neq0$, such that $\hat{g}_1(\lambda_0v_1)\hat{g}_2(\lambda_0v_2)=1$, also satisfies (31) and (32).

Similarly, for $v_1m_1+v_2m_2=0$ we obtain (42) and (43) in the same manner as for the case $v_1m_1+v_2m_2\neq0$ when it is considered that $\lambda_0=0$.

We should notice that the stationary measure of the particle position $x(t)$ is determined by the following relations

$$\rho(x)=\int_0^\infty(\rho(\tau,x,1)+\rho(\tau,x,2))d\tau,\text{for } x\in(a,b), \quad (44)$$

$$\rho[a,2]=\int_0^\infty\rho[\tau,a,2]d\tau,\rho[b,1]=\int_0^\infty\rho[\tau,b,1]d\tau. \quad (45)$$

Example **Markov Case**

Suppose $g_i(t)=\lambda_ie^{-\lambda_it},\lambda_i>0,i=1,2;t\geq0.$ Then,

$$\hat{g}_1(\lambda_0v_1)\hat{g}_2(\lambda_0v_2)=\left(\frac{\lambda_1}{\lambda_1-\lambda_0v_1}\right)\left(\frac{\lambda_2}{\lambda_2-\lambda_0v_2}\right)=1.$$

Therefore, $\lambda_0=\frac{\lambda_1}{v_1}+\frac{\lambda_2}{v_2}$, and this case is the same as the one in the first part of this paper.

Example **Erlang Case**

Let $g_1(t)=\lambda_1e^{-\lambda_1t},g_2(t)=p^2te^{-pt},\lambda_1>0,p>0,$ and $t\geq0.$ Then,

$$\hat{g}_1(\lambda_0v_1)\hat{g}_2(\lambda_0v_2)=\left(\frac{\lambda_1}{\lambda_1-\lambda_0v_1}\right)\left(\frac{p}{p-\lambda_0v_2}\right)^2=1, \quad (46)$$

where we have the conditions $\lambda_1>\lambda_0v_1$ and $p>\lambda_0v_2$. Now, by solving (46) and taking into account the previous conditions, we obtain a unique solution for (46)

$$\lambda_0=\frac{-(\lambda_1v_2+2pv_1)+\sqrt{-4v_1v_2p\lambda_1+(p\lambda_1)^2}}{-2v_1v_2}$$

Since $r_2(t)=\frac{p^2t}{1+pt}\to p>0, as\ t\to+\infty, and\ r_1(t)=\lambda_1,$ then the Theorem 3.2.1 is applicable.

4. Conclusions

The two-state continuous time random walk has been studied by many researchers for the Markov case and only a few have studied for non-Markovian processes [10]. This basic model has many applications in physics, biology, chemistry, and engineering. Most of the former models were oriented to solve the boundary-free particle motion. Recently this basic model has been extended in several directions, such as two and three dimensions, with reflecting and absorbing boundaries. Only a few of these works consider partly reflecting boundaries [5,10], and references therein. However, in none of these previous works a stationary distribution for the particle position is presented, as we did in this paper. We have included the Markov case since it is illustrative and it motivates our analysis of the semi-Markov process.

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Boundary Eigenvalue Problem for Maxwell Equations in a Nonlinear Dielectric Layer

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Abstract

The propagation of TM-polarized electromagnetic waves in a dielectric layer filled with lossless, nonmagnetic, and isotropic medium is considered. The permittivity in the layer defines by Kerr law. We look for eigenvalues of the problem and reduce the issue to the analysis of the corresponding dispersion equation. The equivalence of the boundary eigenvalue problem and the dispersion equation is proved. We show that the solution of the problem exists and from dispersion equation it can be numerically obtained. Using this solution the components of electromagnetic field in the layer can be numerically obtained as well. Transition to the limit in the case of a linear medium in the layer is proved. Some numerical results are presented also.

Keywords: Maxwell Equations, Nonlinear Boundary Eigenvalue Problem, Dispersion Equation, Nonlinear Slab (Film)

1. Introduction

Problems of electromagnetic waves propagation in a nonlinear media are actually important so as these phenomena are widely used in plasma physics, in modern microelectronics, in optics, in laser technology. Mathematical models for some of these problems and certain results are presented in [1-4]. These models yield boundary eigenvalue problems for systems of differential equations. These boundary eigenvalue problems depend on a spectral parameter nonlinearly. Analysis of these problems is very difficult for the reason that it is not possible to apply well-known methods of investigation of spectral problems.

It is necessary to note that such problems are exactly boundary eigenvalue problems. This is due to the fact that the main interest in the problems is finding that values of spectral parameter (eigenvalues) when the wave is propagating in the waveguide. Thus, in such problems it is necessary to pay attention on finding dispersion equations. When we have eigenvalues, the solutions of differential equations can be numerically obtained. With a mathematical view, a dispersion equation is an equation with respect to a spectral parameter, analyzing the equation we can deduce conclusion about existence of solutions of these boundary eigenvalue problems.

The phenomena of propagation of TE-waves were studied rather completely. Results of propagation TE waves in a nonlinear dielectric layer were presented in articles of H.-W. Shurmann, V. S. Serov and Yu. V. Shestopalov [5,6]. Propagation of TE-waves in a nonlinear dielectric waveguide was considered in [7,8] by Yu. G. Smirnov, H.-W. Shurmann and Yu. V. Shestopalov. In both latter articles the problem is considered exactly like boundary eigenvalue problem (nonlinear, of course). Some numerical results are shown in [7]. The article [8] is devoted to strict mathematical results about the problem (solvability, existence of the solution, etc.).

Results about propagation of TM waves in a nonlinear dielectric semi-infinite layer were published in [2,3]. The first integral of the problem under consideration in Section 2 has been found in [4]. This integral represents the conservation law. However, the complete solution of the problem has not been found. The dispersion equation for the problem has not been obtained.

The equations of the TM waves propagation problem in nonlinear layer with Kerr nonlinearity were published for the first time in 1971-1972 in articles of P. N. Eleonskii and V. P. Silin (see, for example, [1]).

2. Statement of the Problem

Given a Cartesian system $Oxyz$, consider electromagnetic waves propagating through a homogeneous isotropic nonmagnetic dielectric layer with Kerr nonlinearity located between two semi-infinite half-spaces $x<0$ and $x>h$. The half spaces are filled with isotropic nonmagnetic media without sources that have constant permittivities $\varepsilon_1 \geq \varepsilon_0$, $\varepsilon_3 \geq \varepsilon_0$, respectively, where

$\varepsilon_0 > 0$ is the permittivity of vacuum. We suppose that in the whole space $\mu = \mu_0$; where μ_0 is the permeability of vacuum.

The electric and magnetic fields are harmonic functions of time *t*:

$$\hat{\mathbf{E}}(x, y, z, t) = \mathbf{E}_+(x, y, z)\cos\omega t + \mathbf{E}_-(x, y, z)\sin\omega t$$

$$\hat{\mathbf{H}}(x, y, z, t) = \mathbf{H}_+(x, y, z)\cos\omega t + \mathbf{H}_-(x, y, z)\sin\omega t$$

and satisfy Maxwell equations

$$\operatorname{rot}\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \quad \operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \tag{1}$$

where

$$\mathbf{E}(x, y, z) = \mathbf{E}_+(x, y, z) + i\mathbf{E}_-(x, y, z)$$
$$\mathbf{H}(x, y, z) = \mathbf{H}_+(x, y, z) + i\mathbf{H}_-(x, y, z)$$

are complex amplitudes.

The permittivity inside the layer is described by the Kerr law $\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2$, where $\varepsilon_2 > \max(\varepsilon_1, \varepsilon_3)$ and a are positive constants. Here, $\mathbf{E} = (E_x, E_y, E_z)^T$, $|\mathbf{E}|^2 = |E_x|^2 + |E_y|^2 + |E_z|^2$ and $|\cdot|$ is modulus of complex function. A solution to the Maxwell equations is sought in the entire space. Hereafter, the time multiplier is omitted.

The electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell Equation (1), the condition that their tangential components at $x = 0$ and $x = h$ are continuous, and the radiation condition at infinity; *i.e.*, the electromagnetic field decays exponentially as $|x| \to \infty$ in the regions $x < 0$ and $x > h$.

Consider the TM waves $\mathbf{E} = (E_x, 0, E_z)^T$ and $\mathbf{H} = (0, H_y, 0)^T$. As a result, Equation (1) become

$$\begin{cases} \dfrac{\partial E_z}{\partial y} = 0; \quad \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = i\omega\mu H_y; \quad \dfrac{\partial E_x}{\partial y} = 0; \\ \dfrac{\partial H_y}{\partial z} = i\omega\varepsilon E_x; \quad \dfrac{\partial H_y}{\partial x} = -i\omega\varepsilon E_z. \end{cases} \tag{2}$$

It follows from (2) that $E_z = E_z(x, z)$ and $E_x = E_x(x, z)$ are independent of *y*. Since H_y is expressed in terms of E_x and E_z, we conclude that H_y is also independent of *y*.

Let $\dfrac{\partial}{\partial x} \equiv (\ldots)'$. Assuming that the field components depend harmonically on *z*, *i.e.*, $H_y = H_y(x, z) = H_y(x)e^{i\gamma z}$, $E_x = E_x(x, z) = E_x(x)e^{i\gamma z}$, and $E_z = E_z(x, z) = E_z(x)e^{i\gamma z}$, we obtain the system of equations

$$\begin{cases} i\gamma E_x(x) - E_z'(x) = i\omega\mu H_y(x); \\ H_y'(x) = -i\omega\varepsilon E_z(x); \\ i\gamma H_y(x) = i\omega\varepsilon E_x(x), \end{cases} \tag{3}$$

which implies that

$$H_y(x) = \frac{1}{i\omega\mu}\left(i\gamma E_x(x) - E_z'(x)\right). \tag{4}$$

Here γ is unknown spectral parameter, *i.e.*, the propagation constant.

Differentiating (4) and using (3) yields

$$\begin{cases} \gamma\left(iE_x(x)\right)' - E_z''(x) = \omega^2\varepsilon\mu E_z(x); \\ \gamma^2\left(iE_x(x)\right) - \gamma E_z'(x) = \omega^2\varepsilon\mu\left(iE_x(x)\right). \end{cases} \tag{5}$$

Introducing $k^2 = \omega^2\mu\varepsilon_0$ with $\mu = \mu_0$ and normalizing the equations according to the formulas $\tilde{x} = kx$, $\dfrac{d}{dx} = k\dfrac{d}{d\tilde{x}}$, $\tilde{\gamma} = \dfrac{\gamma}{k}$, $\tilde{\varepsilon}_j = \dfrac{\varepsilon_j}{\varepsilon_0}$ (j = 1, 2, 3), $\tilde{a} = \dfrac{a}{\varepsilon_0}$.

We introduce the new notation $E_z \equiv Z(\tilde{x})$ and $iE_x \equiv X(\tilde{x})$.

By omitting the tilde, (5) is written in the normalized form as

$$\begin{cases} -Z'' + \gamma X' = \varepsilon Z; \\ -\gamma Z' + \gamma^2 X = \varepsilon X. \end{cases} \tag{6}$$

The real solution $X(x)$ and $Z(x)$ of (6) are sought assuming that γ is real (so that $|\mathbf{E}|^2$ is independent of *z*), where

$$\varepsilon = \begin{cases} \varepsilon_1, \ x < 0; \\ \varepsilon_2 + a\left(X^2 + Z^2\right), \ 0 < x < h; \\ \varepsilon_3, \ x > h. \end{cases} \tag{7}$$

It is also assumed that $X(x)$ and $Z(x)$ are differentiable in the layer:

$$X(x) \in C(-\infty; 0] \cap C[0; h] \cap [h; +\infty) \cap$$
$$\cap C^1(-\infty; 0) \cap C^1(0; h) \cap C^1(h; +\infty);$$

$$Z(x) \in C(-\infty; +\infty) \cap$$
$$\cap C^1(-\infty; 0] \cap C^1[0; h] \cap C^1[h; +\infty) \cap$$
$$\cap C^2(-\infty; 0) \cap C^2(0; h) \cap C^2(h; +\infty).$$

These smoothness conditions follow from continuous conditions of tangential components at $x = 0$ and $x = h$. We search for γ such that $\max(\varepsilon_1, \varepsilon_3) < \gamma^2 < \varepsilon_2$.

The statement of the problem is shown in the **Figure 1**.

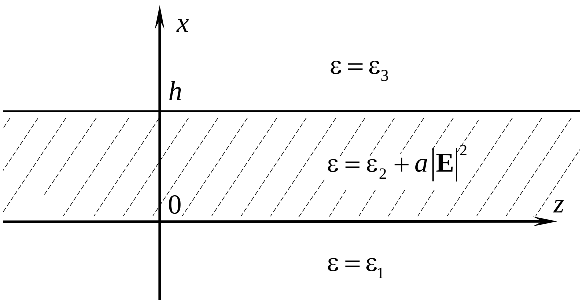

Figure 1. The geometry of the problem.

3. Solution to the System of Differential Equations

For $\varepsilon = \varepsilon_1$ in the half-space $x < 0$, the general solution of (6) is

$$\begin{cases} X(x) = A\exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \\ Z(x) = \dfrac{\sqrt{\gamma^2 - \varepsilon_1}}{\gamma} A\exp\left(x\sqrt{\gamma^2 - \varepsilon_1}\right), \end{cases} \tag{8}$$

where we took into account the condition at infinity.

For $\varepsilon = \varepsilon_3$ in the half-space $x > h$, we have

$$\begin{cases} X(x) = B\exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \\ Z(x) = -\dfrac{\sqrt{\gamma^2 - \varepsilon_3}}{\gamma} B\exp\left(-(x-h)\sqrt{\gamma^2 - \varepsilon_3}\right), \end{cases} \tag{9}$$

according to the condition at infinity. The constants A and B in (8) and (9) are determined by the boundary conditions.

Inside the layer $0 < x < h$, (6) becomes

$$\begin{cases} -\dfrac{d^2Z}{dx^2} + \gamma\dfrac{dX}{dx} = \left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)Z; \\ -\dfrac{dZ}{dx} + \gamma X = \dfrac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)X. \end{cases} \tag{10}$$

We can reduce (10) to the form:

$$\begin{cases} \dfrac{dX}{dx} = \dfrac{2a}{\gamma}\dfrac{\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)}{\varepsilon_2 + 3aX^2 + aZ^2}X^2Z + \\ +\gamma\dfrac{\varepsilon_2 + a\left(X^2 + Z^2\right)}{\varepsilon_2 + 3aX^2 + aZ^2}Z; \\ \dfrac{dZ}{dx} = -\dfrac{1}{\gamma}\left(\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)\right)X. \end{cases} \tag{11}$$

From (11) we have an ordinary differential equation:

$$-\left(\varepsilon_2 + 3aX^2 + aZ^2\right)\frac{dX}{dZ} =$$

$$= 2aXZ + \gamma^2\frac{\varepsilon_2 + a\left(X^2 + Z^2\right)}{\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)}\frac{Z}{X} \tag{12}$$

Multiplying (12) by $\left(\varepsilon_2 - \gamma^2 + a\left(X^2 + Z^2\right)\right)X$ we obtain a total ordinary differential equation. Its solution is easy to write in the form:

$$\frac{\gamma^6C_1 + 3\gamma^2\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)^2 - 2\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)^3}{2\left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)\left(2\gamma^2 - \left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)\right)} =$$

$$= aZ^2 + \varepsilon_2, \tag{13}$$

from item 2 it is known that $\gamma^2 > \varepsilon_0$ and $a > 0$. Equation (13) is true under these conditions.

Define the new variables

$$\tau(x) = \frac{\varepsilon_2 + a\left(X^2(x) + Z^2(x)\right)}{\gamma^2}, \quad \eta(x) = \gamma\frac{X(x)}{Z(x)}\tau(x). \tag{14}$$

Let $\tau_0 = \dfrac{\varepsilon_2}{\gamma^2}$, then $X^2 = \dfrac{\gamma^2}{a}\dfrac{\eta^2(\tau - \tau_0)}{\eta^2 + \gamma^2\tau^2}$, $Z^2 = \dfrac{\gamma^4}{a}\dfrac{\tau^2(\tau - \tau_0)}{\eta^2 + \gamma^2\tau^2}$. In these variables, (11) and (13) become

$$\begin{cases} \dfrac{d\tau}{dx} = 2\gamma^2\dfrac{\tau^2\eta(\tau - \tau_0)(2 - \tau)}{\tau\left(\eta^2 + \gamma^2\tau^2\right) + 2\eta^2(\tau - \tau_0)}, \\ \dfrac{d\eta}{dx} = \dfrac{\gamma^2\tau^2 + \eta^2(\tau - 1)}{\tau}; \end{cases} \tag{15}$$

$$\eta^2 = \frac{\gamma^2\tau^2\left(\tau^2 - C_1\right)}{C_1 + 3\tau^2 - 2\tau^3 - 2\tau(2 - \tau)\tau_0}. \tag{16}$$

Here, (16) is a fourth-degree algebraic equation in τ. Its solution $\tau = \tau(\eta)$ can be explicitly written using the Cardano-Ferrari formulas [9].

4. Boundary Conditions and Dispersion Equation

To derive dispersion equations for the propagation constants, we have to find $\eta(0)$ and $\eta(h)$.

Since the tangential components of **E** and **H** are continuous, we obtain

$$Z(h) = E_z(h+0) = E_z^{(h)}, \quad Z(0) = E_z(0-0) = E_z^{(0)} \tag{17}$$

$$\gamma X(h) - Z'(h) = i\omega\mu H_y(h+0) = H_y^{(h)},$$

$\gamma X(0) - Z'(0) = i\omega\mu H_y(0) = H_y^{(0)}$, where $E_z^{(h)}$ is a known constant. Then

$$H_y^{(h)} = -E_z^{(h)} \frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}}, \quad H_y^{(0)} = E_z^{(0)} \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}}. \quad (18)$$

According to (7) in the layer, we have

$$-Z'(x) + \gamma X(x) = \frac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2(x) + Z^2(x)\right)\right) X(x). \quad (19)$$

Combining (14), (16), (17), and (19) yields

$$\frac{X^2(h)}{\left(E_z^{(h)}\right)^2} = \frac{\tau^2(h) - C_1}{C_1 + 3\tau^2(h) - 2\tau^3(h) - 2\tau(h)(2 - \tau(h))\tau_0}; \quad (20)$$

$$\frac{1}{\gamma}\left(\varepsilon_2 + a\left(X^2(h) + \left(E_z^{(h)}\right)^2\right)\right) X(h) = H_y^{(h)}, \quad (21)$$

where

$$X(h) = \frac{H_y^{(h)}}{\gamma\tau(h)}. \quad (22)$$

Solving (21) for $X(h)$, we obtain

$$X^3(h) + \frac{\varepsilon_2 + a\left(E_z^{(h)}\right)^2}{a} X(h) - \frac{\gamma H_y^{(h)}}{a} = 0. \quad (23)$$

Since the value of $\left(\varepsilon_2 + a\left(E_z^{(h)}\right)^2\right)\Big/a$ is nonnegative, (23) has at least one real root (which we consider).

Thus, $\tau(h) = H_y^{(h)} \big/ \left(\gamma X(h)\right)$. Using (18) and (22), we find

$$X(h) = -\frac{E_z^{(h)}}{\gamma\tau(h)} \frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}}. \quad (24)$$

Combining (20) with (24) gives

$$C_1 = \tau^2(h) - \frac{2\varepsilon_3^2 \tau(h)(2 - \tau(h))(\tau(h) - \tau_0)}{\varepsilon_3^2 + \gamma^2(\gamma^2 - \varepsilon_3)\tau^2(h)}. \quad (25)$$

If $C_1 > 0$, then (16) which is regarded as an equation in $\tau(h)$ has a positive root. It is easy to show that C_1 is strictly positive. Indeed, (25) implies that $C_1 > 0$ for $\tau(h) > 2$, since $\tau(h) \geq \tau_0 > 1$ and $(\gamma^2 - \varepsilon_3) > 0$. Consider the case of $\tau(h) \in [\tau_0, 2)$. Converting the terms in (25) to a common denominator and, if necessary, making the substitution $\tau(h) = \tau_0 + \alpha$ $(0 < \alpha < 1)$, we obtain

$$C_1 = \tau(h) \frac{\gamma^2(\gamma^2 - \varepsilon_3)\tau^3(h) + \varepsilon_3^2\left(2\alpha(\tau(h) - 1) + \tau_0 - \alpha\right)}{\varepsilon_3^2 + \gamma^2(\gamma^2 - \varepsilon_3)\tau^2(h)}$$

with a positive right-hand side.

It is well known that the field components $\varepsilon X(x)$ and $Z(x)$ are continuous at the interface of the media. Then, the function $\gamma \frac{\tau(x) X(x)}{Z(x)}$ is also continuous at the interface of the media at points x such that $Z(x) \neq 0$. Since $\eta(x) = \gamma \frac{\tau(x) X(x)}{Z(x)}$, we use (8) and (9) to obtain

$$\eta(0) = \frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}} > 0; \quad \eta(h) = -\frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}} < 0. \quad (26)$$

Since the right-hand side of the second equation in (15) is positive, $\eta(x)$ is an increasing function on the interval $(0; h)$. Taking into account the signs in (26), we conclude that $\eta(x)$ is not differentiable on the entire interval $(0; h)$ but has a point of discontinuity. Assume that this point is $x^* \in (0; h)$. It follows from (16) that x^* is such that $\tau^* = \tau(x^*)$ is a root of the equation $C_1 + 3(\tau^*)^2 - 2(\tau^*)^3 - 2\tau^*(2 - \tau^*)\tau_0 = 0.$ Moreover, $\eta(x^* - 0) \to +\infty$ and $\eta(x^* + 0) \to -\infty$. Let

$$f \equiv f(\eta) = \frac{\tau}{\gamma^2\tau^2 + \eta^2(\tau - 1)},$$

where $\tau = \tau(\eta)$ is expressed from (16). In the general case, there are several points $x_0, x_1, ..., x_N$ on the interval $[0, h]$ at which $\eta(x)$ becomes infinite, so that

$$\eta(x_0 - 0) = \eta(x_1 - 0) = ... = \eta(x_N - 0) = +\infty,$$
$$\eta(x_0 + 0) = \eta(x_1 + 0) = ... = \eta(x_N + 0) = -\infty. \quad (27)$$

Below, it will be proved that the number of such points is finite for any *h*.

Solutions are sought on each of the intervals $[0, x_0]$, $[x_0, x_1]$, ..., $[x_N, h]$:

$$-\int_{\eta(x)}^{\eta(x_0)} f d\eta = x + c_0, \quad 0 \leq x \leq x_0;$$
$$\int_{\eta(x_i)}^{\eta(x)} f d\eta = x + c_{i+1}, \quad x_i \leq x \leq x_{i+1}, \quad i = \overline{0, N-1};$$
$$\int_{\eta(x_N)}^{\eta(x)} f d\eta = x + c_{N+1}, \quad x_N \leq x \leq h. \quad (28)$$

Taking into account (27) and substituting $x = 0$, $x = x_{i+1}$, $x = x_N$ into the first, second, and third equations in (28), respectively, we find the required constants $c_1, c_2, ..., c_{N+1}$:

$$c_0 = -\int\limits_{\eta(0)}^{+\infty} f d\eta \,;$$

$$c_{i+1} = \int\limits_{-\infty}^{+\infty} f d\eta - x_{i+1}\,, \quad i = \overline{0, N-1};$$

$$c_{N+1} = \int\limits_{-\infty}^{\eta(h)} f d\eta - h\,. \tag{29}$$

In view of (29) Equation (28) become

$$\int\limits_{\eta(x)}^{\eta(x_0)} f d\eta = -x + \int\limits_{\eta(0)}^{+\infty} f d\eta\,, \quad 0 \le x \le x_0\,;$$

$$\int\limits_{\eta(x_i)}^{\eta(x)} f d\eta = x + \int\limits_{-\infty}^{+\infty} f d\eta - x_{i+1}\,, \quad x_i \le x \le x_{i+1}\,, \quad i = \overline{0, N-1};$$

$$\int\limits_{\eta(x_N)}^{\eta(x)} f d\eta = x + \int\limits_{-\infty}^{\eta(h)} f d\eta - h\,, \quad x_N \le x \le h\,. \tag{30}$$

Let $\int\limits_{-\infty}^{+\infty} f d\eta = T$. It follows from (30) that $x_{i+1} - x_i = T > 0$, where $i = \overline{0, N-1}$. Therefore, the number of points at which $\eta(x)$ becomes infinite on the interval $(0; h)$ is finite. Now, setting $x = x_0$, $x = x_i$, $x = x_N$ in the first, second, and third equations in (30), respectively, so that all the integrals on the left-hand sides vanish, we add all the equations in (30) to obtain

$$0 = -x_0 + \int\limits_{\eta(0)}^{+\infty} f d\eta + x_0 + T - x_1 + ... + x_{N-1} +$$

$$+T - x_N + x_N + \int\limits_{-\infty}^{\eta(h)} f d\eta - h \tag{31}$$

Finally, (31) yields

$$-\int\limits_{-\frac{\varepsilon_3}{\sqrt{\gamma^2-\varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2-\varepsilon_1}}} f d\eta + (N+1)T = h\,, \tag{32}$$

where $N \ge 0$ is an integer.

Formula (32) is a dispersion equation that holds for any *h*. It should be noted that, if $N \ne 0$, we have several equations for various values of N. Each of these equations must be solved for γ. All the resulting γ form a set of the propagation constants for which and only for which waves in the layer propagate for given h.

It should be also noted that $\int_{-\infty}^{+\infty} f d\eta$ converges since f can be majorized by the function $\frac{M}{(m-1)\eta^2}$, where $M = \max\limits_{x\in[0,\, h]} \tau(x)$, and $m = \min\limits_{x\in[0,\, h]} \tau(x) > 1$. Since $X(x)$ and $Z(x)$ are bounded, $\tau(x)$ has a finite minimum and maximum.

If we consider first equation of (15) combined with first integral, then the equation can be integrated. The obtained integral is so called hyperelliptic integral (it is one of the simplest type of Abelian integrals). If we extend definitional domain of independent variable x on the whole complex plane, then we can consider the inverse function for these integrals. These functions will be solutions of the system (15). These functions are hyperelliptic functions which belong to set Abelian functions. Abelian functions are meromorphic periodical functions. So as function η algebraically depends on τ therefore η is a meromorphic periodical function. Thus, the break point x^* is a one of the poles of function η. The integral in (32) is a more general Abelian integral [10, 11].

5. Boundary Value Problem and Existence Theorems

The continuity conditions for components of field **E** imply

$$[\varepsilon X]_{x=0} = 0\,, \; [\varepsilon X]_{x=h} = 0\,, \; [Z]_{x=0} = 0\,, \; [Z]_{x=h} = 0\,. \tag{33}$$

We suppose that the functions $X(x)$ and $Z(x)$ also satisfy the conditions

$$X(x) = O\left(\frac{1}{|x|}\right) \text{ and } Z(x) = O\left(\frac{1}{|x|}\right) \text{ as } |x| \to \infty\,. \tag{34}$$

Introduce the notation

$$\mathbf{D} = \begin{pmatrix} d/dx & 0 \\ 0 & d/dx \end{pmatrix},$$

$$\mathbf{F}(X(x), Z(x)) = \begin{pmatrix} X(x) \\ Z(x) \end{pmatrix}, \; \mathbf{G}(F, \gamma) = \begin{pmatrix} G_1(F, \gamma) \\ G_2(F, \gamma) \end{pmatrix},$$

where $X(x)$ and $Z(x)$ are unknown functions, G_1 and G_2 are left-hand sides of system (11). The γ is a spectral parameter. Also we will be considered column-vector $\mathbf{N}(x) = \begin{pmatrix} \varepsilon X(x) \\ Z(x) \end{pmatrix}$. Rewrite the problem using introduced notation.

In the half-space $x < 0$ and $\varepsilon = \varepsilon_1$, we have $\mathbf{N} = \begin{pmatrix} \varepsilon_1 X \\ Z \end{pmatrix}$ and

$$\gamma \mathbf{DF} - \begin{pmatrix} 0 & \gamma^2 \\ \gamma^2 - \varepsilon_1 & 0 \end{pmatrix} \mathbf{F} = 0\,. \tag{35}$$

Inside the layer $0 < x < h$ and $\varepsilon = \varepsilon_2 + a|\mathbf{E}|^2$, we have $\mathbf{N} = \begin{pmatrix} \left(\varepsilon_2 + a\left(X^2 + Z^2\right)\right)X \\ Z \end{pmatrix}$, and the system has the form

$$\mathbf{L}(F, \gamma) \equiv \mathbf{DF} - \mathbf{G}(F, \gamma) = 0. \quad (36)$$

In the half-space $x > h$ and $\varepsilon = \varepsilon_3$, we have $\mathbf{N} = \begin{pmatrix} \varepsilon_3 X \\ Z \end{pmatrix}$ and

$$\gamma \mathbf{DF} - \begin{pmatrix} 0 & \gamma^2 \\ \gamma^2 - \varepsilon_3 & 0 \end{pmatrix} \mathbf{F} = 0 \cdot \quad (37)$$

The continuity conditions (33) imply the following conditions

$$\left[\mathbf{N}(x)\right]_{x=0} = 0, \ \left[\mathbf{N}(x)\right]_{x=h} = 0, \quad (38)$$

where $\left[f(x)\right]_{x=x_0} = \lim\limits_{x \to x_0 - 0} f(x) - \lim\limits_{x \to x_0 + 0} f(x)$, for vector it means transition to the limit for every components of the vector.

Let us formulate boundary problem (conjugation problem). We will search for non-vanishing vector F and corresponding eigenvalues γ so that F satisfies (35)-(37) and conjugation conditions (38). In addition the components of the vector F must obey the condition (34).

The function $\mathbf{L}(F, \gamma)$ from (36) is a nonlinear operator-function, which nonlinearly depends on the spectral parameter. Spectral theory of linear operator-functions, which nonlinearly depend on spectral parameter was built in [12]. As yet there is not a common spectral theory of nonlinear operator-functions which nonlinearly depend on spectral parameter. Therefore, commonly boundary value problems with these operator-functions can not be solved by known methods.

Definition 1. The value $\gamma = \gamma_0$ we call eigenvalue of the problem if the problem (35)-(37) with conditions (34) and (38) has unique nonzero solution $\mathbf{F}$. The solution $\mathbf{F}$, which corresponds to eigenvalues γ_0 we call eigenvector of the problem, and the components $X(x)$ and $Z(x)$ of the eigenvector $\mathbf{F}$ we call eigenfunctions.

There is a well-known definition of eigenvalue of linear operator-function which nonlinear depends from spectral parameter [12]. Definition 1 is a non-classical analog of that well-known definition. Definition under consideration, on the one hand, is a distribution of classical definition for a case of nonlinear operator-function nonlinearly depending on spectral parameter. On the other hand, definition 1 corresponds to physical statement of the problem.

Theorem 1. *Boundary value problems* (35)-(37) *with conditions* (34) *and* (38) *has a solution (eigenvalue) then and only then, when the eigenvalue is a solution of dispersion Equation* (32).

Proof. *Sufficiency*. It is obviously, that, if we have a solution γ of dispersion Equation (32), we can find functions $\tau(x)$ and $\eta(x)$ from system (15) and first integral (16). Using Formula (14), we obtain

$$X(x) = \pm \frac{\gamma\eta}{\sqrt{a}} \sqrt{\frac{\tau - \tau_0}{\eta^2 + \gamma^2\tau^2}}$$

$$Z(x) = \pm \frac{\gamma^2\tau}{\sqrt{a}} \sqrt{\frac{\tau - \tau_0}{\eta^2 + \gamma^2\tau^2}}. \quad (39)$$

The question of sign's choosing is very important. We know the behavior of function $\eta = \gamma\tau \dfrac{X}{Z}$: the function η is monotone increasing, if $x = x^*$ is such that $\eta(x^*) = 0$, then $\eta(x^* - 0) < 0$, $\eta(x^* + 0) > 0$, and if $x = x^{**}$ is such that $\eta(x^{**}) = \pm\infty$, then $\eta(x^{**} - 0) > 0$, $\eta(x^{**} + 0) < 0$. Function η has not other points of sign reversal. The boundary conditions result in $Z(h) = E_z^{(h)}$ (>0). We know, if $\eta > 0$, then functions X and Z have the same signs, but if $\eta < 0$, then X and Z have opposite signs, and keep in mind that X and Z are continuously differentiable functions (in corresponding domains), we chose suitable signs in expressions (39).

Necessity. The way of obtaining dispersion Equation (32) from (15) implies that the eigenvalue of the problem is a solution of dispersion equation.

It is necessary to note that the eigenfunctions correspond to eigenvalue γ_0 can easily find from (11) by using, for example, Runge-Kutta method.

Let us formulate existence and localization theorem is founded on the obtained results. Introduce the notation $J \equiv J(a, \gamma, N)$ which denotes the right-hand side of dispersion Equation (32). It is clear that for each integer nonnegative finite N

$$\inf_{\gamma^2 \in \left(\max(\varepsilon_1, \varepsilon_3), \varepsilon_2\right)} J(a, \gamma, N) > 0,$$

$$\sup_{\gamma^2 \in \left(\max(\varepsilon_1, \varepsilon_3), \varepsilon_2\right)} J(a, \gamma, N) < \infty.$$

Moreover, decreasing of N implies decreasing of the infimum and supremum, and increasing of N implies increasing of the infimum and supremum. It is obviously from the form of dispersion equation.

Theorem 2. *Let*

$$h_1^{(k)} = \inf_{\gamma^2 \in \left(\max(\varepsilon_1, \varepsilon_3), \varepsilon_2\right)} J(a, \gamma, k),$$

$$h_2^{(k)} = \sup_{\gamma^2 \in (\max(\varepsilon_1, \varepsilon_3), \varepsilon_2)} J(a, \gamma, k),$$

and h is such that there are i and j, that $h_2^{(i)} < h$ and $h_2^{(i+1)} > h$; $h_1^{(j)} < h$ and $h_1^{(j+1)} > h$.

Then, there are at least $j - i$ eigenvalues of the problem (35)-(37) with conditions (34) and (38).

Theorem 2 requires some explanations. If for some j infimum is greater than h then for all greater numbers j infimum is all the more greater than h and dispersion equation has not solutions. Similarly, if for some i supremum is smaller than h, then for all smaller numbers i supremum is all the more smaller than h. In this case dispersion equation also has not solutions. These explanations follow from the fact that the infimum and supremum of left-hand of dispersion Equation (32) are finite.

6. Transition to the Limit in the Case of a Linear Medium in the Layer

Consider formal transition to the limit as $a \to 0$ in the case of a linear medium in the layer. The dispersion equation in the linear case is ([13]):

$$\operatorname{tg}\left(h\sqrt{\varepsilon_2 - \gamma^2}\right) = \frac{\varepsilon_2\sqrt{\varepsilon_2 - \gamma^2}\left(\varepsilon_1\sqrt{\gamma^2 - \varepsilon_3} + \varepsilon_3\sqrt{\gamma^2 - \varepsilon_1}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_2 - \gamma^2\right) - \varepsilon_2^2\sqrt{\gamma^2 - \varepsilon_3}\sqrt{\gamma^2 - \varepsilon_1}}. \quad (40)$$

Consider the functions

$$f = \frac{\tau}{\gamma^2\tau^2 + \eta^2(\tau - 1)} \quad \text{and} \quad f_1 = \frac{\varepsilon_2}{\varepsilon_2 - \gamma^2}\frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2 - \gamma^2} + \eta^2}.$$

The function f_1 is derived from f by formal transition to the limit as $a \to 0$ with respect to τ. Since we search for real solutions $X(x)$ and $Z(x)$, the denominator of f_1 cannot vanish. Moreover, as $a \to 0$, the function f tends to f_1 uniformly with respect to $x \in [0, h]$. Under this condition, since f is continuous, we can use classical calculus result to transit to the limit as $a \to 0$ under the integral sign in (32):

$$\left(-\int_{-\frac{\varepsilon_3}{\sqrt{\gamma^2 - \varepsilon_3}}}^{\frac{\varepsilon_1}{\sqrt{\gamma^2 - \varepsilon_1}}} \frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2 - \gamma^2} + \eta^2} d\eta + (N+1)\int_{-\infty}^{+\infty} \frac{1}{\frac{\varepsilon_2^2}{\varepsilon_2 - \gamma^2} + \eta^2} d\eta \right) = \frac{\varepsilon_2 - \gamma^2}{\varepsilon_2} h. \quad (41)$$

The integrals in (41) are analytically evaluated. Finally, (41) yields

$$h\sqrt{\varepsilon_2 - \gamma^2} = \operatorname{arctg}\frac{\varepsilon_2\sqrt{\varepsilon_2 - \gamma^2}\left(\varepsilon_1\sqrt{\gamma^2 - \varepsilon_3} + \varepsilon_3\sqrt{\gamma^2 - \varepsilon_1}\right)}{\varepsilon_1\varepsilon_3\left(\varepsilon_2 - \gamma^2\right) - \varepsilon_2^2\sqrt{\gamma^2 - \varepsilon_3}\sqrt{\gamma^2 - \varepsilon_1}} + (N+1)\pi. \quad (42)$$

Taking the tangent of (42) gives (40).

Results of this paragraph show us that we obtain regular case when we transit to the limit as $a \to 0$. The limit of dispersion Equation (32) for nonlinear medium leads to dispersion Equation (40) for linear medium. The dispersion Equation (40) is well-known classical result in electrodynamics.

Note that the method of finding dispersion equation considered in this section can be applied to more general problem. Namely, to the problem of propagation TM wave in an anisotropic nonlinear layer with Kerr nonlinearity (we studied a case of isotropic nonlinear layer). The statement of the problem differs only one detail. In a case of anisotropic nonlinear layer the permittivity is described by diagonal tensor $\hat{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix}$, where $\varepsilon_{xx} = \varepsilon_{12} + b|E_x|^2 + a|E_z|^2$, $\varepsilon_{zz} = \varepsilon_{21} + a|E_x|^2 + b|E_z|^2$, a, b are nonlinearity coefficients and $\varepsilon_{12} > \max(\varepsilon_1, \varepsilon_3)$ and $\varepsilon_{21} > \max(\varepsilon_1, \varepsilon_3)$. Here, after writing the system of equation in terms of functions $X(x)$ and $Z(x)$ we have to chose new variables $\tau(x)$ and $\eta(x)$ in form $\tau(x) = \frac{\varepsilon_{12} + bX^2 + aZ^2}{\gamma^2}$ and $\eta(x) = \gamma\tau\frac{X}{Z}$, where $X = X(x)$ and $Z = Z(x)$.

Anisotropic case with additional condition $\varepsilon_{12} = \varepsilon_{21} = \varepsilon_2$ was completely investigated in [14].

The first approximation for the propagation constants was presented in [15].

7. Numerical Results

In this section some numerical results are presented. The calculations are illustrated by the plots.

In **Figure 2**, the solid lines show the solutions to the dispersion equation for the case of a linear medium in the layer, and the dashed lines correspond to the nonlinear dispersion equation.

Figure 2 illustrates the dependence of squared propagation constant γ^2 on layer's thickness h. The following parameters are used: $\varepsilon_1 = \varepsilon_3 = 1$, $\varepsilon_2 = 9$,

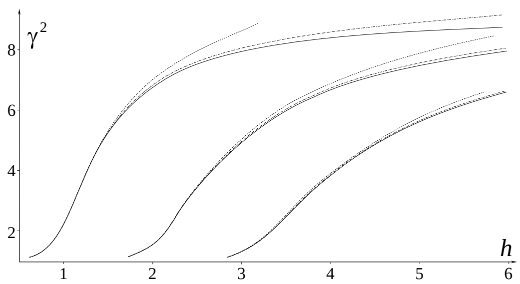

Figure 2. Dispersion curves of the problem.

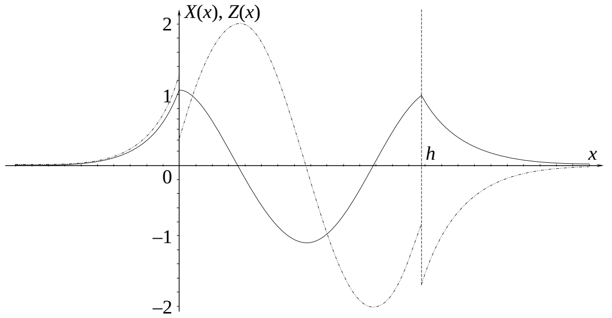

Figure 3. Eigenfunctions for the second eigenvalue of the problem.

$\left(E_z^{(h)}\right)^2 = 1$. In the case of nonlinear dispersion equation, the value $a = 0.1$ is employed.

For the next case the following initial data are chosen: $\varepsilon_1 = 1$, $\varepsilon_2 = 4$, $\varepsilon_3 = 2$, $\left(E_z^{(h)}\right)^2 = 1$, $a = 0.01$ and $h = 5.91$. In this case there are three eigenvalues γ_1, γ_2 and γ_3: $\gamma_1^2 = 3.9$, $\gamma_2^2 = 3.108$ and $\gamma_3^2 = 2.1711$.

In **Figure 3** eigenfunctions $X(x)$ and $Z(x)$ corresponding to the second eigenvalue γ_2 are depicted. The dash-dotted line corresponds to the eigenfunction $X(x)$ and the solid line corresponds to the eigenfunction $Z(x)$.

It is necessary to note that the component $X(x)$ is not continuous at the points $x = 0$ and $x = h$ (at the interfaces) and it has finite jumps at these points. The component $Z(x)$ is continuous but not differentiable at the points $x = 0$ and $x = h$.

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Implied Bond and Derivative Prices Based on Non-Linear Stochastic Interest Rate Models

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Abstract

In this paper we expand the Box Method of Sorwar *et al.* (2007) to value both default free bonds and interest rate contingent claims based on one factor non-linear interest rate models. Further we propose a one-factor non-linear interest rate model that incorporates features suggested by recent research. An example shows the extended Box Method works well in practice.

Keywords: Stochastic, Interest Rates, Derivatives, Box Method

1. Introduction

Stochastic differential equations are the foundations on which modern option pricing methodology is based. However, non-linear stochastic differential equations for interest rate models have been proposed that captures the non-linear dynamics of the spot interest rates. There are two aspects to the modeling of interest rate term structure models and interest rate contingent claims. The first concerns the econometric aspects (see for example, [1]) and the second the numerical implementation of the resulting models. With regard to the numerical aspects of interest rate modeling, there exist three different approaches. The first is the lattice approach introduced by Cox-Ross-Rubinstein (1979) [2]. However, as Barone-Adesi, Dinenis and Sorwar (1997) [3] have demonstrated the lattice approach does not always lead to meaningful bond and hence contingent claim prices. The second approach is the Monte-Carlo simulation approach introduced by Boyle (1977) is mainly used to value path dependent European type contingent claims. To date no single accepted Monte-Carlo simulation scheme has been put forward for the valuation of American type contingent claims. The third approach is the partial differential equation (PDE) approach. With this approach, the partial first and second order derivatives are discretized to produce a system of equations which are then solved iteratively to obtain the bond and contingent claim prices. However, Sorwar *et al.* (2007) have shown that the usual finite difference approach used to discretize the PDE does not always lead to bond and contingent claim prices that correspond with analytical prices where these prices are available.

Sorwar *et al.* (2007) introduced the Box Method from engineering to improve on the standard finite difference approach. Sorwar *et al.* (2007) focused on the CKLS (1992) model. Sorwar *et al.* (2007) did not attempt to value bonds and contingent claims based on non-linear interest rate models. Ait-Sahalia (1996) [4] non-and Conley *et al.* (1997) [5] propose parametric linear one-factor which allows non-linear parameterisation. Our main objective in this paper is to expand the Box Method of Sorwar *et al.* (1997) to price bonds and contingent claims based on both linear and non-linear interest rate models.

The outline of the paper is as follows: Section 2 the general non-linear parametric model and the resulting partial differential equation for default free bonds and contingent claims is outlined. We then derive the Expanded Box Method (EBM) for the valuation of default free bonds and contingent claims. Using US estimates we compute implied bond and contingent claims prices in Section 3. Section 4 contains a summary and conclusion.

2. Expanded Box Method (EBM)

In this section we discuss the valuation of the bond and contingent claim prices based on the extended Ait-Sahalia (1996) [4] and Conley *et al.* (1997) [5] framework. Following Sorwar *et al.* (2007) we let:

$B\left(r_t,t,T^*\right)$: price of a discount bond at time t which

Table 1. lternative Parametric Specifications of the Spot Interest Rate Process $dr_t = \mu(r_t)dt + \sigma^2(r_t)dW_t$.

Drift function $\mu(r)$	Diffusion function $\sigma^2(r)$	Reference
$\alpha_0 + \alpha_1 r$	β_0	Vasicek (1977) [6]
$\alpha_0 + \alpha_1 r$	$\beta_1 r$	Cox-Ingersoll-Ross(1985) [7] Brown-Dybvig(1986) [8] Gibbons-Ramaswamy(1993) [9]
$\alpha_0 + \alpha_1 r$	$\beta_2 r^2$	Courtadon (1982) [10]
$\alpha_0 + \alpha_1 r$	$\beta_2 r^{\beta_3}$	Chen *et al.* (1992)
$\alpha_0 + \alpha_1 r + \alpha_2 r^2 + \frac{\alpha_3}{r}$	$\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}$	Ait-Sahalia (1996) [4]
$\alpha_0 + \alpha_1 r + \alpha_2 r^{\alpha_3} + \frac{\alpha_4}{r^{\alpha_5}}$	$\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}$	

matures at time T^* with the generated spot rate r_t.

$P(t,T^*,T)$: price of a contingent claim at time t which expires at time T based on a discount bond which matures at time T^* subject to suitable boundary conditions.

In a risk-neutral world, the drift rate is adjusted by the market price of risk λr [1] so that the short-term interest process becomes:

$$dr_t = \left(\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}\right)dt + \sqrt{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\, dW_t \tag{1}$$

The resulting partial differential equation is:

$$\frac{1}{2}\left[\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}\right]\frac{\partial^2 U}{\partial r^2} + \left[\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}\right]\frac{\partial U}{\partial r} - rU + \frac{\partial U}{\partial t} = 0 \tag{2}$$

In Equation (2) $U(r_t,t)$ may represent either $B(r_t,t,T^*)$ or $P(t,T^*,T)$ subject to the appropriate boundary conditions (see [10] for more details). Following Sorwar *et al.* (2007) we transform the above pricing equation such that either the bond or the contingent claims evolves from the options expiration date or the bonds maturity date to the present, *i.e.*, we let $\tau = T - t$.

[1] Risk premium is treated differently by researchers. Vasicek (1977) [6] takes $\lambda(r) = \lambda$, Chan *et al.* (1992) [1] take $\lambda(r) = 0$, ox *et al.* (1985), we take $\lambda(r) = \lambda r$.

The above equation then becomes:

$$\frac{\partial^2 U}{\partial r^2} + 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]\frac{\partial U}{\partial r} - \frac{2r}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}U = \frac{2}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\frac{\partial U}{\partial \tau} \tag{3}$$

We now choose a general function $R(r,\alpha,\beta)$ such that:

$$\frac{1}{R}\frac{\partial}{\partial r}\left[R\frac{\partial U}{\partial r}\right] = \frac{\partial^2 U}{\partial r^2} + 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]\frac{\partial U}{\partial r} \tag{4}$$

The above expression simplifies to yield:

$$\frac{1}{R}\frac{\partial R}{\partial r} = 2\left[\frac{\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right] \tag{5}$$

We now integrate from the general value r $(r_{n-1} < r < r_{n+1})$ to the lower limit of integration $r = 0$ to obtain:

$$R(r,\alpha,\beta) = exp\left\{\phi + 2\int_{r_{n-1}}^{r}\left[\frac{\alpha_0 + (\alpha_1 + \lambda) r + \alpha_2 r^{\alpha_3} + \alpha_4 r^{-\alpha_5}}{\beta_0 + \beta_1 r + \beta_2 r^{\beta_3}}\right]dr\right\}$$

where $\phi = ln\, R(0,\alpha,\beta)$. We further note that:

$$\frac{1}{R}\frac{\partial}{\partial r}\left[R\frac{\partial U}{\partial r}\right] = \frac{1}{Q}\frac{\partial}{\partial r}\left[Q\frac{\partial U}{\partial r}\right]$$

where:

$$Q(r,\alpha,\beta)=$$
$$exp\left\{2\int_0^r\left[\frac{\alpha_0+(\alpha_1+\lambda)r+\alpha_2 r^{\alpha_3}+\alpha_4 r^{-\alpha_5}}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}\right]dr\right\}$$

So Equation (3) becomes:

$$\frac{1}{Q}\frac{\partial}{\partial r}\left(Q\frac{\partial U}{\partial r}\right)-\frac{2r}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}U=$$

We now transform the interest rate as:

$$\frac{2}{\beta_0+\beta_1 r+\beta_2 r^{\beta_3}}\frac{\partial U}{\partial \tau} \quad (6)$$

$$s=\frac{cr}{1+cr} \text{ where } c \text{ is a constant.} \quad (7)$$

This leads to the transformation of Equation (6) as:

$$\frac{1}{Q(s)}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)-\frac{2s}{c^2(1-s)^3}\frac{U}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}=\frac{2}{c(1-s)^2}\frac{1}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}\frac{\partial U}{\partial \tau} \quad (8)$$

where:

$$\Psi(s)=c(1-s)^2 Q(s)$$
$$Q(s)=exp\left\{\int_0^s\frac{2}{c(1-s)^2}\left\{\frac{\alpha_0+\frac{(\alpha_1+\lambda)s}{c(1-s)}+\alpha_2\left[\frac{s}{c(1-s)}\right]^{\alpha_3}+\alpha_4\left[\frac{s}{c(1-s)}\right]^{-\alpha_5}}{\beta_0+\frac{\beta_1 s}{c(1-s)}+\beta_2\left[\frac{s}{c(1-s)}\right]^{\beta_3}}\right\}dr\right\}$$

Following the set-up of Sorwar *et al.* (2007) a grid of size $M\times N$ is constructed for values of $U_n^m = U(n\Delta r, m\Delta t)$—the value of U at time increment t_m and interest rate increment s_n, for each method, where:

$$t_m=t_0+m\Delta t \quad m=0,1,....,M$$

$$^2\ \Delta s_{n+1}=\Delta s_n+a \qquad n=1,....,N$$

where a is an arbitrary constant.

Using the Euler backward difference for the time derivative gives: $\frac{\partial U}{\partial \tau}=\frac{U-U_0}{\Delta t}$,

where U_0 and U refers to bond or contingent claims prices at time step *m-1* and *m* respectively.

Integrating Equation (8) from the point

$s_{n-\frac{1}{2}}=\frac{s_n+s_{n-1}}{2}$ to point $s_{n+\frac{1}{2}}=\frac{s_{n+1}+s_n}{2}$, we have:

$$-\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds+2\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{s}{c^2(1-s)^3}Q(s)f(s)Uds+2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)Uds$$
$$=2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)U_0 ds \quad (9)$$

Discretizing each of the above integrals, and rearranging gives us the following matrix equation:

$$\alpha_n U_n^{m-1}=\chi_n U_{n-1}^m+\eta_n U_n^m+\beta_n U_{n+1}^m \quad (10)$$

where:

$$-\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds+$$

$$2\Delta t\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{s}{c^2(1-s)^3}Q(s)f(s)Uds+$$

[2]Where a and Δs_0 are arbitrary constants. A derivation of this expression can be found in Settari and Aziz (1972) [11].

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)Uds$$

$$=2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}}\frac{1}{c(1-s)^2}Q(s)f(s)U_0ds \quad (9)$$

Discretizing each of the above integrals, and rearranging gives us the following matrix equation:

$$\alpha_n U_n^{m-1}=\chi_n U_{n-1}^m+\eta_n U_n^m+\beta_n U_{n+1}^m \quad (10)$$

where:

$$\alpha_n=2I_1$$

$$\chi_n=\frac{-\Delta t}{s_n-s_{n-1}}\frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{Q(s_n)}$$

$$\beta_n=\frac{-\Delta t}{s_{n+1}-s_n}\frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{Q(s_n)}$$

$$\eta_n=\frac{\Delta t}{s_n-s_{n-1}}\frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{Q(s_n)}+\frac{\Delta t}{s_{n+1}-s_n}\frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{Q(s_n)}+2\Delta tI_0+2I_1$$

$$I_0=\frac{s_n f(s_n)}{c^2(1-s_n)^3}\left(s_{n+\frac{1}{2}}-s_{n-\frac{1}{2}}\right)$$

$$I_1=\frac{f(s_n)}{c(1-s_n)^2}\left(s_{n+\frac{1}{2}}-s_{n-\frac{1}{2}}\right)$$

The matrix equation linking all bond prices or contingent claim prices between two successive time steps *m*-1 and *m* is:

$$\begin{pmatrix}\alpha_0U_1^{m-1}\\ \alpha_0U_1^{m-1}\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \alpha_NU_N^{m-1}\end{pmatrix}=$$

$$\begin{pmatrix}\eta_1 & \beta_1 & 0 & 0 & 0 & \cdots & 0\\ \chi_2 & \eta_2 & \beta_2 & 0 & 0 & \cdots & 0\\ 0 & \chi_3 & \eta_3 & \beta_3 & 0 & \cdots & 0\\ \cdots 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \chi_{N-3} & \chi_{N-3} & \beta_{N-3} & 0\\ \vdots & \ddots & \ddots & 0 & \chi_{N-2} & \eta_{N-2} & \beta_{N-2}\\ 0 & \cdots & \cdots & 0 & 0 & \chi_{N-1} & \eta_{N-1}\end{pmatrix}\begin{pmatrix}\alpha_0U_1^m\\ \alpha_0U_1^m\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \alpha_NU_{N-1}^m\end{pmatrix}$$

Sorwar *et al.* [12] used the following SOR iteration process to determine bond and contingent claims prices:

$$z_n^m=\frac{1}{\eta_n}\left(\alpha_nU_n^{m-1}-\chi_nU_{n-1}^{m}-\beta_nU_{n+1}^{m-1}\right) \quad (11)$$

In particular they evaluated bond using the following expression:

$$U_n^m=\omega z_n^m+(1-\omega)U_n^{m-1} \quad (12)$$

Contingent claims were calculated using:

$$U_n^m=max\left[Z,\omega z_n^m+(1-\omega)U_n^{m-1}\right] \quad (13)$$

where Z is the intrinsic value of the contingent claim and for n=1,......,N-1, and $\omega\in(1,2]$ [3].

3. Analysis of Results

In this section we apply the EBM using recent estimates of the non-linear model of Ait-Sahalia (1996) [4] on 7-day Eurodollar deposit spot rate over 1973-1995 to demonstrate the method. Ait-Sahalia (1996, Table 4) [4] obtained the following estimates:

$$\alpha_0=-4.643\times10^{-3},\ \alpha_1=4.333\times10^{-2},\ \alpha_2=-1.143\times10^{-1},\alpha_3=2,$$

$$\alpha_4=1.304\times10^{-4},\ \alpha_5=1.$$

$$\beta_0=1.108\times10^{-4},\ \beta_1=-1.883\times10^{-3},\ \beta_2=9.681\times10^{-3},\ \beta_3=2.073.$$

Table 2 reports the bond prices for maturities ranging from 6 months to 30 years and across interest rates of 2% to 16%. **Table 3** reports both the value of call and put options across a wide range of interest rates. We consider both short and long dated call and put options. The short dated call and put options are based on a 5-year bond with an expiry date of 1 year and is during the last year before the bond matures. Similarly long dated options are based on 10-year bond with an expiry date of 5 years during the last 5 years of the bond. Finally both call and put option prices are calculated across a wide range of exercise prices. The exercise prices are chosen so as to highlight variation of prices for both in-the-money and out-of-the-money options. We assume λ, the market price of risk is zero.

Turning to **Table 2**, we find that at lower interest rate bond prices decay slowly as the term to maturity increases. For example, at 2% interest rate a 1 year maturity bond is valued at 98.1119, whilst a 30 year bond is valued at 74.8290. At high interest rates, the bond price decay is more rapid for example at 16% interest rate, a 1 year maturity bond is valued at 85.2915, whist a 30 year maturity bond is valued at 1.1770. Turning to **Table 3**, we observe the following features. Short expiry call op

[3] ω is determined by numerical experimentation. For all our calculations we took $\omega=1.85$

Table 2. All options written on zero coupon bonds with a face value of $100.00.

	Interest Rate							
Maturity of Bond	**2%**	**4%**	**6%**	**8%**	**10%**	**12%**	**14%**	**16%**
0.5	99.0286	98.0370	96.9855	96.0885	95.1315	94.1844	93.2506	92.3403
1	98.1119	96.1434	94.0805	92.3406	90.5050	88.7059	86.9566	85.2915
5	92.2400	83.3035	74.3413	67.4685	60.8623	54.9010	49.7324	45.6212
10	87.0431	71.9535	56.7017	46.1717	37.2750	30.1193	24.7834	21.3038
15	83.1089	64.1538	44.6651	32.1800	22.9491	16.5267	12.3933	10.1317
20	79.9228	58.6473	36.4723	22.9644	14.3178	9.0809	6.2237	4.8889
25	77.2156	54.6338	30.8731	16.8832	9.0110	5.0032	3.1400	2.3870
30	74.8290	51.6021	27.0075	12.8582	5.7491	2.7679	1.5921	1.1770

Table 3. All options written on zero coupon bonds with a face value of $100.00.

r (%)	**Exercise-Price**	**5 year maturity 1 year expiry**	**5 year maturity 1 year expiry**	**Exercise-Price**	**10 year maturity 5 year expiry**	**10 year maturity 5 year expiry**
	(83.3035)	**call**	**put**	**(71.9535)**	**call**	**put**
4	70	16.0031	0.0000	60	21.9713	0.0007
	75	11.1959	0.0000	65	17.8062	0.0493
	80	6.3895	0.0050	70	13.6418	0.6489
	85	1.9369	1.6966	75	9.5270	3.1894
	90	0.1421	6.6966	80	5.7979	8.0466
	(67.4685)			**(46.1717)**		
8	55	16.6811	0.0000	35	22.5578	0.0000
	60	12.0641	0.0000	40	19.1843	0.0000
	65	7.4471	0.0000	45	15.8109	0.0058
	70	2.8302	2.5315	50	12.4375	3.8283
	75	0.0203	7.5315	55	9.0641	8.8283
12	**(54.9010)**			**(30.1193)**		
	45	14.9341	0.0000	20	19.1395	0.0000
	50	10.4996	0.0000	25	16.3942	0.0000
	55	6.0652	0.1561	30	13.6492	0.0183
	60	1.6310	5.1561	35	10.9042	4.8804
	65	0.0000	10.1561	40	8.1591	9.8804
16	**(45.6212)**			**(21.3038)**		
	35	15.7692	0.0000	10	16.7416	0.0000
	40	11.5046	0.0000	15	14.4606	0.0000
	45	7.2400	0.0005	20	12.1795	0.0001
	50	2.9755	4.3788	25	9.8985	3.6962
	55	0.0129	9.3782	30	7.6174	8.6962

tions decay faster than longer expiry call options; for example at r = 4%; the price of a call option decreases from 16.0031 to 11.1959 when the exercise price in creases from 70 to 75. For a similar 5 year call option the price decreases from 21.9713 to 17.8062, when the exercise price increases from 60 to 65. Furthermore, the call option prices decrease at a slower rate at high interests. This feature becomes more pronounced for longer expiry call options. With regard to put options we find, the prices are very close to zero, when the options are at-the-money or out-of-the-money. Finally, we find that the value of in-the-money put options is dominated by the intrinsic-value.

4. Conclusions

The introduction of non-linear stochastic interest rate models has led to the possibility of valuing interest contingent claims that reflects the characteristics of the yield curve more accurately. In this paper we have expanded the Box Method to value both bond and American type interest rate contingent claims based on single factor non-linear interest rate models. We have found that the

Expanded Box Method works well with the example considered.

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Appendix

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds \approx \Psi\left(s_{n+\frac{1}{2}}\right)\frac{\partial U}{\partial s}\bigg|_{s_{n+\frac{1}{2}}} - \Psi\left(s_{n-\frac{1}{2}}\right)\frac{\partial U}{\partial s}\bigg|_{s_{n-\frac{1}{2}}}$$

Further:

$$\frac{\partial U}{\partial s}\bigg|_{s_{n+\frac{1}{2}}} \approx \frac{U_{n+1}^m - U_n^m}{s_{n+1} - s_n}$$

$$\frac{\partial U}{\partial s}\bigg|_{s_{n-\frac{1}{2}}} \approx \frac{U_n^m - U_{n-1}^m}{s_n - s_{n-1}}$$

Substitution of the above approximation yields:

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{\partial}{\partial s}\left(\Psi(s)\frac{\partial U}{\partial s}\right)ds \approx \frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{s_{n+1}-s_n}U_{n+1}^m -$$

$$\left[\frac{\Psi\left(s_{n+\frac{1}{2}}\right)}{s_{n+1}-s_n} + \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{s_n - s_{n-1}}\right]U_n^m + \frac{\Psi\left(s_{n-\frac{1}{2}}\right)}{s_n - s_{n-1}}U_{n-1}^m$$

$$2\Delta t \int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s)Q(s)U ds \approx$$

$$2\Delta t Q(s_n)U_n^m \int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s)ds$$

We further take:

$$\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{s}{c^2(1-s)^3} f(s)ds \approx \frac{s_n}{c^2(1-s_n)^3} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

Similar approximation yields:

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} f(s)Q(s)U ds \approx$$

$$2Q(s_n)U_n^m \frac{1}{c(1-s_n)^2} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

$$2\int_{s_{n-\frac{1}{2}}}^{s_{n+\frac{1}{2}}} \frac{1}{c(1-s)^2} f(s)Q(s)U_0 ds \approx$$

$$2Q(s_n)U_n^{m-1} \frac{1}{c(1-s_n)^2} f(s_n)\left(s_{n+\frac{1}{2}} - s_{n-\frac{1}{2}}\right)$$

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Modeling of Tsunami Generation and Propagation by a Spreading Curvilinear Seismic Faulting in Linearized Shallow-Water Wave Theory

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Abstract

The processes of tsunami evolution during its generation in search for possible amplification mechanisms resulting from unilateral spreading of the sea floor uplift is investigated. We study the nature of the tsunami build up and propagation during and after realistic curvilinear source models represented by a slowly uplift faulting and a spreading slip-fault model. The models are used to study the tsunami amplitude amplification as a function of the spreading velocity and rise time. Tsunami waveforms within the frame of the linearized shallow water theory for constant water depth are analyzed analytically by transform methods (Laplace in time and Fourier in space) for the movable source models. We analyzed the normalized peak amplitude as a function of the propagated uplift length, width and the average depth of the ocean along the propagation path.

Keywords: Tsunami Modeling, Shallow Water Theory, Water Wave, Bottom Topography, Laplace and Fourier Transforms

1. Introduction

Waves at the surface of a liquid can be generated by various mechanisms such as, wind blowing on the free surface, wave maker, moving disturbance on the bottom or the surface, or even inside the liquid, fall of an object into the liquid, liquid inside a moving container, etc. The generation of tsunamis by a seafloor deformation is an example for the case where the waves are created by a given motion of the bottom. There are different natural phenomena that can lead to a tsunami. For example, one can mention submarine slumps, slides, volcanic explosions, earthquakes, etc.

The sea bottom deformation following an underwater earthquake is a complex phenomenon. This is why, for theoretical or experimental studies, researchers have often used simplified bottom motions such as the vertical motion of a box. Most investigations of tsunami generation and propagation used developed integral solution (in space and time) for an arbitrary bed displacement based on a linearized description of wave motion in either a two or three-dimensional fluid domain of uniform depth.

The complexity of the integral solutions developed from the linear theory even for the simplest model of bed deformation prevented many authors from determining detailed wave behaviour, especially near the source region. However, we construct three-dimensional curvilinear source models involved in the transform methods that may generate a tsunami near the source region.

Many authors have used different analytical solutions and numerical computations to determine the general wave pattern near and far from the source region for a variety of bed motions in a two or three-dimensional fluid domain. Ben-Menahem and Rosenman [1] calculated the two-dimensional radiation pattern from a moving source using linear theory. Tuck and Hwang [2] solved the linear long-wave equation in the presence of a moving bottom and a uniformly sloping beach. According to Synolakis and Bernard [3], Houston and Garcia [4] were the first to use more geophysical realistic initial conditions. Tinti and Bortolucci [5] investigated analytically the generation of tsunamis by submarine slides. They specialized the general solution of the 1D Cauchy linear problems for long water waves to deal with rigid body to explore the characteristics of the generated waves. They studied the body motion in terms of Froude number, wave pattern, wave amplitude and wave energy, see also [6-13].

Since the general features of the waves obtained by these authors for both the near and far fields are based on a linear theory, their applicability is limited to bed deformations and the range of propagation for which nonlinear effects remain small.

Several works done are taken into account the nonlinear effect in the wave motion for modeling the tsunami waves. Kervella *et al*. [14] perform a comparison between three-dimensional linear and nonlinear tsunami generation models. They observed very good agreement from the superposition of the wave profiles computed with the linear and fully nonlinear models. Second, they found that the nonlinear shallow water model was not sufficient to model some of the waves generated by a moving bottom because of the presence of frequency dispersion, hence the suggested that for most events the linear theory is sufficient. Villeneuve [15] derived model equations which combine the linear effect of frequency dispersion and the nonlinear effect of amplitude dispersion including the effects of a moving bed. Liu and Liggett [16] performed comparisons between linear and nonlinear water waves where their study was restricted to simple bottom deformations, namely the generation of transient waves by an upthrust of a rectangular block. Bona *et al*. [17] assessed how well a model equation with weak nonlinearity and dispersion describes the propagation of surface water waves generated at one end of a long channel. In their experiments, they found that the inclusion of a dissipative term was more important than the inclusion of nonlinearity, although the inclusion of nonlinearity was undoubtedly beneficial in describing the observations. Abo Dina *et al*. [18] have adopted a nonlinear theory and constructed a numerical model of tsunami generation and propagation which permits a variable bed displacement with an arbitrary water depth to be included in the model. In this model, he considered nonlinearities and omitted the linear effects of frequency dispersion; hence, no insight into the possible importance of the interaction of nonlinear and linear effects in the far field was possible, see also [19,20]. All the previous studies mention above neglected the details of wave generation in fluid during the source time. One of the reasons is that it is commonly assumed that the source details are not important.

The transient wave generation due to the coupling between the seafloor motion and the free surface has been considered by a few authors only. There are some specific cases where the time scale of the bottom deformation and the horizontal extent of the bottom deformation may become an important factor. Some studies have already been performed to understand wave formation due to different prescribed bottom motions by introducing either some type of rise time or some type of rupture velocity. For example, Todorovska *et al*. [21] studied the generation of waves by a slowly spreading uplift of the bottom in linearized shallow-water wave theory and where able to explain some observations. They studied the tsunami amplitude amplification as a function of the model parameters. They found the effects of the spreading of the ocean floor deformation (faulting, submarine slides or slumps) on the amplitudes and periods of the generated tsunamis are largest when the spreading velocity of uplift and the tsunami velocity are comparable. Trifunac *et al*. [22] mentioned the source parameters for submarine slides and earthquakes including source duration, displacement amplitude, areas and volumes of selected past earthquakes that have or may have generated a tsunami. They contributed the nature of tsunami sources to create tsunami waveforms in the near field and provided a starting point for their elementary mathematical model. Todorovska *et al*. [23] investigated tsunami generation by a slowly spreading uplift of the sea floor in the near field considering the effects of the source finiteness and directivity. They described mathematically various two-dimensional kinematic models of submarine slumps and slides as combinations of spreading constant or slopping uplift functions. There results show that for given constant water depth, the peak amplitude depends on the ratio of the spreading velocity of the sea floor to the long wavelength tsunami velocity, see also their works [24,25]. Hammack [26] generated waves experimentally by raising or lowering a box at one end of a channel. He considered two types of time histories: an exponential and a half-sine bed movement. Dutykh and Dias [27] generated waves theoretically by multiplying the static deformations caused by slip along a fault by various time laws: instantaneous, exponential, trigonometric and linear. Haskell [28] was one of the first authors, who take into account the rupture velocity. In fact he considered both rise time T and rupture velocity *V*.

All the previous approaches done in [21-28] computed tsunami waveforms using linearized shallow water theory and transform methods of solution. We follow the same approach but with a more realistic and more complex source models. This approach is restricted to the water region where the incompressible Euler equations for potential flow can be linearized. In this paper we investigate the tsunami wave in the near and far field using the transform methods (Laplace in time and Fourier in space). We construct mathematically a reasonable curvilinear tsunami source based on available geological, seismological, and tsunami elevation. This model resembles the initial source predicted according to the initial disturbance recorded in [29,30]. We discuss aspects of tsunami generation that should be considered in developing these models, as well as the propagation wave after the formation of the source models have been completed.

We study the fluid wave motion above finite sources, with irregular fault shapes and with variable distribution of the ocean floor uplift, for variable spreading velocities. Here we aim to demonstrate the large scale tsunami generation features computed during the formation of the tsunami source for different ratios between the velocity of the source propagation and the tsunami speed, as well as the overall propagation following the source. Comparison between our results and others obtained for the tsunami model in the near—field is done. According to the results and the numerical estimation, we analyze the normalized peak amplitude as a function of the characteristics size of the source model and the water depth.

2. Mathematical Formulation of the Problem

Consider a three dimensional fluid domain Ω as shown in **Figure 1**. It is supposed to represent the ocean above the fault area. It bounded above by the free surface of the ocean $z = \eta(x, y, t)$ and below by the rigid ocean floor $z = -h(x, y) + \zeta(x, y, t)$, where $\eta(x, y, t)$ is the free surface elevation, $h(x, y)$ is the water depth and $\zeta(x, y, t)$ is the sea floor displacement function. The domain Ω is unbounded in the horizontal directions x and y, and can be written as $\Omega = R^2 \times [-h(x, y) + \zeta(x, y, t), \eta(x, y, t)]$. For simplicity, $h(x, y)$ is assumed to be a constant. Before the earthquake, the fluid is assumed to be at rest, thus the free surface and the solid boundary are defined by $z = 0$ and $z = -h$, respectively. Mathematically, these conditions can be written in the form of initial conditions: $\eta(x, y, 0) = \zeta(x, y, 0) = 0$. At time $t > 0$ the bottom boundary moves in a prescribed manner which is given by $z = -h + \zeta(x, y, t)$. The deformation of the sea bottom is assumed to have all the necessary properties needed to compute its Fourier transform in x,y and its Laplace transform in t. The resulting deformation of the free surface $z = \eta(x,y,t)$ is to be found as part of the solution. It is assumed that the fluid is incompressible and the flow is irrotational. The former implies the existence of a velocity potential $\phi(x, y, z, t)$ which fully describes the flow and the physical process. By definition of ϕ, the fluid velocity vector can be expressed as $\vec{q} = \nabla\phi$. Thus, the potential flow $\phi(x, y, z,t)$ must satisfy the Laplace's equation

$$\nabla^2\phi(x, y, z, t) = 0 \text{ where } (x, y, z) \in \Omega \tag{1}$$

The potential $\phi(x, y, z, t)$ must satisfy the following kinematic and dynamic boundary conditions on the free surface and the solid boundary, respectively:

$$\phi_z = \eta_t + \phi_x\eta_x + \phi_y\phi_y \quad \text{on } z = \eta(x, y, t), \tag{2}$$

$$\phi_z = \zeta_t + \phi_x\zeta_x + \phi_y\zeta_y \quad \text{on } z = -h + \zeta(x, y, t), \tag{3}$$

and

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad \text{on } z = \eta(x, y, t), \tag{4}$$

where g is the acceleration due to gravity. As described above, the initial conditions are given by

$$\phi(x, y, z, 0) = \eta(x, y, 0) = \zeta(x, y, 0) = 0. \tag{5}$$

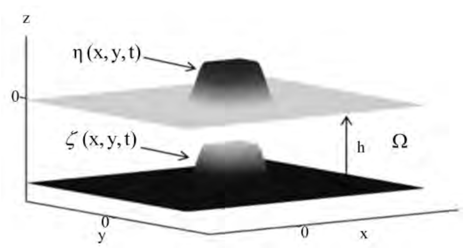


Figure 1. Definition of the fluid domain and coordinate system for a very rapid movement of the assumed source model.

2.1. Linear Shallow Water Theory

Various approximations can be considered for the full water-wave equations. One is the system of Boussinesq equations that retains nonlinearity and dispersion up to a certain order. Boussinesq model is used to study transient varying bottom problems. Fuhrman *et al*. [31] and Zhao *et al*. [32] presented a developed numerical model based on the highly accurate Boussinesq-type formulation subjected to exact expressions for the kinematic and dynamic free surface conditions. Their results show that the model was capable of treating the full life cycle of tsunami evolution, from the initial generation of bottom movements, to the subsequent propagation, and through the final the run-up process. Reasonable computational efficiency has been demonstrated in their work, which made the model attractive for practical coastal engineering studies, where high dispersive and nonlinear accuracy is sought. Another one is the system of nonlinear shallow-water equations that retains nonlinearity but no dispersion. Solving this problem is a difficult task due to the nonlinearities and the a priori unknown free surface. The simplest one is the system of linear shallow-water equations. The concept of shallow water is based on the smallness of the ratio between water depth and wave length. In the case of tsunamis propagating on the surface of deep oceans, one can consider that shallow-water theory is appropriate because the water depth (typically several kilometers) is much smaller than the wave length (typically several hundred kilometers), which is reasonable and usually true for most tsunamis triggered by submarine earthquakes, slumps and slides [26,27]. Hence, the problem can be linearized by neglecting the nonlinear terms in the boundary conditions (2)-(4) and if the boundary conditions are applied on the nondeformed instead of the deformed boundary surfaces (*i.e.*, on $z = -h$ and $z = 0$ instead of $z = -h + \zeta(x, y, t)$ and $z = \eta(x, y, t)$).

The linearized problem in dimensional variables can be written as

$$\nabla^2 \phi(x, y, z, t) = 0 \text{ where } (x, y, z) \in R^2 \times [-h, 0], \quad (6)$$

subjected to the following boundary conditions

$$\phi_z = \eta_t \quad \text{on} \quad z = 0 \quad (7)$$

$$\phi_z = \zeta_t \quad \text{on} \quad z = -h \quad (8)$$

$$\phi_t + g\eta = 0 \quad \text{on} \quad z = 0 \quad . \quad (9)$$

The linearized shallow water solution can be obtained by the Fourier-Laplace transform.

2.2. Solution of the Problem

Our interest is the resulting uplift of the free surface elevation $\eta(x, y, t)$. An analytical analyses is examine to illustrate the generation and propagation of a tsunami for a given bed profile $\zeta(x, y, t)$. Mathematical modeling of waves generated by vertical and lateral displacements of ocean bottom using the combined Fourier-Laplace transform of the Laplace equation analytically is the simplest way of studying tsunami development. All our studies were taken into account constant depths for which the Laplace and Fast Fourier Transform (FFT) methods could be applied. Equations (6)-(9) can be solved by using the method of integral transforms. We apply the Fourier transform in (x, y).

$$\mathfrak{F}[f] = \hat{f}(k_1, k_2) = \int_{R^2} f(x, y) e^{-i(k_1x+k_2y)} dx\, dy$$

with its inverse transform

$$\mathfrak{F}^{-1}[\hat{f}] = f(x,y) = \frac{1}{(2\pi)^2} \int_{R^2} \hat{f}(k_1, k_2) e^{i(k_1x+k_2y)} dk_1\, dk_2$$

and the Laplace transform in time t,

$$\pounds[g] = G(s) = \int_0^\infty g(t) e^{-st} dt$$

For the combined Fourier and Laplace transforms, the following notation is introduced:

$$\mathfrak{F}(\pounds(f(x, y, t)) = \overline{F}(k_1, k_2, s) = \int_{R^2} e^{-i(k_1x+k_2y)} dx\, dy \int_0^\infty f(x, y, t) e^{-st} dt$$

Combining (7) and (9) yields the single free-surface condition

$$\phi_{tt}(x, y, 0, t) + g\phi_z(x, y, 0, t) = 0. \quad (10)$$

After applying the transforms and using the property $\mathfrak{F}[\frac{d^n f}{dx^n}] = (ik)^n \overline{F}(k)$ and the initial conditions (5), Equations (6), (8) and (10) become

$$\overline{\phi}_{zz}(k_1, k_2, z, s) - (k_1^2 + k_2^2)\overline{\phi}(k_1, k_2, z, s) = 0 \quad (11)$$

$$\overline{\phi}_z(k_1, k_2, -h, s) = s\overline{\zeta}(k_1, k_2, s) \quad (12)$$

$$s^2\overline{\phi}(k_1, k_2, 0, s) + g\overline{\phi}_z(k_1, k_2, 0, s) = 0 \quad (13)$$

The transformed free-surface elevation can be obtained from (9) as

$$\overline{\eta}(k_1, k_2, s) = -\frac{s}{g}\overline{\phi}(k_1, k_2, 0, s) \quad (14)$$

A general solution of (11) will be given by

$$\overline{\phi}(k_1, k_2, z, s) = A(k_1, k_2, s)\cosh(kz) + B(k_1, k_2, s)\sinh(kz) \quad (15)$$

where $k = \sqrt{k_1^2 + k_2^2}$. The functions $A(k_1, k_2, s)$ and $B(k_1, k_2, s)$ can be easily found from the boundary conditions (12) and (13),

$$A(k_1, k_2, s) = -\frac{gs\overline{\zeta}(k_1, k_2, s)}{\cosh(kh)[s^2 + gk\tanh(kh)]}$$

$$B(k_1, k_2, s) = \frac{s^3\overline{\zeta}(k_1, k_2, s)}{k\cosh(kh)[s^2 + gk\tanh(kh)]}$$

Substituting the expressions for the functions A and B in the general solution (15) yields

$$\overline{\phi}(k_1, k_2, z, s) = -\frac{gs\overline{\zeta}(k_1, k_2, s)}{\cosh(kh)[s^2 + \omega^2]}\left(\cosh(kz) - \frac{s^2}{gk}\sinh(kz)\right) \quad (16)$$

where $\omega = \sqrt{gk\tanh(kh)}$ is the circular frequency of the wave motion. The free surface elevation $\overline{\eta}(k_1, k_2, s)$ can be obtained from (14) as

$$\overline{\eta}(k_1, k_2, s) = \frac{s^2\overline{\zeta}(k_1, k_2, s)}{\cosh(kh)(s^2 + \omega^2)} \quad (17)$$

A solution for $\eta(x, y, t)$ can be evaluated for specified $\zeta(x, y, t)$ by computing approximately its transform $\overline{\zeta}(k_1, k_2, s)$ then substituting it into (17) and inverting $\overline{\eta}(k_1, k_2, s)$ to obtain $\overline{\eta}(k_1, k_2, t)$. We concern to evaluate $\eta(x, y, t)$ by transforming analytically the assumed source model then inverting the Laplace transform of $\overline{\eta}(k_1, k_2, s)$ to obtain $\overline{\eta}(k_1, k_2, t)$ which is further converted to $\eta(x, y, t)$ by using double inverse Fourier Transform.

The circular frequency ω describes the dispersion relation of tsunamis and implies phase velocity $c = \frac{\omega}{k}$ and group velocity $U = \frac{d\omega}{dk}$. Hence, $c = \sqrt{\frac{g\tanh(kh)}{k}}$, and $U = \frac{1}{2}c(1 + \frac{2kh}{\sinh(2kz)})$.

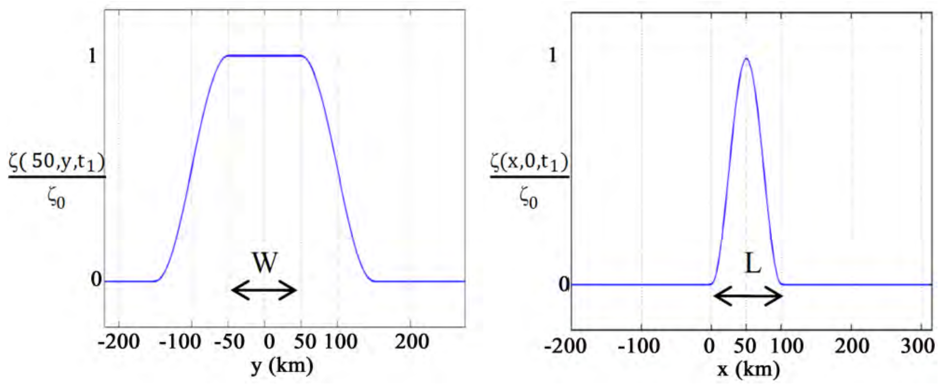
Since, $k = \frac{2\pi}{\lambda}$, hence as $kh \to 0$, both $c \to \sqrt{gh}$ and $U \to \sqrt{gh}$, which implies that the tsunami velocity $v_t = \sqrt{gh}$ for wavelengths λ long compared to the water depth h. The above linearized solution is known as the shallow water solution. We considered two models for the sea floor displacement, namely, a slowly curvilinear vertical faulting with rise time $0 \le t \le t_1$ and a variable single slip-fault, propagating unilaterally in the positive x-direction with time $t_1 \le t \le t^*$, both with finite velocity v. In the y-direction, the models propagate instantaneously. The set of physical parameters used in the problem are given in **Table 1**.

The two models are shown in **Figures 2** and **3**, respectively, and given by:

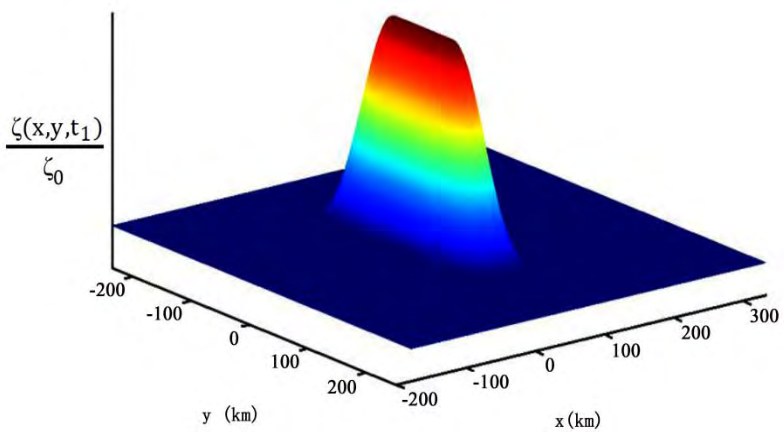
a) Slowly curvilinear uplift faulting

Table 1. Parameters used in the analytical solution of the problem.

Parameters	Value for the uplift faulting	Value for the slip-fault
Source width, W, km	100	100
Propagate length, L, km	100	150
Water depth (uniform), h, km	2	2
Acceleration due to gravity, g,km/sec^2	0.0098	0.0098
Tsunami velocity, $v_t = \sqrt{gh}$, km/sec	0.14	0.14
Rupture velocity, v,km/sec, to obtain maximum surface amplitude	0.14	0.14
Duration of the source process, t, min	$t_1 = \frac{50}{v} = 5.95$	$t^* = \frac{200}{v} = 23.8$

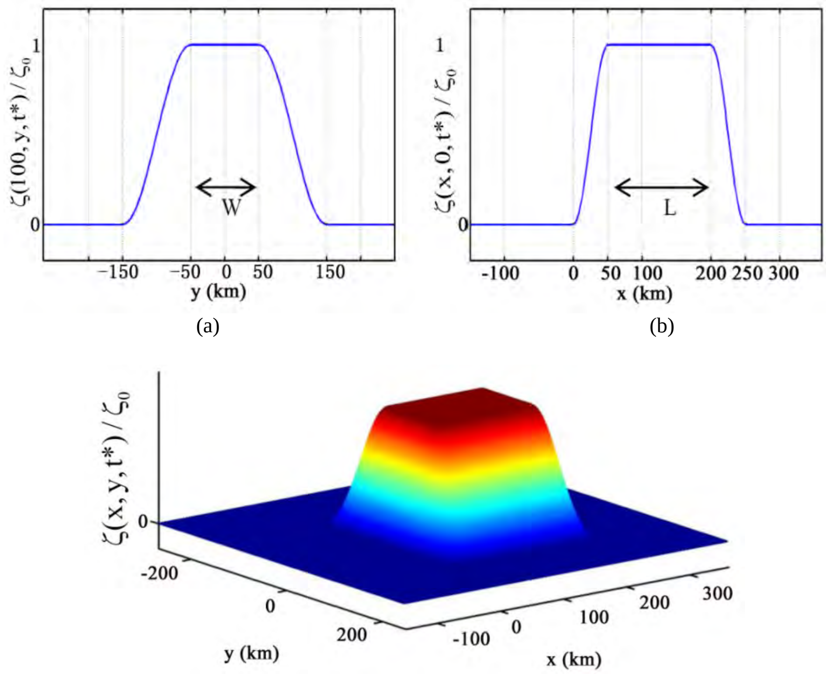


(a) (b)



(c)

Figure 2. Normalized bed deformation representing by a slowly curvilinear uplift faulting at $t_1 = \frac{50}{v}$ (a) Side view along the axis of the symmetry at x = 50; (b) Side view along the axis of the symmetry at y = 0 ; (c) Three- dimensional view.



(c)

Figure 3. Normalized Bed deformation model representing by a single-fault slip at t = t* = 200/v. (a) Side view along the axis of the symmetry at x = 100; (b) Side view along the axis of the symmetry at y = 0; (c) Three-dimensional view.

$$\zeta(x,y,t)=\begin{cases}\zeta_0\dfrac{vt}{2L}(1-\cos\dfrac{\pi}{50}x)[1-\cos\dfrac{\pi}{100}(y+150)], & 0\le x\le 100,\quad -150\le y\le -50\\ \zeta_0\dfrac{vt}{L}(1-\cos\dfrac{\pi}{50}x), & 0\le x\le 100,\quad -50\le y\le 50\\ \zeta_0\dfrac{vt}{2L}(1-\cos\dfrac{\pi}{50}x)[1+\cos\dfrac{\pi}{100}(y-150)], & 0\le x\le 100,\quad 50\le y\le 150\end{cases} \tag{18}$$

For this displacement, the bed rises during $0 \le t \le t_1$ to a maximum displacement ζ_0 in an asymptotic manner.

b) Curvilinear slip-fault

$$\zeta(x,y,t) = \zeta_1(x,y,t) + \zeta_2(x,y,t) + \zeta_3(x,y,t), \tag{19}$$

where

$$\zeta_1(x,y,t)=\begin{cases}\dfrac{\zeta_0}{4}(1-\cos\dfrac{\pi}{50}x)[1-\cos\dfrac{\pi}{100}(y+150)], & 0\le x\le 50,\quad -150\le y\le -50\\ \dfrac{\zeta_0}{2}[1-\cos\dfrac{\pi}{100}(y+150)], & 50\le x\le 50+v(t-t_1),\quad -150\le y\le -50\\ \dfrac{\zeta_0}{4}[1+\cos\dfrac{\pi}{50}(x-(50+v(t-t_1)))][1-\cos\dfrac{\pi}{100}(y+150)], & 50+v(t-t_1)\le x\le 100+v(t-t_1)\\ & -150\le y\le -50\end{cases}$$

$$\zeta_2(x,y,t)=\begin{cases}\dfrac{\zeta_0}{2}(1-\cos\dfrac{\pi}{50}x), & 0\le x\le 50 \quad -50\le y\le 50\\ \zeta_0 & 50\le x\le 50+v(t-t_1),\quad -50\le y\le 50\\ \dfrac{\zeta_0}{2}[1+\cos\dfrac{\pi}{50}(x-(50+v(t-t_1)))], & 50+v(t-t_1)\le x\le 100+v(t-t_1)\quad -50\le y\le 50\end{cases}$$

$$\zeta_3(x,y,t)=\begin{cases}\frac{\zeta_0}{4}(1-\cos\frac{\pi}{50}x)[1+\cos\frac{\pi}{100}(y-50)], & 0\le x\le 50, \quad 50\le y\le 150\\ \frac{\zeta_0}{2}[1+\cos\frac{\pi}{100}(y-50)], & 50\le x\le 50+v(t-t_1), \quad 50\le y\le 150\\ \frac{\zeta_0}{4}[1+\cos\frac{\pi}{50}(x-(50+v(t-t_1)))][1+\cos\frac{\pi}{100}(y-50)], & 50+v(t-t_1)\le x\le 100+v(t-t_1),\\ & 50\le y\le 150\end{cases}$$

For this source model, the free surface elevation takes initially the deformation of the bed shown in **Figure 2** which remain at this elevation $\boldsymbol{\zeta_0}$ for $\mathbf{t \ge t_1}$ and further propagate unilaterally in the positive x- direction with velocity v till it reaches $\mathbf{t^*}$.

Laplace and Fourier transform can now be applied to the bed motion described by (18) and (19). First, beginning with the uplift faulting (18) for $0 \le t \le t_1$ where $t_1 = \frac{50}{v}$ and

$$\mathfrak{F}(\pounds(\zeta(x,y,t)) = \bar{\zeta}(k_1,k_2,s) = \int_{-\infty}^{\infty} e^{-i(k_1x+k_2y)}dx\,dy \int_0^{\infty} \zeta(x,y,t)e^{-st}dt \tag{20}$$

The limits of the above integration are apparent from (18).

Substituting the results of the integration (20) into (17), yields

$$\begin{aligned}&\bar{\eta}(k_1,k_2,s)=\\ &\frac{s}{\cosh(kh)(s^2+\omega^2)}\zeta_0\frac{v}{L}\frac{1}{2s}\left[\frac{(1-e^{-i100k_1})}{ik_1}-\frac{e^{-i100k_1}}{1-(\frac{50}{\pi}k_1)^2}[ik_1(\frac{50}{\pi})^2(e^{i100k_1}-1)]\right]\times\\ &\left[\frac{(e^{i150k_2}-e^{i50k_2})}{ik_2}-\frac{1}{1-(\frac{100}{\pi}k_2)^2}\left[ik_2(\frac{100}{\pi})^2(e^{i50k_2}+e^{i150k_2})\right]+\frac{4\sin(50k_2)}{k_2}+\left[\frac{(e^{-i50k_2}-e^{-i150k_2})}{ik_2}+\frac{1}{1-(\frac{100}{\pi}k_2)^2}\right]\left[ik_2(\frac{100}{\pi})^2(e^{i50k_2}+e^{i150k_2})\right]\right]\end{aligned} \tag{21}$$

The free surface elevation $\bar{\eta}(k_1,k_2,t)$ can be evaluated by using inverse Laplace transform of $\bar{\eta}(k_1,k_2,s)$ as follows:

First, recall that $\pounds^{-1}(\frac{s}{s^2+\omega^2}) = \cos\omega t$ and $\pounds^{-1}(\frac{1}{s})=1$, and the inverse of a product of transforms of two functions is their convolution in time. Hence, $\int_0^t \cos\omega\tau\, d\tau = \frac{\sin\omega t}{\omega}$ and $\bar{\eta}(k_1,k_2,t)$ becomes

$$\begin{aligned}\bar{\eta}(k_1,k_2,t) = &\frac{\sin\omega t}{\omega\cosh(kh)}\frac{\zeta_0 v}{2L}\left[\frac{(1-e^{-i100k_1})}{ik_1}-\frac{e^{-i100k_1}}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(e^{-i100k_1}-1]\right]\right]\times\\ &\left[\frac{e^{i150k_2}-e^{i50k_2}}{ik_2}-\frac{1}{1-(\frac{100}{\pi}k_2)^2}\left[ik_2(\frac{100}{\pi})^2(e^{i50k_2}+e^{i150k_2})\right]+\frac{4\sin(50k_2)}{k_2}+\left[\frac{(e^{-i50k_2}-e^{-i150k_2})}{ik_2}+\frac{1}{1-(\frac{100}{\pi}k_2)^2}\right]\left[ik_2(\frac{100}{\pi})^2(e^{-i150k_2}+e^{-i50k_2})\right]\right]\end{aligned} \tag{22}$$

In case for $t \geq t_1$, $\bar{\eta}(k_1,k_2,t)$ will have the same expression except in the convolution step, the integral become $\int_{t-t_1}^{t} \cos\omega\tau \, d\tau = \frac{\sin\omega t}{\omega} - \frac{\sin\omega(t-t_1)}{\omega}$

Finally, $\eta(x,y,t)$ is evaluated using the double inverse Fourier transform of $\bar{\eta}(k_1,k_2,t)$

$$\eta(x,y,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} e^{ik_2 y}[\int_{-\infty}^{\infty} e^{ik_1 x}\bar{\eta}(k_1,k_2,t)dk_1]dk_2 \quad (23)$$

This inversion is computed by using the FFT. The inverse FFT is a fast algorithm for efficient implementation of the Inverse Discrete Fourier Transform (IDFT) given by

$$f(m,n) = \frac{1}{MN}\sum_{p=0}^{M-1}\sum_{q=0}^{N-1} F(p,q) e^{i(\frac{2\pi}{M})pm} e^{i(\frac{2\pi}{N})qn}, \quad \begin{matrix} p = 0,1,\ldots\ldots, M-1 \\ q = 0,1,\ldots\ldots, N-1 \end{matrix}$$

where f(m,n) is the resulted function of the two spatial variables m and n,corresponding x and y, from the frequency domain function F(p, q) with frequency variables p and q, corresponding k_1 and k_2. This inversion is done efficiently by using the Matlab FFT algorithm.

In order to implement the algorithm efficiently, singularities should be removed by finite limits as follows:

1) As $k \to 0$, implies $k_1 \to 0$, $k_2 \to 0$ and $\omega \to 0$ then $\bar{\eta}(k_1,k_2,t)$ has the following limit

$$\lim_{k\to 0}\bar{\eta}(k_1,k_2,t) = \begin{cases} 200\zeta_0 vt & t \leq t_1 \\ 200\zeta_0 vt_1 & t \geq t_1 \end{cases}, \text{where } t_1 = \frac{50}{v}$$

2) As $k_2 \to 0$, then the singular terms of $\bar{\eta}(k_1,k_2,t)$ have the following limits

$$\lim_{k_2\to 0} k_2\left(\frac{e^{i\,150\,k_2}}{i\,k_2} - \frac{e^{i\,50\,k_2}}{i\,k_2}\right) = 100$$

$$\lim_{k_2\to 0}\left(\frac{4\sin(50k_2)}{k_2}\right) = 200$$

$$\lim_{k_2\to 0}\left(\frac{e^{-i\,50\,k_2}}{i\,k_2} - \frac{e^{-i\,150\,k_2}}{i\,k_2}\right) = 100$$

3) As $k_1 \to 0$, then the singular term of $\bar{\eta}(k_1,k_2,t)$ has the following limit

$$\lim_{k_1\to 0}\left(\frac{1}{i\,k_2} - \frac{e^{-i\,100\,k_1}}{i\,k_1}\right) = 100$$

Using the same steps, $\bar{\eta}(k_1,k_2,t)$ is evaluated by applying the Laplace and Fourier transform to the bed motion described by (19), then substituting into (17) and then inverting the Laplace transform on $\bar{\eta}(k_1,k_2,s)$ to obtain $\bar{\eta}(k_1,k_2,t)$. This is verified for $t_1 \leq t \leq t^*$ where $t^* = \frac{200}{v}$ as follows:

$$\bar{\eta}(k_1,k_2,t) = \bar{\eta}_1(k_1,k_2,t) + \bar{\eta}_2(k_1,k_2,t) + \bar{\eta}_3(k_1,k_2,t) \quad (24)$$

where, $\bar{\eta}_1(k_1,k_2,t)$, $\bar{\eta}_2(k_1,k_2,t)$ and $\bar{\eta}_3(k_1,k_2,t)$ can be written respectively as:

$$\bar{\eta}_1(k_1,k_2,t) =$$

$$\left[\frac{\zeta_0}{4\cosh(kh)}\right]\left[\frac{e^{i\,150\,k_2} - e^{i\,50\,k_2}}{i\,k_2} - \frac{1}{1-(\frac{100}{\pi}k_2)^2}\left[ik_2(\frac{100}{\pi})^2(e^{i\,50\,k_2} + e^{i\,150\,k_2})\right]\right] \times$$

$$\left[\begin{gathered}\left[\frac{1-e^{-i\,50\,k_1}}{i\,k_1} - \frac{e^{-i\,50\,k_1}}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(1+e^{i\,50\,k_1})\right]\right]\cos\omega(t-t_1) + \\ \frac{2ve^{-i50k_1}}{\omega^2-(k_1 v)^2}(\omega\sin\omega(t-t_1) + ik_1 v\cos\omega(t-t_1) - ik_1 ve^{-ik_1 v(t-t_1)}) + \\ \left[\begin{gathered}\frac{e^{-ik_1(50+v(t-t_1))} - e^{-ik_1(100+v(t-t_1))}}{ik_1} + \\ \frac{1}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(e^{-ik_1(100+v(t-t_1))} + e^{-ik_1(50+v(t-t_1))})\right]\end{gathered}\right]\cos\omega(t-t_1)\end{gathered}\right]$$

$$\overline{\eta_2}(k_1,k_2,t)=\left[\frac{\zeta_0}{\cosh(kh)}\right]\left[\frac{\sin(50k_2)}{k_2}\right]\times$$

$$\left[\begin{array}{l}\left[\dfrac{1-e^{-i\,50\,k_1}}{i\,k_1}-\dfrac{e^{-i\,50\,k_1}}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(1+e^{i\,50\,k_1})\right]\right]\cos\omega(t-t_1)+\\ \dfrac{2ve^{-i50k_1}}{\omega^2-(k_1v)^2}(\omega\sin\omega(t-t_1)+ik_1v\cos\omega(t-t_1)-ik_1ve^{-ik_1v(t-t_1)})+\\ \left[\begin{array}{c}\dfrac{e^{-ik_1(50+v(t-t_1))}-e^{-ik_1(100+v(t-t_1))}}{ik_1}+\\ \dfrac{1}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(e^{-ik_1(100+v(t-t_1))}+e^{-ik_1(50+v(t-t_1))}\right]\end{array}\right]\cos\omega(t-t_1)\end{array}\right]$$

and

$$\overline{\eta_3}(k_1,k_2,t)=\left[\frac{\zeta_0}{4\cosh(kh)}\right]\left[\frac{e^{-i\,50\,k_2}-e^{-i\,150\,k_2}}{i\,k_2}+\frac{1}{1-(\frac{100}{\pi}k_2)^2}\left[ik_2(\frac{100}{\pi})^2(e^{-i\,150\,k_2}+e^{-i\,50\,k_2}\right]\right]\times$$

$$\left[\begin{array}{l}\left[\dfrac{1-e^{-i\,50\,k_1}}{i\,k_1}-\dfrac{e^{-i\,50\,k_1}}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(1+e^{i\,50\,k_1})\right]\right]\cos\omega(t-t_1)+\\ \dfrac{2ve^{-i50k_1}}{\omega^2-(k_1v)^2}(\omega\sin\omega(t-t_1)+ik_1v\cos\omega(t-t_1)-ik_1ve^{-ik_1v(t-t_1)})+\\ \left[\begin{array}{c}\dfrac{e^{-ik_1(50+v(t-t_1))}-e^{-ik_1(100+v(t-t_1))}}{ik_1}+\\ \dfrac{1}{1-(\frac{50}{\pi}k_1)^2}\left[ik_1(\frac{50}{\pi})^2(e^{-ik_1(100+v(t-t_1))}+e^{-ik_1(50+v(t-t_1))}\right]\end{array}\right]\cos\omega(t-t_1)\end{array}\right]$$

Substituting $\overline{\eta_1}(k_1,k_2,t),\overline{\eta_2}(k_1,k_2,t)$ and $\overline{\eta_3}(k_1,k_2,t)$ into (24) gives $\eta(k_1,k_2,t)$ for $t_1 \le t \le t^*$

In case for $t \ge t^*$, $\overline{\eta}(k_1,k_2,t)$ will have the expression (24) except the term resulting from the convolution theorem, *i.e.*

$$\int_{(t-t_1)-t^*}^{t}\cos\omega\tau e^{-ik_1v(t-t_1)-\tau}d\tau=\frac{1}{\omega^2-(k_1v)^2}\left[\begin{array}{l}\omega\sin\omega(t-t_1)+ik_1v\cos\omega(t-t_1)\\ -e^{-ik_1vt^*}\left(\omega\sin\omega((t-t_1)-t^*)+ik_1v\cos\omega((t-t_1)-t^*)\right)\end{array}\right],$$

instead of

$$\int_{(t-t_1)-t^*}^{t}\cos\omega\tau e^{-ik_1v(t-\tau)}d\tau=\frac{1}{\omega^2-(k_1v)^2}(\omega\sin\omega(t-t_1)+ik_1v\cos\omega(t-t_1)-ik_1ve^{-ik_1v(t-t_1)})$$

Finally, $\eta(x, y, t)$ is computed using inverse Fast Fourier transform of $\eta(k_1, k_2, t)$. Again, the singular points should be removed to compute $\eta(x, y, t)$ efficiently

1) As k→0, then $\bar{\eta}(k_1, k_2, t)$ has the following limit

$$\lim_{k\to 0} \bar{\eta}(k_1, k_2, t) = \begin{cases} 2\zeta_0(5000 + 100v(t - t_1) & (t - t_1) \le t^* \\ 2\zeta_0(5000 + 100vt^*) & (t - t_1) \ge t^* \end{cases}$$

where $t^* = \frac{200}{v}$.

2) As k_2→0, then the singular terms of $\bar{\eta}(k_1, k_2, t)$ have the following limits

$$\lim_{k_2\to 0}\left(\frac{e^{i\,150\,k_2}}{i\,k_2} - \frac{e^{i\,50\,k_2}}{i\,k_2}\right) = 100$$

$$\lim_{k_2\to 0}\left(\frac{\sin(50k_2)}{k_2}\right) = 50$$

$$\lim_{k_2\to 0}\left(\frac{e^{-i\,50\,k_2}}{i\,k_2} - \frac{e^{-i\,150\,k_2}}{i\,k_2}\right) = 100$$

3) As k_1→0, then the singular terms of $\bar{\eta}(k_1, k_2, t)$ have the following limits

$$\lim_{k_1\to 0}\left(\frac{1}{i\,k_1} - \frac{e^{-i\,50\,k_1}}{i\,k_1}\right) = 50$$

$$\lim_{k_1\to 0}\left(\frac{e^{-i\,k_1(50+v(t-t_1))}}{i\,k_1} - \frac{e^{-i\,k_1(100+v(t-t_1))}}{i\,k_1}\right) = 50$$

We investigated the water wave motion in the near and far-field by considering two kinematic in sequence representation of the sea floor faulting, one with vertical uplift motion with time followed by unilateral spreading in x-direction, both with constant velocity v. Clearly, from the mathematical derivation done above, $\eta(x, y, t)$ depends continuously on the source ζ (x, y, t). Hence, from the mathematical point of view, this problem is said to be well-posed for modeling the physical processes of the tsunami wave.

3. Results and Discussion

We are interest in illustrating the nature of the tsunami build up and propagation during and after the uplift process. In addition, searching for explanations for abnormally large tsunami amplitudes, we demonstrate the waveform amplification resulting from source spreading and wave focusing in the near-field. We first examine the significance of the spreading velocity of the ocean floor uplift by comparing displacement waveforms along the x-axis and in 3-dimendional frame of work for various values of the ratio $\frac{v}{v_t}$.

3.1. Effect of $\frac{v}{v_t}$ on Tsunami Waveform

In this section, we study the focusing and the amplification of the tsunami amplitude, determined by the velocity of spreading. The results in **Figure 4** show that, at the time when the source process is completed and for rapid lateral spreading ($\frac{v}{v_t}$ ≥ 10), the displacement of the free surface above the source resembles the displacement of the ocean floor. For velocities of spreading smaller than v_t ($\frac{v}{v_t}$ < 1), the tsunami amplitudes in the direction of the source propagation become small with high frequencies. As the velocity of the spreading approaches vt, the tsunami waveform has progressively larger amplitude, with high frequency content, in the direction of the slip spreading, **Figure 5**. These large amplitudes are caused by wave focusing (*i.e.* during slow earthquakes). It was observed from past tsunami, that slow earthquakes (0.1 < v < 1 km/sec) may consist of one or several high velocity rupture events, which thus produce the usual train of high frequency waves, with long delays between the successive events, accompanied by the source, which can contribute large amplitude and low-frequency excitation, see [33]. Examples of such slow earthquakes are the June 6, 1960, Chile earthquakes which ruptured as a series of earthquakes for about an hour, [34], and the February 21, 1978, Banda Sea earthquake, [35]. In our case, we concern about studying the amplification of the tsunami amplitude, determined by the velocity of spreading of a single-fault source and it will explain in details later on.

It is difficult to estimate, at present, how often this type of amplification may occur during actual slow submarine process, because of the lack of detailed knowledge about the ground deformations in the source area of past tsunamis. Therefore, we presented here only the basic ideas and illustrate the possible range of amplification factors by means of a realistic curvilinear slip-fault.

The amplification shown in Figure 4 and 5 depends on the spreading velocity v and the time t taken to spread the motion over the entire source region. This observation can be verified by comparing these results with the tsunami waveforms obtained by using a simple kinematic source model represented by a sliding Heaviside step function. This case was studied by Todorovska and Trifunac [21]. They considered a square source model characterized by W = L = 50 km, with uniform final elevation ζ_0, and the velocity of lateral spreading of the ocean floor uplift was constant. We expand the propagation length L to 150 km and W to 100 km for the Heaviside step function in order to make comparison with the results we obtained. We assumed the beginning of the spreading from x = 0 to x = 150 km for both cases.

$$\zeta(x, y, t) = \zeta_0 H\left(t - \frac{x}{v}\right),\ (x, y) \in \left[(0, L) \times \left(\frac{-w}{2}, \frac{w}{2}\right)\right] \tag{25}$$

Applying the Laplace-Fourier transform on ζ (x, y, t), then substituting it in (17) and using the inverse Lap-

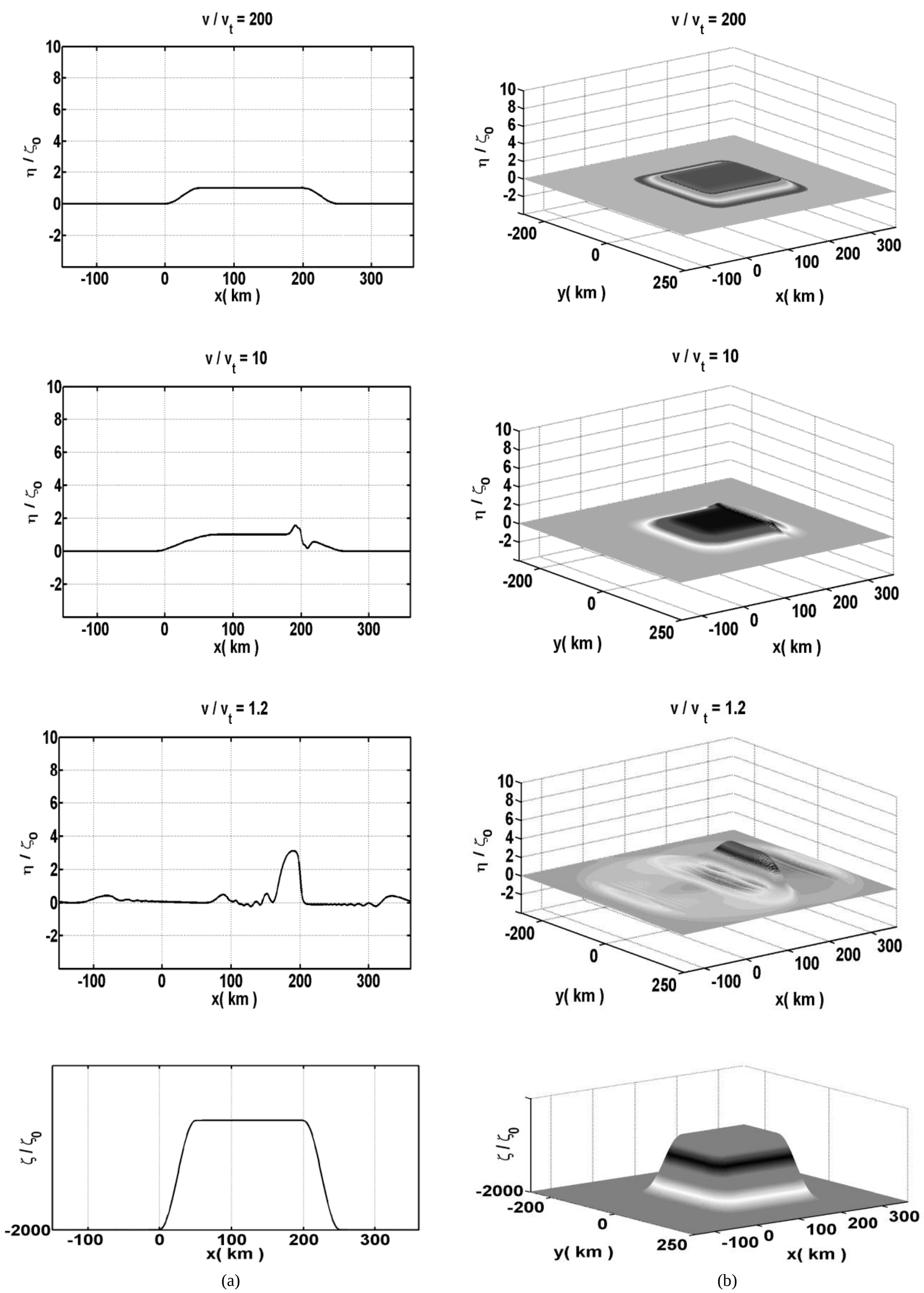
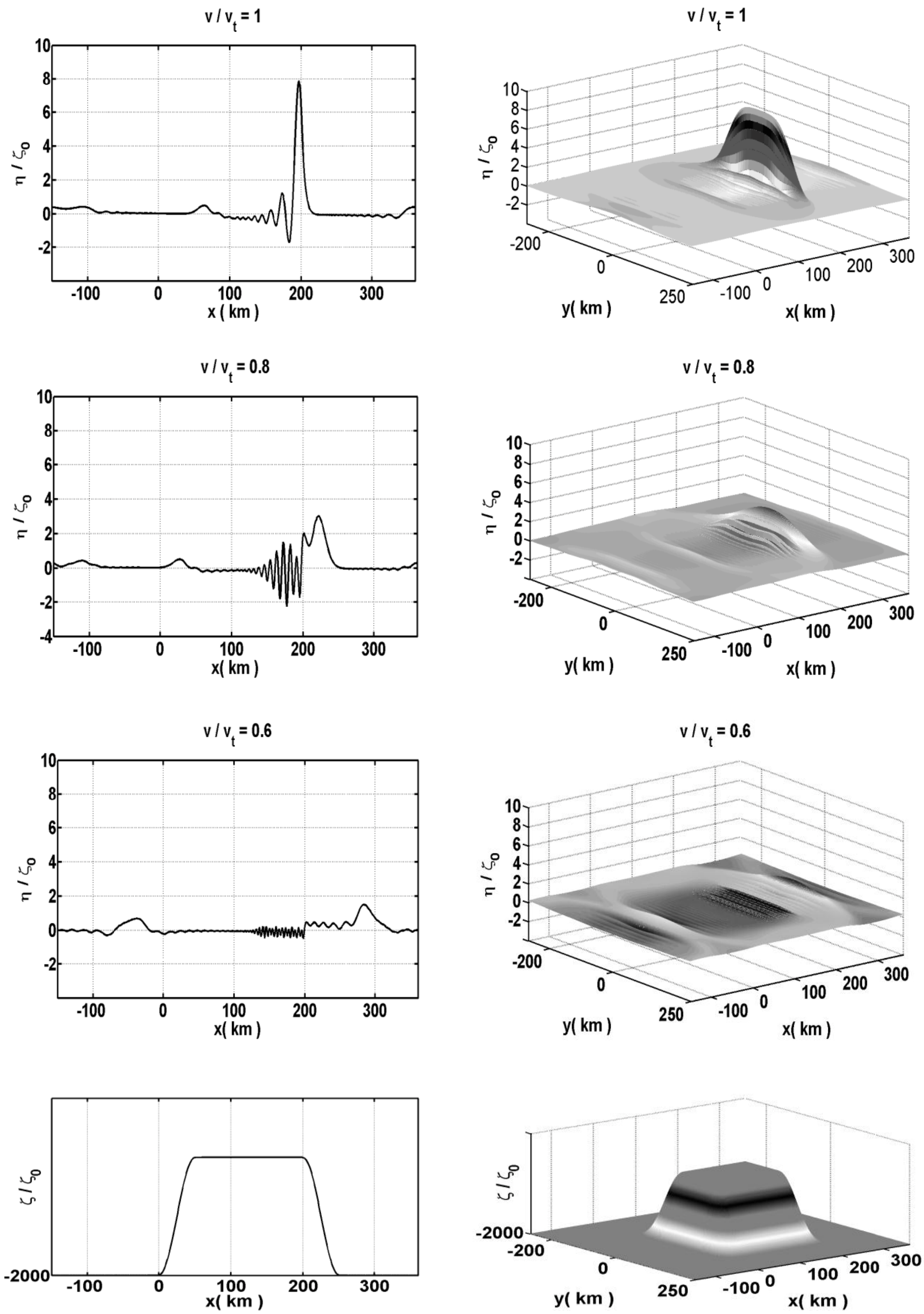


Figure 4. Dimensionless free-surface deformation $\eta(x, y, t^*)/\zeta_0$ for $v/v_t > 1$ at h = 2 km, L = 150 km, W = 100 km, v_t =0.14 km/sec and t^* = 200/v sec. (a) Side view along the axis of the symmetry at y = 0; (b) Three dimensional view.



(a) (b)

Figure 5. Dimensionless free-surface deformation $\eta(x, y, t^*)/\zeta_0$ for $v/v_t \leq 1$ at h = 2 km, L = 150 km, W = 100 km, v_t = 0.14 km/sec and t^* = 200/v sec. (a) Side view along the axis of the symmetry at y = 0; (b) Three dimensional view.

lace transform of $\bar{\eta}(k_1, k_2, s)$ yields for $t \le t^*$.

$$\bar{\eta}(k_1, k_2, t) = \left[\frac{2\sin(50k_2)}{k_2}\right]\left[\frac{\zeta_0 v}{\cosh(kh)}\left(\frac{1}{\omega^2 - (k_1 v)^2}(\omega \sin \omega t + ik_1 v \cos \omega t - ik_1 v e^{-ik_1 vt}\right)\right] \quad (26)$$

Hence, $\eta(x, y, t)$ can be computed using inverse FFT of $\bar{\eta}(k_1, k_2, t)$.

The removable singularities in this case are given by the following limits as

1) As k→0, then $\bar{\eta}(k_1, k_2, t)$ has the following limit

$$\lim_{k\to 0} \bar{\eta}(k_1, k_2, t) = \begin{cases} \zeta_0 100vt & t \le t^* \\ \zeta_0 100vt^* & t \ge t^*, \end{cases}$$

where $t^* = \dfrac{150}{v}$

2) As k_2→0, then the singular term of $\bar{\eta}(k_1, k_2, t)$ has the following limit

$$\lim_{k_2\to 0}\left(\frac{2\sin(50k_2)}{k_2}\right) = 100$$

No singular terms needed to remove as k_1→0.

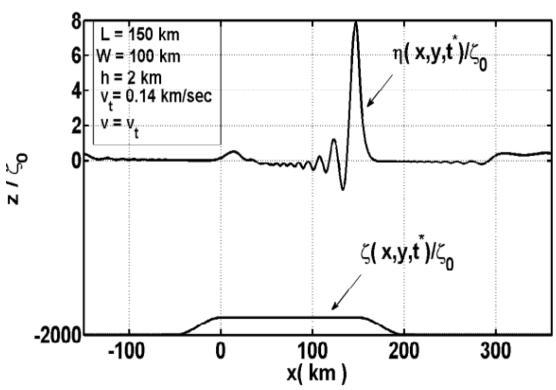
Table 2 shows the comparison for the peak tsunami amplitude η_{max}/ζ_0 at certain values of the spreading velocity, for the curvilinear slip-fault source and the simple rectangular-slide with L = 150 km and W = 100 km.

Figure 6 illustrates the tsunami waveforms generated by the slip-fault model and the rectangular-slide model with L = 150 km and W = 100 km when the maximum amplitude amplification occurs at $v = v_t$.

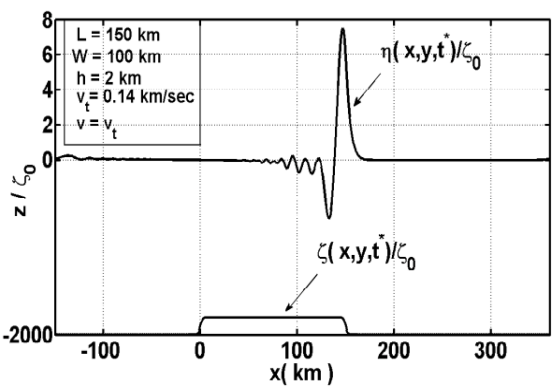
It can be observed from **Table 2** that, at certain value of v, the waveform generated by the slip-fault model and the rectangular-slide has little different peak amplitudes. This happens as a result of wave focusing covers a wider area (curvilinear region) above the source which needed more time for amplification. For example, at $v = v_t$, where the time for the slip-fault takes 23.8 min to complete the entire source region with peak amplitude equal to 7.906 m, while for the rectangular-slide the time duration takes 17.8 min to reach a maximum amplitude equal to 7.524 m.

Table 2. Peak tsunami amplification for the curvilinear slip-fault model and the rectangular-slide at different values of the spreading velocity v.

spreading velocity v (km/sec)	η_{max}/ζ_0 **(slip-fault)**	η_{max}/ζ_0 **(rectangular-slide)**
$v = 200v_t = 28$	1	1
$v = 20v_t = 2.8$	1.481	1.003
$v = 10v_t = 1.4$	1.575	1.010
$v = 5v_t = 0.7$	1.615	1.042
$v = 2v_t = 0.28$	1.880	1.333
$v = 1.2v_t = 0.168$	3.145	3.204
$v = v_t = 0.14$	7.906	7.524
$v = 0.8v_t = 0.112$	3.014	1.958
$v = 0.6v_t = 0.084$	1.498	0.569



(a)



(b)

Figure 6. Maximum tsunami amplitude at spreading velocity $v = v_t$, generated by: (a) Slip-fault source; (b) Rectangular-slide source.

For $v > v_t$, the maximum amplitudes calculated are small comparable to the peak amplitude at $v = v_t$ as shown in **Figure 4** and **Table 2.** These values are logic due to rapid movements of the bed ($v = 10v_t$ and $v = 200v_t$), which progressively approximated the shape of the bed deformation. This phenomenon agrees with the physical process of tsunami wave produced by earthquakes. For $v = 0.6v_t$ and $v = 0.8v_t$ where the tsunami is faster than the uplift, the initial wave escapes ahead of the currently uplifted water and the amplitudes of the tsunami waves above the source become smaller with higher frequency content continues and hence no amplification will occur as shown in **Figure 5(a)**. At the instant the source motion has been stopped, it may cover larger area than the area of the source due to dispersion as demonstrated in Figure **5(b)**.

3.2. Tsunami Generation and Propagation-Evolution in Time

We study the effects of variations of the uplift as a function of time in the slowly uplift faulting and the spreading slip fault sources in generating and propagating of tsunamis above and away from the sources. The generation of tsunamis by vertical displacements of the ocean floor depends on the characteristic size (length L and width W) of the displaced area and on the time t, taken to spread the motion over the entire source region. The ratio L/t then defines the average spreading (or fault rupture) velocity v, assuming unilateral spreading of the slip-fault along length L. We choose the velocity of the sea floor spreads similar to the long wave tsunami velocity v_t (*i.e.* maximum amplification).We illustrate the final uplifted area for the slowly uplift faulting by L = 100 km and W = 100 km, and for the slip-fault by length of propagation L = 150 km and W = 100 km with constant spreading velocity v. **Figures 7** and **9(a)** show the tsunami generated waveforms at times t = 0.4t*, 0.6t*, 0.8t*, t*. It is seen how the amplitude of the wave builds up progressively as t increases. **Figures 8** and **9(b)** illustrate the propagation process of the tsunami wave away from the source for times between t = 2t* and t = 3.5t*. **Table 3** represents the determined values η_{max}/ζ_0 by the slip-fault source during the times t = 0.2t*, 0.4t*, 0.6t*, 0.8t*, t*. At $v = v_t$, the wave will be focusing and the amplification may occurs above the spreading edge of the slip as shown in **Figure 9(a)**. This amplification occurs above the source progressively, as the source evolves, by adding uplifted fluid to the fluid displaced previously by uplifts of preceding source segments. This explains why the amplification is larger for wider area of uplift source, than for small source area.

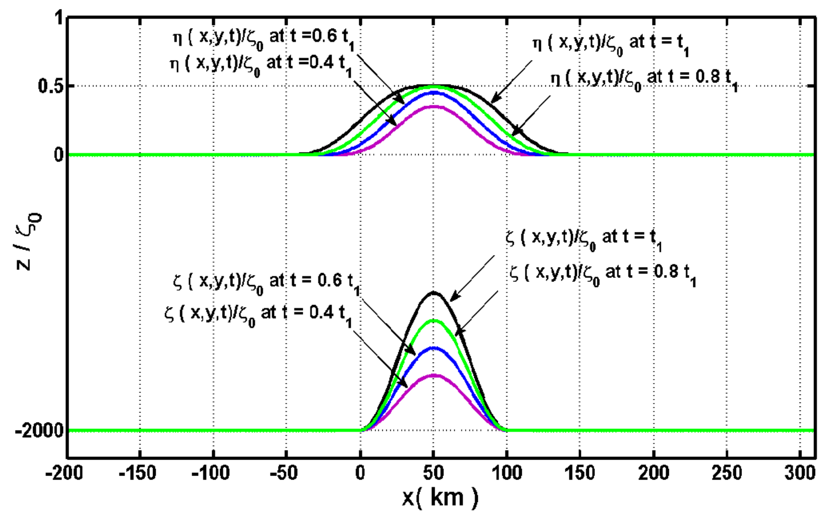
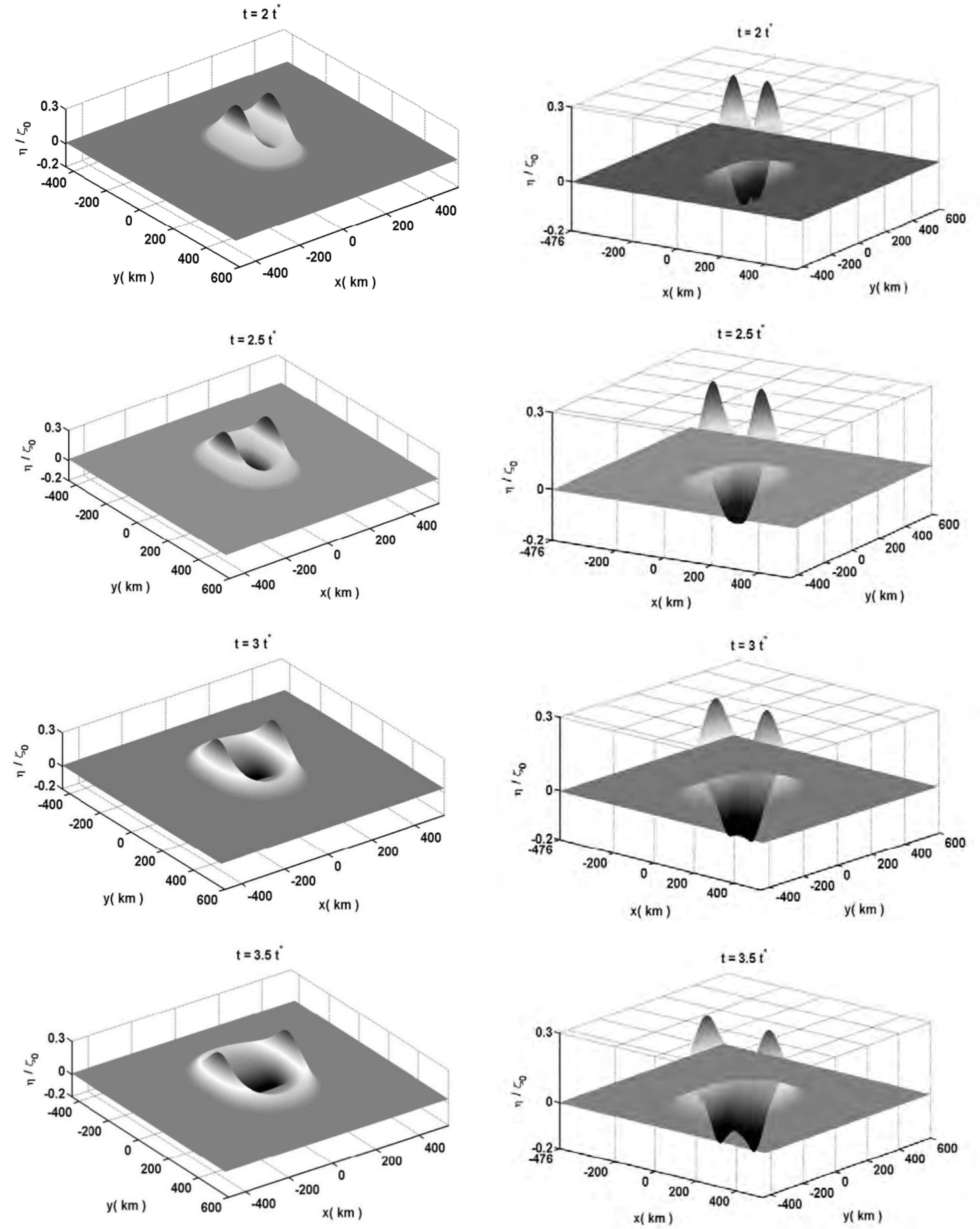


Figure 7. Tsunami generated waveforms by a curvilinear uplift source with characteristic size L = 100 km and W = 100 km at different uplift times t at $v = v_t$ and $t^* = t_1 = \frac{50}{v}$ sec.

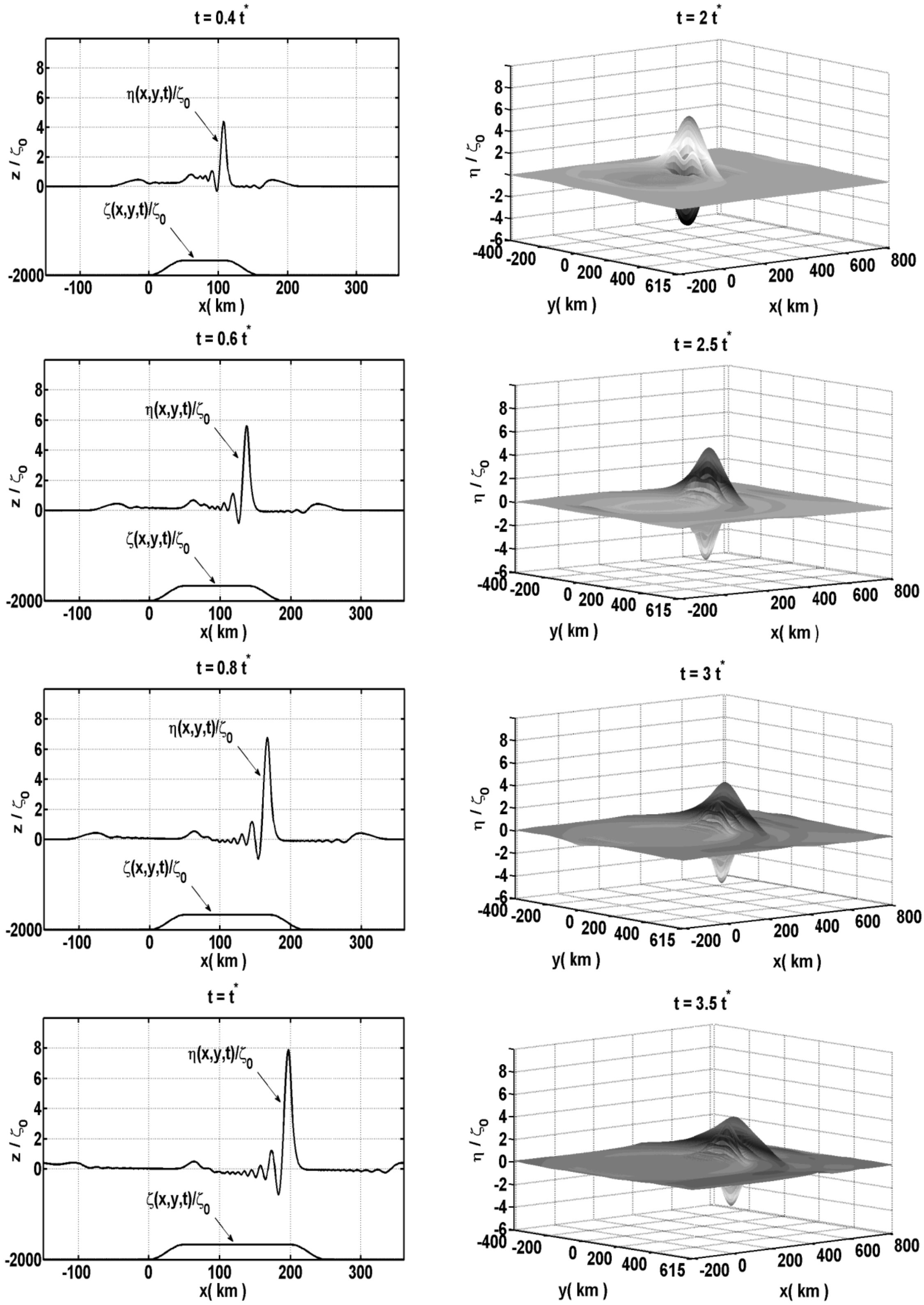
Table 3.Values of η_{max} (x, y, t) /ζ_0 at different rise time t and at $v = v_t$.

Rise time t	η_{max}/ζ_0
t = 0.2 t*	3.065
t = 0.4 t*	4.385
t = 0.6 t*	5.637
t = 0.8 t*	6.800
t = t*	7.906



(a) (b)

Figure 8. Tsunami propagated waveforms following the curvilinear uplift source with characteristic size L = 100 km and W = 100 km at different rise times t and $t^* = \frac{50}{v}$ sec. (a) Top view; (b) Bottom view.



(a) (b)

Figure 9. Tsunami waveform by the slip-fault source with $v = v_t$, L = 150 km and W = 100 km at different rise time t where $t^* = \frac{200}{v}$. (a) Side view of the tsunami generated wave at y = 0; (b) Three dimensional view of the tsunami propagation.

For very rapid movements of the bed, the water surface displacement initially approximated the shape of the bed deformation and then divided into two wave trains propagating in opposite directions. The maximum amplitude of the largest wave leaving the generation region for these bed motions was found never to exceed one half of the maximum bed displacement. For slower motions of the bed like the case we study, the maximum wave amplitudes decreased [31]. Similar features relating to the maximum amplitudes of waves propagating from the generation region in a three-dimensional fluid domain were discussed by Nakamura [10], Momoi [11] and Kajiura [12]. The results shown in **Figure 7** demonstrated the water surface displacement generated by slow motions of the bed. The waveforms approximated the shape of the bed deformation with smaller amplitude than the sea bed which then divided into two wave trains propagating in opposite directions as shown in **Figure 8**. The maximum amplitude of the largest wave leaving the generation region for this bed motions was one half of the maximum initial wave displacement. This agrees with the typical phenomena of past tsunami wave generated by slowly earthquakes.

In case of the curvilinear slip-fault, we assume the waveform was initiated by end stage of the uplift source, and then propagated in the x-direction with constant velocity v. It is seen from **Figure 9(a)**, how the amplitude of the wave builds up as progressively more water is lifted below the leading wave depending on its variation in time and the space in the source area. The parameter that governs the amplification of the near-field water waves by focusing is the ratio of the spreading length L over the water depth, L/h, as shown in **Figure 9(a)** and **Table 4**. As the spreading length in the slip-fault increases, the amplitude of the tsunami wave becomes higher. As the wave propagates, the wave height decreases and the slope of the front of the wave becomes smaller, causing a train of small waves forms behind the main wave, as shown in Figure 9b. The maximum wave amplitude decreases with time, due to the geometric spreading and also due to the dispersion. At t = 3.5t*, the wave front is at about x = 567 km and η/ζ_0 decreases to 3.313. This happens because the amplification of the waveforms in the far-field does not depend on the source velocity, but only on the volume of the displaced water by the source process which become an important factor in modeling the generation of tsunami. This was clear from the singular points removed for the two source models, where the finite limits of the free surface depends on the characteristic volume of the source models.

Table 3 shows the variation in the amplification factor η_{max}/ζ_0 for various values of the rise time t at h = 2 km and t = t* = L/v. It can observe that, the peak amplitude increases as the rise time increases. This is due to the amplification mainly depends on the length of propagation L.

Table 4 shows the variation in the amplification factor η_{max}/ζ_0 for various values of the propagated uplift length L and width W at h = 2 km and $t = t_1 + L/v$. Note that at L = 0, no propagation occurs and the waveform takes initially the shape and amplitude of the curvilinear uplift fault (*i.e.* $\eta_{max}/\zeta_0 = 1$). It is seen from **Table 4** that for L/h between 0 and 500, η_{max}/ζ_0 varies from 1 to 27.82. It can be observed that the amplification increases with the increase in L/h (0-500) and with the increase in W/L (0.25-5). The increase in the propagation length produces larger amplification than the increase in the width. This happens from the assumption that the source model propagated instantaneously in the y-direction. This leaves us to study the generation of tsunami by spreading curvilinear slides and slumps in two directions in Future work.

Figure 10 represents the values of the normalized peak wave amplitude η_{max}/ζ_0 at $v = v_t$ and h = 2 km for different values for W/L obtained from **Table 4**.

Table 5 presents the effect of the water depth h on the amplification factor η_{max}/ζ_0 for various values of the ratios W/L and at constant propagation L = 150 km. It is seen that for h between 0.5 and 6 km, η_{max}/ζ_0 is varies from 19.77 to 3.565. The values determined in Table 5 shows that the maximum amplitude amplification increases with the decrease in h. This phenomenon happens because the speed of the tsunami is related to the water depth which produces small wavelength as the velocity decreases and hence the height of the wave grows as the change of total energy of the tsunami remains constant. Mathematically, wave energy is proportional to both the length of the wave and the height squared. Therefore, if the energy remains constant and

Table 4. Values of η_{max}/ζ_0 at $v = v_t$ and h = 2 km with various values of L and W and $t = t_1 + L/v$.

L/h	**W/L**				
(h = 2 km)	**0.25**	**0.5**	**1.0**	**2**	**5**
0	1.0000	1.0000	1.0000	1.0000	1.0000
5	1.9190	1.9230	1.9290	1.9320	1.9330
10	2.5410	2.5550	2.5720	2.5840	2.5840
25	3.7820	3.8250	3.8780	3.9260	3.9270
50	5.9160	5.9920	6.0330	6.1210	6.1220
100	9.3310	9.5360	9.5390	9.5610	9.5940
250	16.940	17.500	17.530	17.540	17.540
500	27.360	27.780	27.790	27.820	27.820

Table 5. Values of η_{max}/ζ_0 at v = gh, L = 150 km and t* = 200/v for various values of the W/L.

h (km)	W/L				
	0.25	0.5	1.0	2	5
0.5	19.690	19.730	19.730	19.770	19.770
1	12.410	12.510	12.510	12.560	12.560
2	7.7610	7.8990	7.9100	7.9730	7.9740
3	5.8600	6.0190	6.0430	6.1160	6.1180
4	4.7800	4.9490	4.9880	5.0680	5.0710
5	4.0670	4.2400	4.2920	4.3790	4.3820
6	3.5650	3.7290	3.7920	3.8840	3.8880

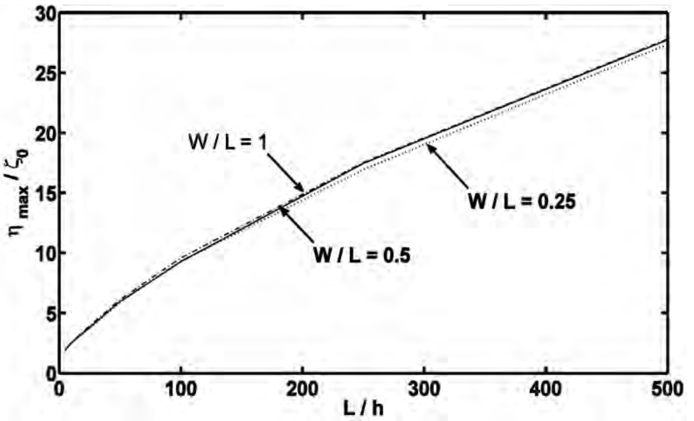


Figure 10. The normalized peak wave amplitude η_{max}/ζ_0 versus the dimensionless parameter L/h for v = v_t, W/L ≥ 0.25 and h = 2 km.

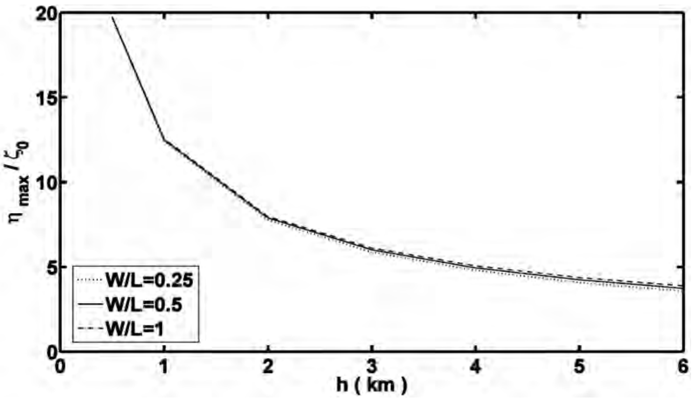


Figure 11. The normalized peak wave amplitude η_{max}/ζ_0 versus the water depth h for v = v_t, W/L ≥ 0.25 and propagation length L = 150 km.

the wavelength decreases, then the height must increase. These results agree with the aspect obtained by Hayir [36] who determined the effects of ocean depth on tsunami amplitudes for simple kinematic source models.

Figure 11 represents the values of the normalized peak wave amplitude η_{max}/ζ_0 for different values of water depth h and the ratios W/L at constant propagation L = 150 km.

4. Conclusions

In this paper, we presented a review of the main physical characteristics of a realistic tsunami sources. We considered two curvilinear source models represented by a slowly uplift faulting followed by a slip-fault model. We studied the effect of the source propagation and wave focusing on the amplitudes of the tsunami generated and propagated by the dynamic source models considered.The results showed that the amplitude amplification of up to 8 order of magnitude occurs in the direction of source propagation when the velocity of the source is close to the long period tsunami velocity. This means that amplification occurs above the source, progressively, as the source evolves, by adding uplifted fluid to the fluid displaced previously by uplifts of preceding source segments. This amplification depends on the characteristic size of the displaced area and the time it takes to spread the motion over the entire source region. It is observed that near the source, the wave has large amplitude with short wavelength pulse, while as the tsunami further departed away from the source the amplitude of this pulse decreased due to dispersion. This happens because in the far-field the peak tsunami amplitude does not depend on the source velocity, but only on the volume of the displaced water by the source process. The results show that, the largest peak of the tsunami amplitude at time t* occurs when $v = v_t$ due to wave focusing. It is seen that for source model we considered that spread rapidly ($\frac{v}{v_t} \geq 10$), the displacement of the free surface resembles the displacement of the ocean floor at time t* (*i.e.*, $\frac{\eta}{\zeta_0} \approx 1$). For ($\frac{v}{v_t} < 1$), the peak amplitude decreases due to dispersion with the present of high frequency contents in the wave. These results are in complete agreement with the aspect of the tsunami generated by a slowly spreading uplift of the ocean bottom presented by Todorovsk & Trifunac [21] who considered a very simple kinematic source model. From this observation, we numerically analyzed the dependence of the peak amplification of the tsunami waveforms by changing the length of propagation, the width of the source and the water depth. It was found that the maximum amplitude amplification is proportion to the propagation length and the width of the source model and inversely proportional with the water depth. The presented analysis suggests that some abnormally large tsunamis could be explained in part by a slowly spreading uplift of the sea floor. Our results should help to enable quantitative tsunami forecasts and warnings based on recoverable seismic data and to increase the possibilities for the use of tsunami data to study earthquakes, particularly historical events for which adequate seismic data do not exist. The estimated near-field tsunami generated under the effect of variable velocity of curvilinear slides and slumps spreading in two directions are underway.

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APPENDIX

The equations for conservation of mass and momentum for an inviscid, incompressible fluid are

$$\text{Mass:} \qquad \nabla.\mathbf{u} = 0 \qquad \text{(A.1)}$$

$$\text{Momentum:} \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\nabla\left(\frac{\mathbf{P}}{\rho} + gz\right) \qquad \text{(A.2)}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity vector (u, v, w) of the fluid, $\mathbf{P}(\mathbf{x}, t)$ is the pressure vector, ρ the density, g the gravitational acceleration, and $\mathbf{x} = (x, y, t)$ with the z axis pointing vertically upward.

By assuming that the flow is irrotational, hence the velocity field $\mathbf{u}$ can be written as the gradiant of the potential function $\mathbf{u} = \nabla\boldsymbol{\phi}$, where $\boldsymbol{\phi}$ is the velocity potential. Then the continuity equation becomes the Laplace's equation

$$\nabla^2\phi = 0 . \qquad \text{(A.3)}$$

Mathematically, a general solution does not exist for gravity waves and approximations must be made for even simple waves. One of the important problems in water wave theory is to establish the limits of validity of the various solutions that are due to the simplifying assumptions. The mathematical treatments of water wave motion use all the mathematical physics dealing with linear and nonlinear problems. The main difficulty in the study of water motion is that the free surface boundary is unknown. The coordinate axis that will be used to describe wave motion will be located at the vertical displacement $z = \eta(x, y, t)$. The bottom of the water body will be at $z = -h + \zeta(x, t)$.

If the velocity potential is known, then the pressure field can be found from (A.2). By using the vector identity

$$\mathbf{u}.\nabla\mathbf{u} = \nabla\frac{u^2}{2} - \mathbf{u}\times(\nabla\times\mathbf{u}) . \qquad \text{(A.4)}$$

From irrotationality (*i.e.* $\nabla\times\mathbf{u} = 0$), (A.2) may be rewritten as

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right] = -\nabla(\frac{\mathbf{P}}{\rho} + gz) . \qquad \text{(A.5)}$$

Upon integration with respect to the space variables, we obtain

$$-\frac{P}{\rho} = gz + \frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + C(t) \quad , \qquad \text{(A.6)}$$

where C(t) is an arbitrary function of t and can usually be omitted by redefining ϕ without affecting the velocity field. Equation (A.6) is called the Bernoulli equation. The first term gz, on the right-hand side of (A.6) is the hydrostatic contribution, whereas the hydrodynamic contribution to the total pressure P.

Two types of boundaries interest us: the air-water interface which will also be called the free surface and the wetted surface of an impenetrable solid (bottom surface). Along these two boundaries the fluid is assumed to move only tangentially. Let the instantaneous equation of the boundary be

$$F(x, t) = z - \eta(x, y, t) \quad , \qquad \text{(A.7)}$$

where η is the height measured from z = 0, and let the velocity of a geometrical point x of the moving free surface be q.

After short time dt, the free surface is described by

$$F\left(x + qdt, t + dt\right) = 0 = F\left(x, t\right) + \left(\frac{\partial F}{\partial t} + q\nabla F\right)dt + O(dt)^2 \qquad \text{(A.8)}$$

In view of (A.7), it follows that

$$\frac{\partial F}{\partial t} + q.\nabla F = 0 . \qquad \text{(A.9)}$$

For small but arbitrary dt. The assumption of tangential motion requires $u.\nabla F = q.\nabla F$.

This in turn implies that

$$\frac{\partial F}{\partial t} + u.\nabla F = 0 \quad \text{on} \quad z = \eta(x, y, t) \quad , \qquad \text{(A.10)}$$

or equivalently

$$\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y} = \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = \eta(x, y, t) \qquad \text{(A.11)}$$

On the sea bottom $\zeta(x, y, t)$ at depth h, (A.7) becomes $z + h - \zeta(x, y, t) = 0$ and (A.9) may be written by the same way as:

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y} = \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = -h + \zeta(x, y, t) . \qquad \text{(A.12)}$$

Equations (A.11) and (A.12) are referred to the *kinematic boundary conditions*.

On the air-water interface, both η and ϕ are unknown and it is necessary to add a dynamic boundary condition concerning forces.

The wavelength is so long that surface tension is unimportant, and hence the pressure at the beneath the free surface must equal to the atmospheric pressure P_a above. It can be taken $P = 0$ using the simple transformation $P \to P - P_a$ which does not change the basic Euler equations which depend upon ∇P. Hence Bernoulli Equation (A.6) at the free surface gives the boundary condition

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0 \quad \text{on} \quad z = \eta(x, y, t) . \qquad \text{(A.13)}$$

This is known as the *dynamic boundary condition* at the free surface.

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Numerical Approximation of Real Finite Nonnegative Function by the Modulus of Discrete Fourier Transform

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Abstract

The numerical algorithms for finding the lines of branching and branching-off solutions of nonlinear problem on mean-square approximation of a real finite nonnegative function with respect to two variables by the modulus of double discrete Fourier transform dependent on two parameters, are constructed and justified.

Keywords: Mean-Square Approximation, Discrete Fourier Transform, Two-Dimensional Nonlinear Integral Equation, Nonuniqueness And Branching of Solutions.

1. Introduction

The mean-square approximation of real finite nonnegative function with respect to two variables by the modulus of double discrete Fourier transform dependent on physical parameters, is widely used, in particular, at modeling and solution of the synthesis problems of different types of antenna arrays, signal processing etc. [1-3]. Nonuniqueness and branching of solutions are essential features of nonlinear approximation problem which remains unexplored. The problem on finding the set of branching points, in turn, is not adequately explored nonlinear spectral two-parametric problem. The methods of investigation and numerical finding the solutions of one-parametric spectral problems at presence of discrete spectrum [4-8] are most well-developed. The existence of coherent components of spectrum, which are spectral lines for the case of real parameters [9], is essential difference of nonlinear two-parametric spectral problems.

In the work a variational problem on the best mean-square approximation of a real finite nonnegative function by the module of double discrete Fourier transform is reduced to finding the solutions of Hammerstein type nonlinear two-dimensional integral equation. Using the Schauder principle the existence of solutions is proved. The existence theorem of coherent components of spectrum of holomorphic matrix functions dependent on two spectral parameters is proved. It justifies the application of implicit functions methods to multiparametric spectral problems [9]. The applicability of this theorem to the analysis of spectrum of two-dimensional integral homogeneous equation to which is reduced the problem on finding the lines of possible branching of solutions of the Hammerstein equation, is shown. Algorithms for numerical finding the optimum solutions of an approximation problem are constructed and justified. Numerical examples are presented.

2. Problem Formulation, Basic Equations and Relations

Consider the special case of double discrete Fourier transform

$$f(s_1,s_2)=\sum_{n=-N_1}^{N_2}\sum_{m=-M_1(n)}^{M_2(n)} I_{nm}\exp\left[i\left(\tilde{c}_1 x_{nm}s_1+\tilde{c}_2 y_{nm}s_2\right)\right]$$

setting here $x_{nm}=n\Delta_x$ $(n=-N\div N)$, $y_{nm}=m\Delta_y$ $(m=-M\div M)$; $c_1=\tilde{c}_1\Delta_x$, $c_2=\tilde{c}_2\Delta_y$. If it is necessary for the accepted assumptions we shall consider the formula

$$f(s_1,s_2)=A\mathbf{I}\equiv\sum_{n=-N}^{N}\sum_{m=-M}^{M} I_{nm}\exp\left[i\left(c_1 n s_1+c_2 m s_2\right)\right] \quad (1)$$

as a linear operator, acting from complex finite-dimensional space $H_I=\mathbb{C}^{N_2\times M_2}$ ($N_2=2N+1$, $M_2=2M+1$) into the space of complex-valued continuous functions with respect to two real variables determined in the domain

$$\Omega=\left\{(s_1,s_2):|s_1|\le\pi/c_1,\ |s_2|\le\pi/c_2\right\}.$$

Here c_1, c_2 are any real non-dimensional numerical parameters belonging to

$$\Lambda_c = \left\{(c_1, c_2): 0 < c_1 \le a,\ 0 < c_2 \le b\right\} \cdot$$

The function $f(s_1, s_2)$ is $2\pi/c_1$ - periodic function on argument s_1 and $2\pi/c_2$ - periodic on s_2.

In considered spaces we introduce scalar products and generable by them norms

$$(\mathbf{I}_1, \mathbf{I}_2)_{H_I} = \frac{4\pi^2}{c_1 c_2} \sum_{n=-N}^{N} \sum_{m=-M}^{M} I_{nm} \overline{I_{nm}},$$

$$\|\mathbf{I}\| = (\mathbf{I}, \mathbf{I})_{H_I}^{1/2}, \tag{2}$$

$$(f_1, f_2)_{C^{(2)}_{(\Omega)}} = \iint_{\Omega} f_1(s_1, s_2) \overline{f_2(s_1, s_2)} ds_1 ds_2,$$

$$\|f\| = (f, f)_{C^{(2)}_{(\Omega)}}^{1/2} \cdot \tag{3}$$

Denote an augmented space of continuous functions with entered scalar product and norm (3) as $\mathbf{C}^{(2)}_{(\Omega)}$ and notice that its augmentation coincides with the Hilbert space $L_2(\Omega)$ [10].

By direct check we are sure that such equality

$$\|A\mathbf{I}\|^2 = \iint_{\Omega} |f(s_1, s_2)|^2 ds_1 ds_2 = \sum_{n,m} |I_{nm}|^2 = \|\mathbf{I}\|^2 \tag{4}$$

is valid. From here follows, that A is isometric operator in sense (4).

Using the entered scalar products (2) and (3) we find the conjugate operator required later on

$$A^* f = \frac{c_1 c_2}{4\pi^2} \iint_{\Omega} f(s_1, s_2) \exp\left[-i(c_1 n s_1 + c_2 m s_2)\right] ds_1 ds_2$$

$$(n = -N \div N,\ m = -M \div M). \tag{5}$$

Let such function be given

$$\tilde{F}(s_1, s_2) = \begin{cases} F(s_1, s_2), & (s_1, s_2) \in \bar{G} \subseteq \Omega, \\ 0, & (s_1, s_2) \in \Omega \setminus \bar{G}, \end{cases} \tag{6}$$

where $F(s_1, s_2)$ is a real continuous and nonnegative in the domain $\bar{G}$ function.

Consider a problem on the best mean-square approximation of the function $F(s_1, s_2)$ in the domain Ω by the module of double discrete Fourier transform (1) owing to select of coefficients of the vector $\mathbf{I}$. We shall formulate it as a minimization problem of the functional

$$\sigma(\mathbf{I}) = \|F - |A\mathbf{I}|\|^2_{\mathbf{C}^{(2)}_{(\Omega)}} \equiv \|F - |f|\|^2_{\mathbf{C}^{(2)}_{(\Omega)}} \tag{7}$$

in the Hilbertian space H_I. Taking into account (4) and (5), we write the functional $\sigma(\mathbf{I})$ in a simplified form

$$\sigma(\mathbf{I}) = \|F\|^2_{\mathbf{C}^{(2)}_{(\Omega)}} - 2(F, |A\mathbf{I}|)^2_{\mathbf{C}^{(2)}_{(\Omega)}} + \|\mathbf{I}\|^2_{H_I} \cdot \tag{8}$$

On the basis of necessary condition of functional minimum we obtain a nonlinear system of equations relating to the components of vector $\mathbf{I}$ in the space H_I that are represented in the vector and expanded forms, respectively:

$$\mathbf{I} = A^* \left\{F \exp\left[i \arg(A\mathbf{I})\right]\right\}, \tag{9}$$

$$I_{nm} = \frac{c_1 c_2}{4\pi^2} \iint_{\Omega} F(s_1, s_2) \exp\left\{i \left[\arg\left(\sum_{k=-N}^{N} \sum_{l=-M}^{M} I_{nm} \times \right.\right.\right.$$

$$\left.\left.\left. \times \exp\left[i(c_1 k s_1 + c_2 l s_2)\right]\right) - (c_1 n s_1 + c_2 m s_2)\right]\right\} ds_1 ds_2$$

$$(n = -N \div N,\ m = -M \div M). \tag{9'}$$

Acting on both parts of (9) by operator A we obtain equivalent to (9) the Hammerstein type nonlinear integral equation relating to f:

$$f(Q) = \mathbf{B}f \equiv \iint_{\Omega} K(Q, Q', \mathbf{c}) F(Q') \exp[i \arg f(Q')] dQ', \tag{10}$$

where $Q' = (s_1', s_2')$, $dQ' = ds_1' ds_2'$, $\mathbf{c} = (c_1, c_2)$;

$$K(Q, Q', \mathbf{c}) = K_1(s_1, s_1', c_1) \cdot K_2(s_2, s_2', c_2), \tag{11}$$

$$K_1(s_1, s_1', c_1) = \frac{c_1}{2\pi} \sum_{n=-N}^{N} \exp\left[i c_1 n (s_1 - s_1')\right] \equiv$$

$$\equiv \frac{c_1}{2\pi} \frac{\sin\left[\frac{N_2 c_1}{2}(s_1 - s_1')\right]}{\frac{c_1}{2}(s_1 - s_1')},$$

$$K_2(s_2, s_2', c_2) = \frac{c_2}{2\pi} \sum_{m=-M}^{M} \exp\left[i c_2 m (s_2 - s_2')\right] \equiv$$

$$\equiv \frac{c_2}{2\pi} \frac{\sin\left[\frac{M_2 c_2}{2}(s_2 - s_2')\right]}{\frac{c_2}{2}(s_2 - s_2')}.$$

Note, that the kernel (11) of Equation (10) is degenerate and real.

We shall consider one of the properties of function $\exp(i \arg f(Q'))$ entering into (10) at $f(Q') \to 0$. Obviously that the function

$$\exp(i \arg f(Q')) = \frac{f(Q')}{|f(Q')|} \equiv \frac{u(Q') + i v(Q')}{\left(u^2(Q') + v^2(Q')\right)^{1/2}}$$

is continuous if $u(Q') = \operatorname{Re} f(Q')$ and $v(Q') = \operatorname{Im} f(Q')$ are continuous functions, where $|\exp(i \arg f(Q'))| = 1$ for any $f(Q')$. If $u(Q') \to 0$ and $v(Q') \to 0$ simultaneously then $f(Q') \to 0$ is a complex zero. Its argument is undetermined accordingly to definition of complex zero [11, p. 20]. On this basis we redefine $\exp\left(i\arg f(Q')\right)$ at $u(Q')\to 0$ and $v(Q')\to 0$ as a function which has module equal to unit and undetermined argument.

The equivalence of (9) and (10) follows from the following lemma.

Lemma 1. *Between solutions of Equations* (9) *and* (10) *there exists bijection, i.e., if* $\mathbf{I}_*$ *is a solution of* (9) *then* $f_* = A\mathbf{I}_*$ *is the solution of (*10); *on the contrary, if* f_* *is the solution of* (10) *then*

$$\mathbf{I}_* = A^*\left\{F\exp\left[i\arg(f_*)\right]\right\} \tag{12}$$

is the solution of (9).

Proof. Let $\mathbf{I}_*$ be a solution of (9). Then $\mathbf{I}_* - A^* \left\{F\exp\left[i\arg\left(A\mathbf{I}_*\right)\right]\right\}\equiv 0$. Acting on this identity by the linear operator A, we have $A\mathbf{I}_* - AA^*\left\{F\exp\left[i\arg\left(A\mathbf{I}_*\right)\right]\right\}\equiv 0$. Since the operator A acts from the space $H_I = \mathbb{C}^{N_2\times M_2}$ into the space $\mathbf{C}^{(2)}_{(\Omega)}$ and accordingly into the space $\tilde{H}_f = L_2(\Omega)$, and the set of its nulls consists of only null element from the last identity follows, what $A\mathbf{I}_* = f_* \in \tilde{H}_f$ is a solution of (10).

On the contrary, let $f_* \in \tilde{H}_f$ solves the Equation (10). The operator A^* acts from the space $\tilde{H}_f^* = L_2^*(\Omega)$ into the space $H_I = \mathbb{C}^{N_2\times M_2}$ [10] and the Hilbertian space L_2^* coincides with the space L_2 [10]. From here follows, that A^* acts from the space $\tilde{H}_f = L_2(\Omega)$ into the space $H_I = \mathbb{C}^{N_2\times M_2}$. Taking into account that F is a finite function determined by (6), and f_* is continuous, the function $F\exp\left(i\arg(f_*)\right)$ is quadratic integrability in the domain Ω, *i.e.* $F\exp\left(i\arg(f_*)\right)\in H_f$. Thus $A^*\left(F\exp\left(i\arg(f_*)\right)\right) = \mathbf{I}_* \in H_I$ and the right part of (10) is the result of action of operator A on an element $\mathbf{I}_*$, *i.e.* $A\mathbf{I}_* = AA^*\left(F\exp\left(i\arg(f_*)\right)\right) = f_*$. Writing this equality as $A\left(\mathbf{I}_* - A^*\left(F\exp\left(i\arg(A\mathbf{I}_*)\right)\right)\right) = 0$ and taking into account that a set of operator nulls consists of only a null element we obtain $\mathbf{I}_* = A^*\left(F\exp\left(i\arg(A\mathbf{I}_*)\right)\right)$. So, $\mathbf{I}_* = A^*\left(F\exp\left(i\arg(f_*)\right)\right)$ solves the Equation (9). Lemma is proved.

Thus owing to the equivalence of (9) and (10) we consider simpler of them, namely (10). The Equation (9) is a more complicated equation in sense that in its right part the operator A is in an index of the power of exponent.

Besides taking into account that a set of values of operator A is a set of continuous functions in the domain Ω belonging to the space $L_2(\Omega)$ and this set is a compact in the space $L_2(\Omega)$ [12], we shall investigate solutions of (10) in the space $\mathbf{C}(\Omega)$.

Formulate the important properties of (10), which are checked directly.

1) If function $f(Q)$ is a solution of (10) then the conjugate complex function $\overline{f(Q)}$ is also the solution of (10).

2) If function $f(Q)$ is a solution of (10), then $\exp(i\beta)f(Q)$ is also the solution of (10) (β is any real constant).

3) For even on two arguments (or on one argument) functions $F(s_1,s_2)$ the nonlinear operator $\mathbf{B}$ that is in the right part of (10), is an invariant concerning the type of parity of the function $\arg f(s_1,s_2)$ on two arguments (or on that argument on which $F(s_1,s_2)$ is an even function).

Below taking into account the property 2) for uniqueness of solutions we set the parameter $\beta = 0$.

Consider the operator

$$Df \equiv \iint_\Omega K\left(Q,Q',\mathbf{c}\right) f(Q')dQ' \tag{13}$$

and corresponding to it quadratic form

$$\left(Df, f\right) = \iint_\Omega\iint_\Omega K(Q,Q',\mathbf{c}) f(Q')dQ'\overline{f(Q)}dQ =$$

$$= \sum_{n=-N}^{N}\sum_{m=-M}^{M}\left|\iint_\Omega \exp\left[i\left(c_1 ns_1 + c_2 ms_2\right)\right]\overline{f(s_1,s_2)}\,ds_1 ds_2\right|^2 =$$

$$= \left(\frac{2\pi}{c_1}\right)\left(\frac{2\pi}{c_2}\right)\|\mathbf{I}\|^2 \ge 0\,.$$

Obviously that this inequality modifies into equality only as $\mathbf{I} = 0$. From here follows that the kernel $K\left(Q,Q',\mathbf{c}\right)$ is positively defined [13]. Accordingly operator D is positive on nonnegative functions cone K of the space $C(\Omega)$ [14]. According to it D leaves invariant the cone K, *i.e.* $D\mathsf{K}\subset\mathsf{K}$.

Complex decomplexified space $\mathbf{C}(\Omega)$ [10] we consider as a direct sum of two real spaces of continuous functions $\mathbf{C}(\Omega) = C(\Omega)\oplus C(\Omega)$ in the domain Ω. The elements of this space are written as $f = (u,v)^T \in \mathbf{C}(\Omega)$, $u\in C(\Omega)$, $v\in C(\Omega)$. Norms in these spaces have the form:

$$\|u\|_{C(\Omega)} = \max_{Q\in\Omega}|u(Q)|,\ \ \|v\|_{C(\Omega)} = \max_{Q\in\Omega}|v(Q)|,$$

$$\| f \|_{\mathbf{C}(\Omega)} = \max\left(\|u\|_{C(\Omega)}, \|v\|_{C(\Omega)} \right).$$

The Equation (10) in the decomplexified space $\mathbf{C}(\Omega)$ we reduce to equivalent to it system of the nonlinear equations

$$u(Q) = B_1(u,v) \equiv \iint_\Omega K\left(Q,Q',\mathbf{c}\right) F(Q') \frac{u(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ',$$

$$v(Q) = B_2(u,v) \equiv \iint_\Omega K\left(Q,Q',\mathbf{c}\right) F(Q') \frac{v(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ'. \tag{14}$$

Denote the closed convex set of continuous functions as $S_R \subset \mathbf{C}(\Omega)$ supposing that

$$S_R = S_{R_u} \oplus S_{R_v}, \quad S_{R_u} = \left\{ u : \|u\|_{C(\Omega)} \le R \right\},$$

$$S_{R_v} = \left\{ v : \|v\|_{C(\Omega)} \le R \right\},$$

$$R = \max_{Q\in\Omega} \iint_\Omega \left| K(Q,Q',\mathbf{c}) \right| F(Q') dQ'.$$

Theorem 1. *The operator* $\mathbf{B} = (B_1, B_2)^T$ *determined by the Formula* (14) *maps a closed convex set* S_R *of the Banach space* $\mathbf{C}(\Omega)$ *in itself and it is completely continuous.*

Proof. At first we show that $\mathbf{B} : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$. Let $f = (u,v)^T$ be any function belonging to $\mathbf{C}(\Omega)$. At $\left(c_1, c_2\right) \in \Lambda_c$ the kernel $K\left(Q,Q',\mathbf{c}\right)$ is a continuous function with respect to its arguments in the closed domain $\Omega\times\Omega$. Then accordingly to the Cantor theorem [15] $K\left(Q,Q',\mathbf{c}\right)$ is a uniformly continuous function in $\Omega\times\Omega$. From here follows: for any points (Q_1,Q_1'), (Q_2,Q_2') such that whenever $\left|\left(Q_1,Q_1'\right)-\left(Q_2,Q_2'\right)\right| < \delta$, then $\left|K\left(Q_1,Q_1',\mathbf{c}\right) - K\left(Q_2,Q_2',\mathbf{c}\right)\right| < \dfrac{\varepsilon}{a}$, where $a = \iint_\Omega F(Q')dQ'$. On this basis we obtain

$$\left|u(Q_1)-u(Q_2)\right| = \left| \iint_\Omega F(Q')\left[K\left(Q_1,Q',\mathbf{c}\right) - K\left(Q_2,Q',\mathbf{c}\right)\right] \times \frac{u(Q')}{\sqrt{u^2(Q')+v^2(Q')}} dQ' \right| \le \frac{\varepsilon}{a} \iint_G F(Q')dQ' = \varepsilon, \tag{15}$$

$$\text{since} \quad \max_{Q'\in\bar\Omega} \left| \frac{u(Q')}{\sqrt{u^2(Q')+v^2(Q')}} \right| \le 1.$$

Analogously we have that $\left|v(Q_1)-v(Q_2)\right| \le \varepsilon$ whenever $\left|\left(Q_1,Q_1'\right)-\left(Q_2,Q_2'\right)\right| < \delta$, *i.e.* $(u,v)^T \in \mathbf{C}(\Omega)$ and $\mathbf{B} : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$.

To prove the property of a complete continuity of the operator $\mathbf{B} = (B_1, B_2)^T$ it is necessary to prove its compactness and continuity [12]. Show a continuity $\mathbf{B} = (B_1, B_2)^T$. Let $f_1 = (u_1, v_1)^T \in S_R$ be any fixed function and $f_2 = (u_2, v_2)^T$ be any function belonging to S_R. It is necessary to show that $\left\|\mathbf{B}f_1 - \mathbf{B}f_2\right\|_{\mathbf{C}(\Omega)} \to 0$ as $\left\| f_1 - f_2 \right\|_{\mathbf{C}(\Omega)} \to 0$. Set $u_2 = u_1 + \Delta u$, $v_2 = v_1 + \Delta v$. Taking into account these equalities we obtain

$$\frac{u_2}{\sqrt{u_2^2+v_2^2}} = \frac{u_1+\Delta u}{\sqrt{u_1^2+v_1^2}\sqrt{1+\dfrac{2u_1\Delta u + 2v_1\Delta v + \Delta u^2 + \Delta v^2}{u_1^2+v_1^2}}}.$$

At $\|\Delta u\|_{C(\Omega)} \to 0$, $\|\Delta v\|_{C(\Omega)} \to 0$ we have

$$\lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \left\| \frac{u_1(Q)}{\sqrt{u_1^2(Q)+v_1^2(Q)}} - \frac{u_2(Q)}{\sqrt{u_2^2(Q)+v_2^2(Q)}} \right\|_{\mathbf{C}(\Omega)} \le$$

$$\le \lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \max_{Q\in\Omega} \left\{ \left| \frac{u_1(Q)}{\sqrt{u_1^2(Q)+v_1^2(Q)}} \left(1 - \frac{1}{P\left(u_1(Q), v_1(Q)\right)} \right) \right| + \left| \frac{\Delta u(Q)}{\sqrt{u_1^2(Q)+v_1^2(Q)} P\left(u_1(Q), v_1(Q)\right)} \right| \right\} = 0, \tag{16}$$

where

$$P\left(u_1(Q), v_1(Q)\right) = \sqrt{1+\frac{2u_1(Q)\Delta u(Q) + 2v_1(Q)\Delta v(Q) + \Delta u^2(Q) + \Delta v^2(Q)}{u_1^2(Q)+v_1^2(Q)}},$$

since

$$\lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \max_{Q\in\Omega} \left| P\left(u_1(Q), v_1(Q)\right) \right| = 1.$$

Similarly we obtain

$$\lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \max_{Q\in\bar\Omega} \left| \frac{v_1(Q)}{\sqrt{u_1^2(Q)+v_1^2(Q)}} - \frac{v_2(Q)}{\sqrt{u_2^2(Q)+v_2^2(Q)}} \right| = 0. \tag{17}$$

Thus, from (16) and (17) follows

$$\lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \left\| B_1(u_1, v_1) - B_1(u_2, v_2) \right\|_{\mathbf{C}(\Omega)} =$$

$$= \lim_{\substack{\|\Delta u\|\to 0,\\ \|\Delta v\|\to 0}} \max_{Q\in\bar\Omega} \left| \iint_\Omega F(Q') K\left(Q,Q',\mathbf{c}\right) \times \left(\frac{u_1(Q')}{\sqrt{u_1^2(Q')+v_1^2(Q')}} - \frac{u_2(Q')}{\sqrt{u_2^2(Q')+v_2^2 dQ'(Q')}} \right) dQ' \right| = 0.$$

Analogously

$$\lim_{\substack{\|\Delta u\|_{C(\Omega)}\to 0,\\ \|\Delta v\|_{C(\Omega)}\to 0}} \left\| B_2(u_1,v_1) - B_2(u_2,v_2) \right\|_{\mathbf{C}(\Omega)} = 0 .$$

So, $\mathbf{B} = (B_1, B_2)^T$ is continuous operator from $\mathbf{C}(\Omega)$ into $\mathbf{C}(\Omega)$.

We show that a set of functions $S_g = \mathbf{B}S_R$ satisfies conditions of the Arzela theorem [12], *i.e.* we show that functions of the set S_g are uniformly bounded and equipotentially continuous. Furthermore $\mathbf{B}S_R \subset S_R$. Let $g = (w,\omega)^T = \mathbf{B}f \equiv (B_1(u,v), B_2(u,v))^T$, where $f = (u,v)^T$ is any function of the set S_R. Then as $\left|(Q_1,Q') - (Q_2,Q')\right| < \delta$ analogously with (15) we have

$$\begin{pmatrix} |w(Q_1) - w(Q_2)| \\ |\omega(Q_1) - \omega(Q_2)| \end{pmatrix} \le \begin{pmatrix} \dfrac{\varepsilon}{a} \iint\limits_{\Omega} F(Q')dQ' \\ \dfrac{\varepsilon}{a} \iint\limits_{\Omega} F(Q')dQ' \end{pmatrix} = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix}.$$

Thus functions of the set $S_g = \mathbf{B}S_R$ are equipotentially continuous.

The uniform boundedness of the set $S_g = \mathbf{B}S_R$ follows from an inequality

$$\|g\|_{\mathbf{C}(\Omega)} = \max\left\{ \max_{Q\in\Omega} \left| \iint\limits_{\Omega} F(Q')K(Q,Q',\mathbf{c}) \times \right.\right.$$

$$\left. \times \frac{u(Q')}{\sqrt{u^2(Q') + v^2(Q')}} dQ' \right| \le$$

$$\left.\le \max_{Q\in\Omega} \left| \iint\limits_{\Omega} F(Q')K(Q,Q',\mathbf{c}) \frac{v(Q')}{\sqrt{u^2(Q') + v^2(Q')}} dQ' \right| \right\} \le R ,$$

where $f = (u,v)^T$ is any function of the set S_R and $g = \mathbf{B}f \equiv (B_1(u,v), B_2(u,v))^T$. From the last inequality we have also $\mathbf{B}S_R \subset S_R$. So, the operator $\mathbf{B} = (B_1, B_2)^T$ is completely continuous mapping a closed convex set $S_R \subset \mathbf{C}(\Omega)$ into itself.

Theorem is proved.

From the Theorem 1 follows satisfaction of conditions of the Schauder principle [16] according to which the operator $\mathbf{B} = (B_1, B_2)^T$ has a fixed point $f_* = (u_*, v_*)^T$ belonging to the set S_R. This point is a solution of a system of Equation (14) and Equation (10), respectively. Substituting $f_* = (u_*, v_*)^T$ into (12), we obtain a solution of (9) being a stationary point of the functional (7).

The solutions of a system of equations analogous with (14) in a case of one-dimensional domains Ω were investigated for the synthesis problem of linear antenna array in particular in [17]. The obtained there results show that for equations of the type (10) and (14) non-uniqueness and branching of solutions dependent on the size of physical parameter are characteristic. Directly the results [17] cannot be transferred on the two-dimensional two-parametric problem (8) and (14). Here, as unlike the points of branching [17], the branching lines of solutions exist and a problem on finding the lines of branching is a nonlinear two-parametrical spectral problem.

Easily to be convinced that function

$$f_0(Q,\mathbf{c}) = \iint\limits_{G} F(Q')K(Q,Q',\mathbf{c})dQ' \tag{18}$$

is one of solutions of (10) in the class of real functions. Since, as shown before, the operator D determined by (13), is positive on the nonnegative functions cone $\mathsf{K} \in \mathrm{C}(\Omega)$, $D\mathsf{K} \subset \mathsf{K}$ and $F \subset \mathsf{K}$, then $f_0 = DF$ also is a nonnegative function in the domain Ω.

To find the lines of branching and complex solutions of (10), branching-off from real solution $f_0(Q,\mathbf{c})$, we consider a problem on finding such set of values of parameters $\mathbf{c}^{(0)} = \left(c_1^{(0)}, c_2^{(0)}\right)$ and all distinct from $f_0(Q,\mathbf{c})$ solutions of the system (14) which for $\left|\mathbf{c} - \mathbf{c}^{(0)}\right| \to 0$ (where $c_1 \ge c_1^{(0)}$, $c_2 \ge c_2^{(0)}$) satisfy conditions

$$\max_{Q\in G} \left| u(Q,\mathbf{c}) - f\left(Q,\mathbf{c}^{(0)}\right) \right| \to 0 , \quad \max_{Q\in G} \left| v(Q,\mathbf{c}) \right| \to 0 . \tag{19}$$

These conditions indicate the need to find small continuous in G solutions

$$w(Q,\mathbf{c}) = u(Q,\mathbf{c}) - f_0\left(Q,\mathbf{c}^{(0)}\right), \quad \omega(Q,\mathbf{c}) = v(Q,\mathbf{c}),$$

which converge uniformly to zero as $\mathbf{c} \to \mathbf{c}^{(0)}$.

Set

$$c_1 = c_1^{(0)} + \mu , \quad c_2 = c_2^{(0)} + \nu \tag{20}$$

and desired solutions we find in the form

$$u(Q,\mathbf{c}) = f_0\left(Q,\mathbf{c}^{(0)}\right) + w(Q,\mu,\nu), \quad v(Q,\mathbf{c}) = \omega(Q,\mu,\nu) . \tag{21}$$

Further we omit dependence of the functions $w(Q,\mu,\nu)$ and $\omega(Q,\mu,\nu)$ on parameters μ and ν.

Notice the properties of integrand in the system (14). They are continuous functions with respect to the arguments. After substitution (20) and (21) into (14) the integrand develop in equiconvergent power series by functional arguments w and ω, numerical parameters μ and ν in the vicinity of a point $\left(\mathbf{c}^{(0)}, f_0\left(Q,\mathbf{c}^{(0)}\right), 0\right)$:

$$F(Q')K(Q,Q',\mathbf{c}) \frac{u(Q')}{\sqrt{u^2(Q') + v^2(Q')}} =$$

$$= \sum_{m+n+p+q\ge 0} A_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right) w^m(Q')\omega^n(Q')\mu^p\nu^q,$$

$$F(Q')K\left(Q,Q',\mathbf{c}\right)\frac{v(Q')}{\sqrt{u^2(Q')+v^2(Q')}} =$$

$$= \sum_{m+n+p+q\ge 1} B_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right) w^m(Q')\omega^n(Q')\mu^p\nu^q \,. \quad (22)$$

Here $A_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right)$, $B_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right)$ are coefficients of expansion continuously dependent on the arguments. Substituting (20) and (22) into (14) and taking into account that $f_0\left(Q',\mathbf{c}^{(0)}\right)$ solves the system (14) we obtain a system of nonlinear equations with respect to small solutions w, ω:

$$u(Q) = a_{10}\left(Q,\mathbf{c}^{(0)}\right)\mu + a_{01}\left(Q,\mathbf{c}^{(0)}\right)\nu +$$

$$+ \sum_{m+n+p+q\ge 2} \mu^p\nu^q \iint_{\Omega} A_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right) w^m(Q')\omega^n(Q')dQ', \quad (23)$$

$$\omega(Q) - \iint_{\Omega} F(Q)K\left(Q,Q',\mathbf{c}^{(0)}\right)\frac{\omega(Q')}{f_0\left(Q',\mathbf{c}^{(0)}\right)}dQ' =$$

$$= \sum_{m+n+p+q\ge 2} \mu^p\nu^q \iint_{\Omega} B_{mnpq}\left(Q,Q',\mathbf{c}^{(0)}\right) w^m(Q')\omega^n(Q')dQ', \quad (24)$$

where

$$a_{10}\left(Q,\mathbf{c}^{(0)}\right) = \iint_G A_{0010}\left(Q,Q',\mathbf{c}^{(0)}\right)dQ',$$

$$a_{01}\left(Q,\mathbf{c}^{(0)}\right) = \iint_G A_{0001}\left(Q,Q',\mathbf{c}^{(0)}\right)dQ'.$$

3. Nonlinear Two-Parametric Spectral Problem

For further application of methods of the branching theory of solutions of nonlinear equations [18] to a system (23) and (24) it is necessary to find solutions of distinct from trivial of the linear homogeneous integral equation obtained equating to zero the left part of (24)

$$\varphi(Q) = T(c_1,c_2)\varphi \equiv \iint_G \frac{F(Q')}{f_0(Q',c_1,c_2)} K\left(Q,Q',c_1,c_2\right)\varphi(Q')dQ' \quad (25)$$

under condition $f_0(Q',\mathbf{c}) > 0$. Indicate that the operator $T(\mathbf{c}):\mathbf{C}(\Omega)\to\mathbf{C}(\Omega)$ is completely continuous. Proof of this property is similar to the proof of a complete continuity of the operator $\mathbf{B}=(B_1,B_1)^T$ in the Theorem 1.

According to [18] such values of parameters $(c_1^{(0)},c_2^{(0)})\in\mathbb{R}^2$ at which linear homogeneous Equation (25) has distinct from identical zero solutions are points of possible branching of solutions of a system of nonlinear Equations (23) and (24). The eigenfunctions of (25) are used at construction branching-off solutions of (23) and (24).

The spectral parameters c_1 and c_2 are included non- linearly into the kernel of the integral operator. Therefore a problem on finding the distinct from $f_0(Q,c_1,c_2)$ solutions of (25) is a nonlinear two-parametric spectral problem. It consists in finding such values of real parameters $\left(c_1,c_2\right)\in\Lambda_c$ at which (25) has distinct from identical zero solutions.

In operational form a nonlinear two-parametric problem is presented as

$$\mathrm{A}\,(c_1,c_2)x \equiv \left(E - T(c_1,c_2)\right)x = 0\,. \quad (26)$$

Here E is an identical operator and $T(c_1,c_2)$ is a linear integrated operator acting in the Banach space $C(\Omega)$. It is necessary to find eigenvalue $\mathbf{c} = \left(c_1^{(0)},c_2^{(0)}\right)\in\mathbf{\Lambda}_c$ and corresponding eigenvectors $x^{(0)}\in C\left(\Omega\right)$ ($x^{(0)}\ne 0$) such that $\mathrm{A}\,(c_1^{(0)},c_2^{(0)})x^{(0)} = 0$.

By direct check we ascertain that for any values of parameters $\left(c_1,c_2\right)\in\mathbf{\Lambda}_c$ the function

$$\hat{\varphi}_0(Q,\mathbf{c}) = \iint_{\Omega} F(Q')K\left(Q,Q',\mathbf{c}\right)dQ' \quad (27)$$

is one of eigenfunctions.

Write a conjugate to (25) equation required in later

$$\psi(Q) = T^*(\mathbf{c})\psi \equiv \frac{F(Q)}{f_0(Q,\mathbf{c})}\iint_{\Omega} K\left(Q,Q',\mathbf{c}\right)\psi(Q')dQ' \,. \quad (28)$$

At arbitrary $\left(c_1,c_2\right)\in\mathbf{\Lambda}_c$ the function

$$\hat{\psi}_0(Q) = F(Q) \quad (29)$$

is one of eigenfunctions of (28)

The existence of distinct from identical zero solutions of (25) at arbitrary $\left(c_1,c_2\right)\in\mathbf{\Lambda}_c$ testifies to the existence of coherent components of a spectrum conterminous with the domain $\mathbf{\Lambda}_c$.

For finding the distinct from $\hat{\varphi}_0(Q,\mathbf{c})$ solutions we exclude from the kernel of integral Equation (25) the eigen function (27), namely: consider the equation

$$\varphi(Q,\mathbf{c}) = \tilde{T}(\mathbf{c})\varphi \equiv \iint_{\Omega} \mathrm{K}\left(Q,Q',\mathbf{c}\right)\varphi(Q')dQ', \quad (30)$$

where

$$\mathrm{K}\left(Q,Q',\mathbf{c}\right) = \frac{F(Q')}{f_0\left(Q',\mathbf{c}\right)}K\left(Q,Q',\mathbf{c}\right) - \psi_0(Q)\varphi_0\left(Q',\mathbf{c}\right), \quad (31)$$

$$\psi_0(Q) = \frac{\hat{\psi}_0(Q)}{\left\|\hat{\psi}_0\right\|_{L_2}},\quad \varphi_0\left(Q',\mathbf{c}\right) = \frac{\hat{\varphi}_0\left(Q',\mathbf{c}\right)}{\left\|\hat{\varphi}_0\right\|_{L_2}}. \quad (32)$$

From Schmidt Lemma [18] follows that $\varphi_0(Q,\mathbf{c})$ will not be an eigenfunction of this equation for any values $(c_1,c_2)\in\Lambda_c$. Thus from a spectrum of operator there is excluded coherent component coinciding with the domain Λ_c and corresponding to the function $\varphi_0(Q,\mathbf{c})$.

Using the property of degeneration of the kernel $\mathrm{K}(Q,Q',c_1,c_2)$, we reduce (25) to equivalent system of linear algebraic equations having coefficients analytically dependent on parameters c_1, c_2. We write (25) as

$$\varphi(s_1,s_2)=\sum_{n=-N}^{N}\sum_{m=-M}^{M}x_{nm}\exp\left[i\left(c_1ns_1+c_2ms_2\right)\right]-$$
$$-x_0\psi_0(s_1,s_2)\,, \qquad (33)$$

where x_{nm}, x_0 are constants determined by the formulas

$$x_{nm}=\frac{c_1}{2\pi}\frac{c_2}{2\pi}\iint_{\Omega}\frac{F(s_1',s_2')}{f_0(s_1',s_2',c_1,c_2)}\exp\left[-i\left(c_1ns_1'+c_2ms_2'\right)\right]\times$$
$$\times\varphi(s_1',s_2')ds_1'ds_2'\ \left(n=-N\div N,\ m=-M\div M\right),$$
$$x_0=\iint_{\Omega}\varphi_0(s_1',s_2',c_1,c_2)\varphi(s_1',s_2')ds_1'ds_2'\,.$$

From the Formula (33) follows, that the function $\varphi(s_1,s_2)$ will become known, if will be found x_{nm}, x_0.

Multiplication of both parts of (33) by $\frac{F(s_1',s_2')}{f_0(s_1',s_2',c_1,c_2)}\exp\left[-i\left(c_1ks_1'+c_2ls_2'\right)\right]$ at $k=-N\div N$, $l=-M\div M$ and by $\varphi_0(s_1',s_2')$, and integration over Ω gives a homogeneous system of the linear algebraic equations for finding x_{nm}, x_0

$$x_{kl}=\sum_{n=-N}^{N}\sum_{m=-M}^{M}a_{nm}^{(kl)}\left(c_1,c_2\right)x_{nm}\quad\begin{pmatrix}k=-N\div N,\\ l=-M\div M\end{pmatrix}. \qquad (34)$$

Here

$$a_{nm}^{(kl)}\left(c_1,c_2\right)=\left\{t_{nm}^{(kl)}\left(c_1,c_2\right)-\frac{b^{(kl)}\left(c_1,c_2\right)}{1+d_0\left(c_1,c_2\right)}d_{nm}\left(c_1,c_2\right)\right\},$$
$$t_{nm}^{(kl)}\left(c_1,c_2\right)=\frac{c_1c_2}{4\pi^2}\iint_{\Omega}\frac{F(s_1,s_2)}{f_0(s_1,s_2,c_1,c_2)}\times$$
$$\times\exp\left\{i\left[c_1\left(n-k\right)s_1+c_2\left(m-l\right)s_2\right]\right\}ds_1ds_2\,,$$
$$b^{(kl)}\left(c_1,c_2\right)=\frac{c_1c_2}{4\pi^2}\iint_{\Omega}\frac{F(s_1,s_2)}{f_0(s_1,s_2,c_1,c_2)}\psi_0(s_1,s_2)\times,$$
$$\times\exp\left[-i\left(c_1ks_1+c_2ls_2\right)\right]ds_1ds_2\,,$$
$$d_{nm}\left(c_1,c_2\right)=\iint_{\Omega}\varphi_0(s_1,s_2)\exp\left[i\left(c_1ns_1+c_2ms_2\right)\right]ds_1ds_2\,,$$
$$d_0\left(c_1,c_2\right)=\iint_{\Omega}\psi_0(s_1,s_2)\varphi_0(s_1,s_2,c_1,c_2)ds_1ds_2\,.$$

For coefficients of the matrix $\mathbf{A}_M\left(c_1,c_1\right)=\left\|a_{nm}^{(kl)}\left(c_1,c_1\right)\right\|_{\substack{k,n=-N\div N\\ m,l=-M\div M}}$ the equality $a_{mn}^{(lk)}\left(c_1,c_1\right)=\overline{a_{nm}^{(kl)}\left(c_1,c_1\right)}$ is valid, *i.e.* $\mathbf{A}_M$ is the Hermitian or self-adjoint matrix.

Write the equivalent to (26) nonlinear two-parametrical spectral problem, corresponding to a system of Equation (34), as

$$\mathrm{A}_M(c_1,c_2)\mathbf{x}\equiv\left(\mathbf{E}_M-\mathbf{A}_M(c_1,c_2)\right)\mathbf{x}=0\,, \qquad (35)$$

where $\mathbf{E}_M$ is a unit matrix of dimension $N_2\times M_2$.

In order that the system (34) should have distinct from zero solutions, it is necessary

$$\Psi(c_1,c_2)=\det\left(\mathbf{E}_M-\mathbf{A}_M(c_1,c_2)\right)=0\,. \qquad (36)$$

It is easy to be convinced, that $\Psi(c_1,c_2)$ is a real function. Really as $\mathbf{T}_M(c_1,c_2)$ is the Hermitian matrix then it is obvious that $\left(\mathbf{E}-\mathbf{A}_M(c_1,c_2)\right)$ is also the Hermitian matrix. It is known [19] that the determinant of the Hermitian matrix is a real number. So, $\Psi(c_1,c_2)$ is a real function with respect to the real arguments c_1 and c_2.

Therefore, the problem on finding the set of eigenvalues of (25) or equivalent linear algebraic system (34) is reduced to finding the nulls of the function $\Psi(c_1,c_2)$.

Consider a necessary later on auxiliary one-dimensional spectral problem (as a special case of the problem (35)) on the ray $c_2=\gamma c_1$ (γ is a real coefficient, $\left(c_1,\,c_2\right)\in\Lambda_c$). Introduce into consideration the matrix-function $\tilde{\mathrm{A}}_M(c_1)\equiv\mathrm{A}_M\left(c_1,\gamma c_1\right)$ and connected with it the one-dimensional spectral problem

$$\tilde{\mathrm{A}}_M\left(c_1,\gamma c_1\right)\mathbf{x}=\left(\mathbf{E}_M-\mathbf{A}_M(c_1,\gamma c_1)\right)\mathbf{x}=0\,. \qquad (37)$$

It is easy to be convinced, that from the properties of coefficients of matrix $\mathbf{A}_M(c_1,c_2)$ follows, that the matrix function $\mathrm{A}_M(c_1,c_2)$ is continuous and differentiable on the variables in any open and limited domain $\Lambda\subset\Lambda_c\subset\mathbb{R}^2$. In other words $\mathrm{A}_M(c_1,c_2)$ is a holomorphic matrix-function, if c_1,c_2 to continue into the domain of complex variables.

Corresponding to (37), Equation (36) has the form

$$\Psi(c_1,\gamma c_1)=\det\left(\mathbf{E}_M-\mathbf{A}_M(c_1,\gamma c_1)\right)=0\,. \qquad (38)$$

We denote the spectrums of the problems (35) and (37) as $s(\mathrm{A})$ and $s(\tilde{\mathrm{A}})$, respectively, and the parameter domain c_1 as $\Lambda_{c_1}=\left\{c_1:0<c_1\le a\right\}$. Then for properties of the spectrum of (35) the Theorem 1 from [9] is

applied which relatively to (35) is formulated thus:

Theorem 2. Let at each $\mathbf{c} = (c_1, c_2) \in \Lambda_c$ the matrix $\mathcal{A}_M(c_1, c_2) \in \mathcal{L}(\mathbb{C}^{N_2 \times M_2}, \mathbb{C}^{N_2 \times M_2})$ be the Fredholm operator with a zero index, the matrix-function $\mathcal{A}(\cdot,\cdot)$: $\Lambda_c \to \mathcal{L}\left(\mathbb{C}^{N_2 \times M_2}, \mathbb{C}^{N_2 \times M_2}\right)$ be holomorphic in the domain Λ_c and $s(\tilde{\mathcal{A}}) \neq \Lambda_{c_1}$. Moreover, let function $\Psi(c_1, c_2)$ be continuously differentiable in Λ_c. Then:

1) Each point of a spectrum $c_1^{(0)} \in s(\tilde{\mathcal{A}})$ is isolated and it is eigenvalue of the matrix-function $\tilde{\mathcal{A}}(c_1) \equiv \mathcal{A}(c_1, \gamma c_1)$, to it is corresponding the finite-dimensional eigensubspace $N\left(\tilde{\mathcal{A}}\left(c_1^{(0)}\right)\right)$ and finite-dimensional root subspace;

2) Each point $\mathbf{c}^{(0)} = \left(c_1^{(0)}, \gamma c_1^{(0)}\right) \in \Lambda_c$ is a point of spectrum of the matrix-function $\mathcal{A}(\lambda_1, \lambda_2)$;

3) If $\Psi'_{c_2}\left(c_1^{(0)}, c_2^{(0)}\right) \neq 0$ then in some vicinity of the point $c_1^{(0)}$ there is a unique continuous differentiable function $c_2 = c_2(c_1)$ solving the Equation (36), *i.e.* in some bicircular domain $\Lambda_0 = \left\{(c_1, c_2) : \left|c_1 - c_1^{(0)}\right| < \varepsilon_1, \left|c_2 - c_2^{(0)}\right| < \varepsilon_2\right\}$ there exists a connected component of spectrum of the matrix-function $\mathcal{A}(c_1, c_2)$ (where ε_1, ε_2 are small real constants).

Proof of this theorem concerning the nonlinear two-parametrical spectral problem of the type (35) for more general case (when the operators E and $T(c_1, c_2)$ act in the infinite dimensional Banach space) is presented in [9]. For satisfaction of conditions of Theorem 1 from [9] it is necessary to show that the matrix-function $\mathcal{A}(c_1, c_2)$ is the Fredholm matrix at $(c_1, c_2) \in \Lambda_c$. This property follows from the known equality [19] $\dim(\ker\mathcal{A}) = \dim(\ker\mathcal{A}^*)$.

The existence of connected components of spectrum of the matrix-function $\mathcal{A}(c_1, c_2)$, under condition of $\Psi'_{c_2}\left(c_1^{(0)}, c_2^{(0)}\right) \neq 0$, follows from the existence theorem of implicitly given function [20, 21].

Let $c_1^{(i)}$ be a root of (38). Then $\left(c_1^{(i)}, c_2^{(i)} = \gamma c_1^{(i)}\right) \in \Lambda_c$ is eigenvalue of the problem (33). Consider the equation $\Psi(c_1, c_2) = 0$ as a problem on finding the implicitly given function $c_2 = c_2(c_1)$ in the vicinity of a point $c_1^{(i)}$ for which the conditions of existence theorem [21] are satisfied. Hence we have the Cauchy problem

$$\frac{dc_2}{dc_1} = -\frac{\Psi'_{c_1}(c_1, c_2)}{\Psi'_{c_2}(c_1, c_2)}, \tag{39}$$

$$c_2^{(i)}\left(c_1^{(i)}\right) = \gamma c_1^{(i)}. \tag{40}$$

Solving numerically (39) and (40) in some vicinity of a point $c_1^{(i)}$, we find the i-th connected component of spectrum (spectral line) of the matrix-function $\mathcal{A}_M(c_1, c_2)$.

By found solutions of the Cauchy problem at the fixed values $\left(c_1^{(i)}, c_2^{(i)}\right)$ the eigenfunctions of (25) are determined through the eigenvectors of the matrix $\mathbf{A}_M\left(c_1^{(i)}, c_2^{(i)}\right)$ obtained by the known methods. Thus four-dimensional matrix $\mathbf{A}_M$ is reduced to two-dimensional one by means of corresponding renumbering of elements.

4. Numerical Algorithm of Finding the Solutions of a Nonlinear Equation

Show one of iterative processes for numerical finding the solutions of the system (14) based on the successive approximations method [2]:

$$u_{n+1}(Q) = B_1(u_n, v_n) \equiv \iint_\Omega K(Q, Q', \mathbf{c}) F(Q') \times \times \frac{u_n(Q')}{\sqrt{u_n^2(Q') + v_n^2(Q')}} dQ',$$

$$v_{n+1}(Q) = B_2(u_n, v_n) \equiv \iint_\Omega K(Q, Q', \mathbf{c}) F(Q') \times \times \frac{v_n(Q')}{\sqrt{u_n^2(Q') + v_n^2(Q')}} dQ' \quad (n = 0, 1, \ldots). \tag{41}$$

After substituting the function $\arg f_n(Q) = \mathrm{arctg}\left(v_n(Q) / u_n(Q)\right)$ (obtained on the basis of successive approximations (41)) into (12), we denote the obtained sequence of function values as $\{\mathbf{I}_n\}$. For the sequence $\{\mathbf{I}_n\}$ the Theorem 4.2.1 from [3] is fulfilled. From here follows, that the sequence $\{\mathbf{I}_n\}$ is a relaxation one for the functional (7) and numerical sequence $\{\sigma(\mathbf{I}_n)\}$ is convergent.

At realization of the iterative process (41) in the case of even on both arguments function $F(s_1, s_2)$ and symmetric domains G and Ω it is expedient to use the property of invariance of integral operators $B_1(u, v)$, $B_2(u, v)$ in the system (14) concerning the type of parity of functions $u(s_1, s_2)$, $v(s_1, s_2)$. The functions u, v having certain type of evenness on corresponding arguments belong to the appropriate invariant sets U_{ij}, $V_{k\ell}$ of the space $\mathbf{C}(\Omega)$. Here the indices i, j, k, ℓ have values 0 or 1. In particular, if $u(s_1,s_2)\in U_{01}$ then $u(-s_1,s_2)=u(s_1,s_2)$ and $u(s_1,-s_2)=-u(s_1,s_2)$. By direct check we are convinced that such inclusions take place:

$$B_1(U_{ij}\cup V_{k\ell})\subset U_{ij},\qquad B_2(U_{ij}\cup V_{k\ell})\subset V_{k\ell},$$

$$\mathbf{B}(U_{ij}\cup V_{k\ell})\subset U_{ij}\cup V_{k\ell}.$$

The possibility of existence of fixed points of the operator **B** belonging to appropriate invariant set (*i.e.* solutions of system (14) and, respectively, Equation (10)) follows from these relations.

5. Numerical Example

Consider an example of approximation of the function $F(s_1,s_2)=\cos(\pi s_1/2)|\sin(\pi s_2)|$ (**Figure 1**), given in the domain $\bar{G}=\{(s_1,s_2):|s_1|\le 1,|s_2|\le 1\}\subset\Omega$, for $N_2\times M_2=11\times 11$ and values of parameters $c_1=1.6$ and $c_2=1.2$ belonging to the ray $c_2=0.75c_1$. The possible branching lines of solutions of the system (14) and accordingly the Equation (10), as solutions of two-dimensional spectral problem (25), are shown in **Figure 2**. Here the first branching lines are denoted by numbers 1 and 2. To the solutions branching-off at the points of these lines there correspond the odd on s_2 functions $\arg f(s_1,s_2)$ and the coefficients of transformation $I_{n,m}$ ($n=-N\div N,\ m=-M\div M$) are real, but nonsymmetrical concerning to the plane XOZ.

In **Figure 3** in logarithmic scale are presented values of the functional σ obtained on the solutions of two types at values of parameter $c_2=0.75c_1$: the curve 1 corresponds to solutions in a class of real functions $f_0(Q)$, curve 2 – to the branching-off solution with odd on s_2 argument $\arg f(s_1,s_2)$. From analysis of **Figure 3** follows that at the point $c_1\approx 0.77$ from real solution branch-off more effective complex-conjugate between themselves solutions, on which the functional σ accepts smaller values, than on the real solution. If to introduce into consideration parameter $C_2=Mc_2$ characterizing the quantity of basic functions in transformation (1), the identical efficiency of approximation (identical values of the functional σ on real and branching-off solutions) is reached with use of the branching-off solution at decrease of the quantity of basic functions on the value $\Delta C_2=0.75\Delta c_1$.

An amplitude (**a**) and argument (**b**) of approximate function are given in **Figure 4** for $c_1=1.6$ and $c_2=1.2$. The amplitude values of the Fourier Transform coefficients corresponding to this solution are shown in **Figure 5**. As we see in figure, the values of amplitudes of coefficients are nonsymmetrical concerning the plane

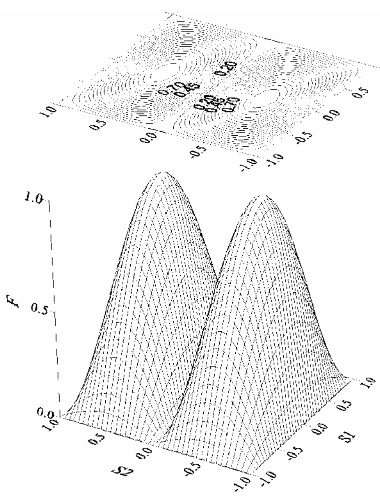


Figure 1. The function $F(s_1,s_2)=\cos(\pi s_1/2)|\sin(\pi s_2)|$ given in the domain $\bar{G}=\{(s_1,s_2):|s_1|\le 1,|s_2|\le 1\}\subset\Omega$.

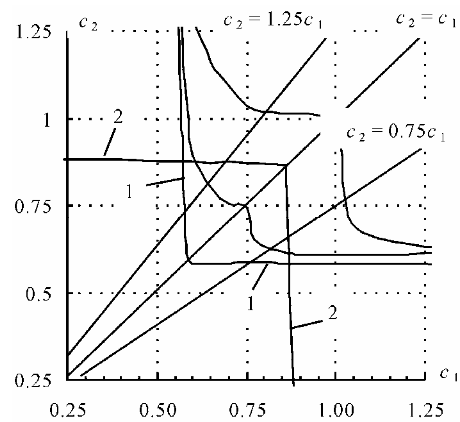


Figure 2. The branching lines of solutions

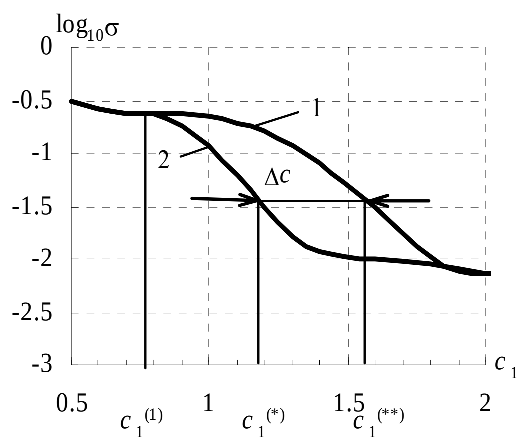
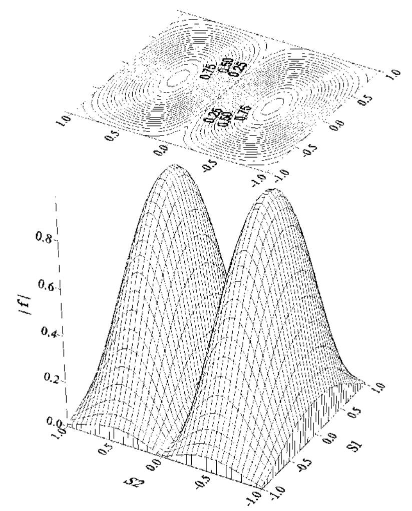
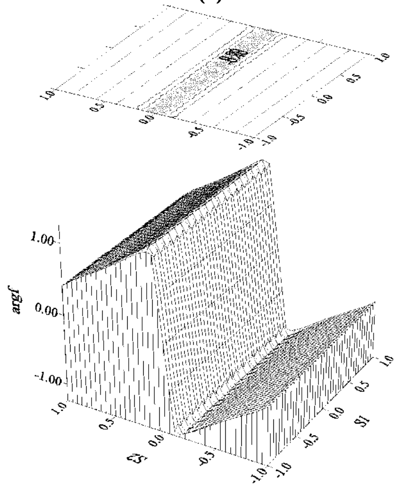


Figure 3. The values of functional on initial and branching-off solutions.



(a)



(b)

Figure 4. The modulus (a) and argument (b) of approximation function.

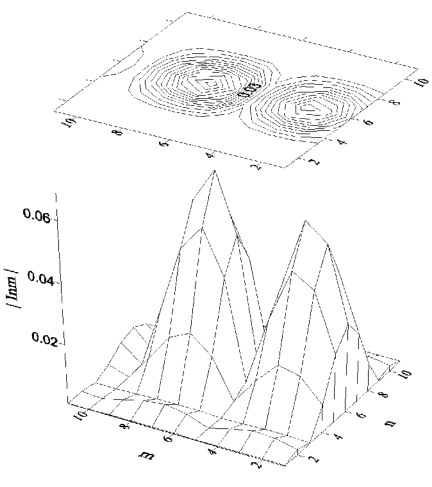


Figure 5. The optimum amplitude of Fouier transform coefficients.

YOZ, but the amplitude of approximate function (**Figure 4(a)**) is symmetric.

For comparison of approximate functions, corresponding to different solutions of (10), the curves corresponding to different types of the presented solutions in the section $s_1 \equiv 0$ are given in **Figure 6**. The curve 1 corresponds to the given function $F(0, s_2)$, the curve 2 – to branching-off solution, the curve 3 − to real solution $f_0(0, s_2)$. Obviously that the branching-off solution better (in meaning of the functional σ) approaches the prescribed function by the module.

6. Conclusions

Mark the basic features and problems arising at investigation of the considered class of tasks:

The basic difficulty to solve this class of problems is study of nonuniqueness and branching of existing solutions dependent on the parameters c_1, c_2 entering into the discrete Fourier Transform.

As follows from investigations, presented, in particular, in [3,17] (for a special case, when $F(s_1, s_2) = F_1(s_1) \cdot F_2(s_2)$), the quantity of the existing solutions grows considerably with increase of the parameters c_1, c_2. Let us indicate, that in many practical applications, in particular, in the synthesis problems of radiating systems, it is important to obtain the best approximation to the given function $F(s_1, s_2)$ at rather small values of parameters c_1, c_2. This allows limiting by investigation of several first points (lines) of branching.

To find the branching points (lines) of solutions of (8), it is necessary, as opposed to [3, 17], to solve not enough studied multiparametric spectral problem. The offered in this work approaches allow to find the solutions of a nonlinear two-parametric spectral problem for homogeneous integral equations with degenerate kernels analytically dependent on two spectral parameters.

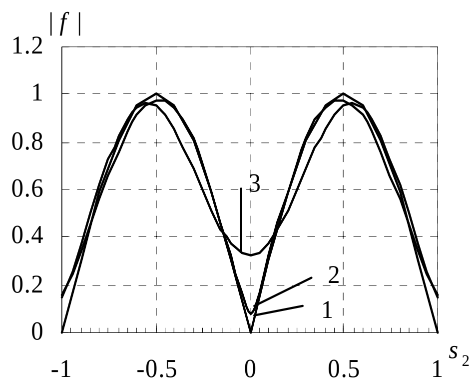


Figure 6. The given (curve 1) and approximation functions in the section $s_2 = 0$ corresponding to branching-off (curve 2) and real (curve 3) solutions.

When finding the solutions to a system of Equation (14) by successive approximations method, to obtain the solutions of a certain type of parity of the function $\arg f(s_1, s_2)$ it is necessary to choose an initial approximation $\arg f_0(s_1, s_2)$ of the same type of parity according to (42).

To obtain the irrefragable answer concerning the branching-off solutions for certain values of parameters c_1, c_2 it is necessary to use the branching theory of solutions [18]. It is the object of special investigations.

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The Structure of Reflective Function of Higher Dimensional Differential System

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Abstract

In this article, we discuss the structure of reflective function of the higher dimensional differential systems and apply the results to study the existence of periodic solutions of these systems.

Keywords: Reflecting Function, Periodic Solution, Higher Dimensional System

1. Introduction

As we know, to study the property of the solutions of differential system

$$x' = X(t,x) \tag{1}$$

is very important not only for the theory of ordinary differential equation but also for practical reasons. If $X(t+2\omega, x) = X(t,x)$ (ω is a positive constant), to study the solutions' behavior of (1), we could use, as introduced in [1], the Poincare mapping. But it is very difficult to find the Poincare mapping for many systems which cannot be integrated in quadratures. In the 1980's the Russian mathematician Mironenko [2] first established the theory of reflective functions (RF). Since then a quite new method to study (1) has been found.

In the present section, we introduce the concept of the reflective function, which will be used throughout the rest of this article.

Now consider the system (1) with a continuously differentiable right-hand side and with a general solution $\psi(t;t_0,x_0)$. For each such system, the reflective function (RF) of (1) is defined as $F(t,x(t)) = \psi(-t;t,x)$. Then for any solution $x(t)$ of (1), we have $F(t,x(t)) = x(-t)$. If system (1) is $2\omega-$periodic with respect to t, and $F(t,x)$ is its RF, then $F(-\omega,x) = \psi(\omega;-\omega,x)$ is the Poincare [1-2] mapping of (1) over the period $[-\omega,\omega]$. So, for any solution $x(t)$ of (1) defined on $[-\omega,\omega]$, it will be 2ω--periodic if and only if $x(-\omega)$ is a fixed point of the Poincare mapping $T(x) = F(-\omega,x)$. A function $F(t,x)$ is a reflective function of system (1) if and only if it is a solution of the partial differential equation (called a basic relation, BR)

$$F_t' + F_x' X(t,x) + X(-t,x) = 0 \tag{2}$$

with the initial condition $F(0,x) = x$. It implies that for non-integrable periodic systems we also can find out its Poincare mapping. If, for example, $X(t,x) + X(-t,x) = 0$, then $T(x) = x$.

If $F(t,x)$ is the RF of (1), then it is also the RF of the system

$$x' = X(t,x) + F_x^{-1}R(t,x) - R(-t,F(t,x)),$$

where $R(t,x)$ is an arbitrary vector function such that the solutions of the above systems are uniquely determined by their initial conditions. Therefore, all these $2\omega-$periodic systems have a common Poincare mapping over the period $[-\omega,\omega]$, and the behavior of the periodic solutions of these systems are the same.

To find out the reflective function is very important for studying the qualitative behavior of solutions of differential systems. The literatures [5-8] have discussed the structure of the reflective function of some second order quadric systems and linear systems and obtained many good results.

Now, we consider the higher dimensional polynomial differential system

$$\begin{cases} x' = p_1 + p_2 y + p_3 z = P(t,x,y,z) \\ y' = q_1 + q_2 y + q_3 z + q_4 y^2 + q_5 yz + q_6 z^2 = Q(t,x,y,z) \\ z' = r_1 + r_2 y + r_3 z + r_4 y^2 + r_5 yz + r_6 z^2 = R(t,x,y,z) \end{cases} \tag{3}$$

where

$$p_i = p_i(t,x), q_j = q_j(t,x), r_j = r_j(t,x)$$
$$(i = 1,2,3;\ j = 1,2,\ldots,6)$$

are continuously differentiable functions in R^2, and $p_2^2 + p_3^2 \neq 0$ (in some deleted neighborhood of $t = 0$

and $|t|$ being small enough, $p_2^2+p_3^2\neq 0$ is different from zero）, and there exists a unique solution for the initial value problem of (3). And suppose that $F(t,x,y,z)=(F_1(t,x,y,z),F_2(t,x,y,z),F_3(t,x,y,z))^T$ is the RF of (3).

In this paper, we will discuss the structure of $F_i(t,x,y,z)$ $(i=2,3)$ when $F_1(t,x,y,z)=f(t,x)$. At the same time, we obtain the good results that $F_i(t,x,y,z)=f_{i1}(t,x)+f_{i2}(t,x)y+f_{i3}(t,x)z$ $(i=2,3)$. The obtained results are used for research of problems of existence of periodic solution of the system (3) and establish the sufficient conditions under which the first component of the solution of (3) is even function.

In the following, we will denote $\overline{p}_i=p_i(-t,x);\ \overline{q}_j=q_j(-t,x);\ \overline{r}_j=r_j(t,x);\ F_i=F_i(t,x,y,z)$, $i=1,2,3,\ j=1,2,...,6$. The notation $p_i(t,x)\neq 0$ means that, in some deleted neighborhood of $t=0$ and $|t|$ being small enough, $p_i(t,x)$ is different from zero,

$$DA=\frac{\partial A}{\partial t}+\frac{\partial A}{\partial x}P(t,x,y,z)+\frac{\partial A}{\partial y}Q(t,x,y,z)+\frac{\partial A}{\partial z}R(t,x,y,z).$$

2. Main Results

Without loss of generality, we suppose that $f(t,x)=x$. Otherwise, we take the transformation $\xi=f(t,x)$, $\eta=y,\ \zeta=z$.

Now, let's consider the system (3).

Lemma 1. For the system (3), suppose that $F_1=x$. Then

$$p_i(t,x)=0,\ i=1,2,3. \tag{4}$$

Proof. Using the relation (2), we get

$$P(t,x,y,z)+P(-t,x,F_2,F_3)=0,$$

i.e.

$$p_1+\overline{p}_1+p_2y+p_3z+\overline{p}_2F_2+\overline{p}_3F_3=0\cdot \tag{5}$$

Putting $t=0$, we get

$$p_1(0,x)+p_2(0,x)y+p_3(0,x)z\equiv 0,\ \forall x,y,z\ .$$

It implies that the relation (4) is valid.

In the following discussion, we always assume (4) holds without further mention.

Case 1. $p_3\neq 0$.

From (5), we get

$$F_3=\lambda_1+\lambda_2F_2, \tag{6}$$

where

$$\lambda_1=-\frac{p_1+\overline{p}_1}{\overline{p}_3}-\frac{p_2}{\overline{p}_3}y-\frac{p_3}{\overline{p}_3}z=\lambda_{11}+\lambda_{12}y+\lambda_{13}z,$$

$$\lambda_2=-\frac{\overline{p}_2}{\overline{p}_3}.$$

Differentiating relation (6) respect to t implies

$$A_0+A_1F_2+A_2F_2^{\,2}=0, \tag{7}$$

where

$$A_0=D\lambda_1-\lambda_2\overline{q}_1+\overline{r}_1+\lambda_1(\overline{r}_3-\lambda_2\overline{q}_3)+\lambda_1^{\,2}(\overline{r}_6-\lambda_2\overline{q}_6)$$
$$=a_{01}+a_{02}y+a_{03}z+a_{04}y^2+a_{05}yz+a_{06}z^2;$$

$$A_1=D\lambda_2-\lambda_2\overline{q}_2+\overline{r}_2+\lambda_2(\overline{r}_3-\lambda_2\overline{q}_3)+\lambda_1(\overline{r}_5-\lambda_2\overline{q}_5)+2\lambda_1\lambda_2(\overline{r}_6-\lambda_2\overline{q}_6)=a_{11}+a_{12}y+a_{13}z;$$

$$A_2=A_2(t,x)=-\lambda_2\overline{q}_4+\overline{r}_4+\lambda_2(\overline{r}_5-\lambda_2\overline{q}_5)+\lambda_2^2(\overline{r}_6-\lambda_2\overline{q}_6),$$

In which

$$a_{01}=\lambda_{11t}^{'}+\lambda_{11x}^{'}p_1+\lambda_{12}q_1+\lambda_{13}r_1+\overline{r}_1-\lambda_2\overline{q}_1+$$
$$+\lambda_{11}(\overline{r}_3-\lambda_2\overline{q}_3)+(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{11}^2;$$

$$a_{02}=\lambda_{12t}^{'}+\lambda_{12x}^{'}p_2+\lambda_{12x}^{'}p_1+\lambda_{12}q_2+\lambda_{13}r_2+$$
$$+\lambda_{12}(\overline{r}_3-\lambda_2\overline{q}_3)+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{11}\lambda_{12};$$

$$a_{03}=\lambda_{13t}^{'}+\lambda_{11x}^{'}p_3+\lambda_{13x}^{'}p_3+\lambda_{12}q_3+\lambda_{13}r_3+$$
$$+\lambda_{13}(\overline{r}_3-\lambda_2\overline{q}_3)+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{11}\lambda_{13};$$

$$a_{04}=\lambda_{12x}^{'}p_2+\lambda_{12}q_4+\lambda_{13}r_4+(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{12}^2;$$

$$a_{05}=\lambda_{12x}^{'}p_3+\lambda_{13x}^{'}p_2+\lambda_{12}q_5+\lambda_{13}r_5+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{12}\lambda_{13};$$

$$a_{06}=\lambda_{13x}^{'}p_3+\lambda_{12}q_6+\lambda_{13}r_6+(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_{13}^2;$$

$$a_{11}=\lambda_{2t}^{'}+\lambda_{2x}^{'}p_1+\overline{r}_2-\lambda_2\overline{q}_2+\lambda_2(\overline{r}_3-\lambda_2\overline{q}_3)+$$
$$+\lambda_{11}(\overline{r}_5-\lambda_2\overline{q}_5)+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_2\lambda_{11};$$

$$a_{12}=\lambda_{2x}^{'}p_2+\lambda_{12}(\overline{r}_5-\lambda_2\overline{q}_5)+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_2\lambda_{12};$$

$$a_{13}=\lambda_{2x}^{'}p_3+\lambda_{13}(\overline{r}_5-\lambda_2\overline{q}_5)+2(\overline{r}_6-\lambda_2\overline{q}_6)\lambda_2\lambda_{13};$$

$$\lambda_{11}=-\frac{p_1+\overline{p}_1}{\overline{p}_3},\ \lambda_{12}=-\frac{p_2}{\overline{p}_3},\ \lambda_{13}=-\frac{p_3}{\overline{p}_3}.$$

Lemma 2. Let $F_1=x,\ A_2\neq 0$ and $\lim\limits_{t\to 0}\frac{a_{0i}}{A_2}$ $(i=1,2,...,6)$ exist. Then $\lim\limits_{t\to 0}\frac{a_{0j}}{A_2}=0$ $(j=1,3,6)$, $\lim\limits_{t\to 0}\frac{a_{02}+a_{11}}{A_2}=0$,

$$\lim_{t\to 0}\frac{a_{04}+a_{12}}{A_2}=-1,\ \lim_{t\to 0}\frac{a_{05}+a_{13}}{A_2}=0.$$

Proof. Using the relation (7), we have

$$\lim_{t\to 0}\frac{a_{01}+a_{02}y+a_{03}z+a_{04}y^2+a_{05}yz+a_{06}z^2}{A_2}+$$

$$\lim_{t\to 0}\frac{a_{11}+a_{12}y+a_{13}z}{A_2}F_2+\lim_{t\to 0}F_2^{\,2}=0,$$

As $F_2(0,x,y,z)=y$, it follows that the results of Lemma 2 are true.

Theorem 1. Let the conditions of Lemma 1 and Lemma 2 satisfy and $\lim\limits_{t\to 0}\frac{a_{12}}{A_2}+2>0$. Then

$$F_i=f_{i1}(t,x)+f_{i2}(t,x)y+f_{i3}(t,x)z\quad (i=2,3).$$

Proof. As $A_2 \neq 0$. From (7), it follows

$$F_2^2 = -\frac{A_0 + A_1 F_2}{A_2},\ F_2^3 = \frac{A_1 A_0}{A_2^2} + \frac{A_1^2 - A_0 A_2}{A_2^2} F_2 \qquad (8)$$

Differentiating relation (7) respect to t implies

$$DA_0 + DA_1 F_2 + DA_2 F_2^2 - A_1 Q(-t, x, F_2, F_3)$$
$$-2A_2 Q(-t, x, F_2, F_3) F_2 = 0.$$

Substituting (6) into the above, we get

$$B_0 + B_1 F_2 + B_3 F_2^2 + B_4 F_2^3 = 0, \qquad (9)$$

where

$$B_0 = DA_0 - A_1\mu_1,\quad B_1 = DA_1 - A_1\mu_2 - 2A_2\mu_1,$$
$$B_2 = DA_2 - A_1\mu_3 - 2A_2\mu_2,\quad B_4 = -2A_2\mu_3,$$

in which

$$\mu_1 = \bar{q}_1 + \bar{q}_3\lambda_1 + \bar{q}_6\lambda_1^2, \mu_2 = \bar{q}_2 + \bar{q}_3\lambda_2 + \bar{q}_5\lambda_1 + 2\bar{q}_6\lambda_2\lambda_1,$$
$$\mu_3 = \bar{q}_4 + \bar{q}_5\lambda_2 + \bar{q}_6\lambda_2^2.$$

Substituting (8) into (9), we have

$$C_0 + C_1 F_2 = 0, \qquad (10)$$

where

$$C_0 = B_0 - B_2\frac{A_0}{A_2} + B_3\frac{A_1 A_0}{A_2^2},$$
$$C_1 = B_1 - B_2\frac{A_1}{A_2} + B_3\frac{A_1^2 - A_0 A_2}{A_2^2}.$$

1) If $C_1 \equiv 0$, from (10) follows $C_0 \equiv 0$. By simple computation, we obtain

$$D\frac{A_0}{A_2} = \mu_1\frac{A_1}{A_2} - 2\mu_2\frac{A_0}{A_2} + \mu_3\frac{A_0 A_1}{A_2^2}, \qquad (11)$$

$$D\frac{A_1}{A_2} = 2\mu_1 - \mu_2\frac{A_1}{A_2} + \mu_3\left(\frac{A_1^2}{A_2^2} - 2\frac{A_0}{A_2}\right). \qquad (12)$$

Let $\Delta = (\frac{A_1}{A_2})^2 - 4\frac{A_0}{A_2}$. Using (11) (12) we get

$$D\Delta = 2\left(\mu_3\frac{A_1}{A_2} - \mu_2\right)\Delta. \qquad (13)$$

Since

$$\Delta = \left(\frac{A_1}{A_2}\right)^2 - 4\frac{A_0}{A_2} = \frac{1}{A_2^2}((a_{11} + a_{12}y + a_{13}z)^2 -$$
$$4A_2(a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2))$$
$$= d_1 + d_2 y + d_3 z + d_4 y^2 + d_5 yz + d_6 z^2$$
$$= d_4\left(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4}\right)^2 + W,$$

where

$$W = \frac{1}{4d_4}[(4d_4 d_6 - d_5^2)z^2 + 2(2d_3 d_4 - d_2 d_5)z + 4d_1 d_4 - d_2^2],$$
$$d_1 = \frac{1}{A_2^2}(a_{11}^2 - 4A_2 a_{01}), d_2 = \frac{1}{A_2^2}(2a_{11}a_{12} - 4A_2 a_{02}),$$
$$d_3 = \frac{1}{A_2^2}(2a_{11}a_{13} - 4A_2 a_{03}), d_4 = \frac{1}{A_2^2}(a_{12}^2 - 4A_2 a_{04}),$$
$$d_5 = \frac{1}{A_2^2}(2a_{13}a_{12} - 4A_2 a_{05}), d_6 = \frac{1}{A_2^2}(a_{13}^2 - 4A_2 a_{06}).$$

By **Lemma 2** we get

$$\lim_{t\to 0}(4d_4 d_6 - d_5^2) = 0, \lim_{t\to 0}(3d_3 d_4 - d_2 d_5) = 0,$$
$$\lim_{t\to 0}(4d_4 d_1 - d_2^2) = 0,$$

thus, $\lim_{t\to 0} W(t, x, z) = 0.$ In the identity (13) taking

$y = \phi = -\frac{d_5}{2d_4}z - \frac{d_2}{2d_4}$. We obtain

$$W_t' + W_x' P(t, x, \phi) + W_z' Q(t, x, \phi, z) =$$
$$2W\left(\frac{A_1(t, x, \phi, z)}{A_2(t, x)}\mu_3(t, x) - \mu_2(t, x, \phi, z)\right).$$

By the uniqueness of solution of initial problem of linear partial differential equation, we get $W(t, x, z) \equiv 0$. Therefore

$$\Delta = d_4\left(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4}\right)^2.$$

Using the relation (7), we obtain

$$F_2 = -\frac{a_{11} + a_{12}y + a_{13}z}{2A_2} + \frac{1}{2}\sqrt{d_4}\left(y + \frac{d_5}{2d_4}z + \frac{d_2}{2d_4}\right)$$
$$= f_{21}(t, x) + f_{22}(t, x)y + f_{23}(t, x)z.$$

By the relation (6), we get

$$F_3 = \lambda_1 + \lambda_2 F_2 = f_{31}(t, x) + f_{32}(t, x)y + f_{33}(t, x)z.$$

2) If $C_1 \neq 0$. From (10) follows $F_2 = -\frac{C_0}{C_1}$. By the express of A_i, B_j $(i = 0,1,2, j = 0,1,2,3)$, we know that C_1 is a quadratic polynomial respect of y, z, C_0 is a cubic polynomial respect to y, z. Substituting $F_2 = -C_0/C_1$ into relation (7), we get $C_1(C_0 A_1 - C_1 A_0) = A_2 C_0^2$. It implies that C_1 is divided by C_0 or A_2, and $F_2 = \sum_{i+j=0}^{3} f_{2ij}(t, x) y^i z^j$, substituting it into (7) and equating the coefficients of like powers of y and z implies $f_{2ij}(t, x) = 0, i + j > 1$. Thus,

$$F_i = f_{i1}(t, x) + f_{i2}(t, x)y + f_{i3}(t, x)z,\ i = 2,3.$$

Summarizing the above, the proof is completed.
Obviously, from the relation (7) implies

Theorem 2. Let

$F_1 = x,\ p_i(0,x) = 0\ (i = 1,2,3),\ A_2 = 0,\ A_1 \neq 0$.

Then

$$F_2 = -\frac{A_0}{A_1} = -\frac{a_{01} + a_{02}y + a_{03}z + a_{04}y^2 + a_{05}yz + a_{06}z^2}{a_{11} + a_{12}y + a_{13}z},$$

$$F_3 = \lambda_1 + \lambda_2 F_2.$$

Case 2. $p_3(t,x) \equiv 0,\ p_2 \neq 0$.

Applying identity (5) yields

$$F_2 = -\frac{p_1 + \overline{p}_1}{\overline{p}_2} - \frac{p_2}{\overline{p}_2} y = \zeta_1 + \zeta_2 y,$$

where $\zeta_1 = -\frac{p_1 + \overline{p}_1}{\overline{p}_2},\ \zeta_2 = -\frac{p_2}{\overline{p}_2}$.

Differentiating this identity respect to t implies

$M_0 + M_1 F_3 + M_2 F_3^2 = 0$,

where

$$M_0 = D(\zeta_1 + \zeta_2 y) + \overline{q}_1 + \overline{q}_2 F_2 + \overline{q}_4 F_3^2 =$$
$$m_{01} + m_{02}y + m_{03}z + m_{04}y^2 + m_{05}yz + m_{06}z^2,$$
$$M_1 = \overline{q}_3 + \overline{q}_5 F_2 = m_{11} + m_{12}y,\ M_2 = \overline{q}_6,$$
$$m_{01} = \zeta_{1t}' + \zeta_{1x}' p_1 + \zeta_2 q_1 + \overline{q}_1 + \overline{q}_2\zeta_1 + \overline{q}_4\zeta_1^2;$$
$$m_{02} = \zeta_{1x}' p_2 + \zeta_{2x}' p_1 + \zeta_{2t}' + \zeta_2 q_2 + \overline{q}_2\zeta_2 + 2\overline{q}_4\zeta_1\zeta_2;$$
$$m_{04} = \zeta_{2x}' p_2 + \zeta_2 q_4 + \overline{q}_4\zeta_2^2;$$
$$m_{03} = \zeta_2 q_3,\ m_{05} = \zeta_2 q_5,\ m_{06} = \zeta_2 q_6,$$
$$m_{11} = \overline{q}_3 + \overline{q}_5\zeta_1,\ m_{12} = \overline{q}_5\zeta_2.$$

Similarly, we obtain the following conclusion:

Lemma 3. Let $p_3 = 0,\ p_2 \neq 0,\ q_6 \neq 0,\ F_1 = x$ and

$\lim\limits_{t\to 0}\frac{m_{0i}}{q_6}\ (i = 1,2,...,5)$ and $\lim\limits_{t\to 0}\frac{p_1}{\overline{p}_2}$ exist. Then

$$\lim_{t\to 0}\frac{m_{0j}}{q_6} = 0\ (j = 1,2,4), \lim_{t\to 0}\frac{p_1 + \overline{p}_1}{\overline{p}_2} = 0,\ \lim_{t\to 0}\frac{p_2}{\overline{p}_2} = -1$$

$$\lim_{t\to 0}\frac{m_{03} + m_{11}}{q_6} = 0, \lim_{t\to 0}\frac{m_{05} + m_{12}}{q_6} = 0,\ \lim_{t\to 0}\frac{q_6}{\overline{q}_6} = -1.$$

Theorem 3. Let the conditions of Lemma 1 and Lemma 3 satisfy. Then

$$F_2 = \zeta_1(t,x) + \zeta_2(t,x)y,$$
$$F_3 = f_{31}(t,x) + f_{32}(t,x)y + f_{33}(t,x)z.$$

Theorem 4. Let

$p_i(0,x) = 0\ (i = 1,2), q_3(t,x) \equiv 0,\ q_6(t,x) \equiv 0$
$m_{11}^2 + m_{12}^2 \neq 0, F_1 = x$
, Then

$$F_2 = \zeta_1(t,x) + \zeta_2(t,x)y,$$
$$F_3 = -\frac{m_{01} + m_{02}y + m_{03}z + m_{04}y^2 + m_{05}yz}{m_{11} + m_{12}y}.$$

Theorem 5. For the system (3), if the following conditions satisfy

$$p_1 + \overline{p}_1 + \overline{p}_2 f_{21} + \overline{p}_3 f_{31} = 0,\ p_2 + \overline{p}_2 f_{22} + \overline{p}_3 f_{32} = 0,$$

$$p_3 + \overline{p}_2 f_{23} + \overline{p}_3 f_{33} = 0,\ f_{21}(0,x) = 0,\ f_{31}(0,x) = 0,$$

$$\begin{pmatrix} f_{21t}' \\ f_{31t}' \end{pmatrix} + \begin{pmatrix} f_{21x}' \\ f_{31x}' \end{pmatrix} p_1 + \begin{pmatrix} f_{22} \\ f_{23} \end{pmatrix} q_1 + \begin{pmatrix} f_{23} \\ f_{33} \end{pmatrix} r_1 + \begin{pmatrix} Q(-t,x,f_{21},f_{31}) \\ R(-t,x,f_{21},f_{31}) \end{pmatrix}$$
$$= 0,$$

$$\begin{pmatrix} f_{22t}' & f_{23t}' \\ f_{32t}' & f_{33t}' \end{pmatrix} + \begin{pmatrix} f_{22x}' & f_{23x}' \\ f_{32x}' & f_{33x}' \end{pmatrix} p_1 + \begin{pmatrix} f_{21x}' p_2 & f_{21x}' p_3 \\ f_{31x}' p_2 & f_{32x}' p_3 \end{pmatrix} +$$

$$\begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_2 & q_3 \\ r_2 & r_3 \end{pmatrix} + \begin{pmatrix} \overline{Q}_y' & \overline{Q}_z' \\ \overline{R}_y' & \overline{R}_z' \end{pmatrix}_{(t,x,0,0)} = 0,$$

$$\begin{pmatrix} f_{22x}' p_2 & f_{23x}' p_3 \\ f_{32x}' p_2 & f_{33x}' p_3 \end{pmatrix} + \begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_4 & q_6 \\ r_4 & r_6 \end{pmatrix} +$$

$$\begin{pmatrix} \overline{q}_4 & \overline{q}_6 \\ \overline{r}_4 & \overline{r}_6 \end{pmatrix}\begin{pmatrix} f_{22}^2 & f_{23}^2 \\ f_{32}^2 & f_{33}^2 \end{pmatrix} + \begin{pmatrix} f_{22} f_{23}\overline{q}_5 & f_{23} f_{33}\overline{q}_5 \\ f_{22} f_{23}\overline{r}_5 & f_{23} f_{33}\overline{r}_5 \end{pmatrix} = 0,$$

$$\begin{pmatrix} f_{22x}' & f_{23x}' \\ f_{32x}' & f_{33x}' \end{pmatrix}\begin{pmatrix} p_3 \\ p_2 \end{pmatrix} + \begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} q_5 \\ r_5 \end{pmatrix} +$$

$$\begin{pmatrix} 2\overline{q}_4 f_{22} f_{23} + \overline{q}_5(f_{22} f_{33} + f_{23} f_{32}) + 2\overline{q}_6 f_{32} f_{33} \\ 2\overline{r}_4 f_{22} f_{23} + \overline{r}_5(f_{22} f_{33} + f_{23} f_{32}) + 2\overline{r}_6 f_{32} f_{33} \end{pmatrix} = 0,$$

$$\begin{pmatrix} f_{22} & f_{23} \\ f_{32} & f_{33} \end{pmatrix}\begin{pmatrix} \overline{f}_{22} & \overline{f}_{23} \\ \overline{f}_{32} & \overline{f}_{33} \end{pmatrix} = \begin{pmatrix} f_{22}(0,x) & f_{23}(0,x) \\ f_{32}(0,x) & f_{33}(0,x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $F = \begin{pmatrix} x \\ f_{21} + f_{22}y + f_{22}z \\ f_{31} + f_{32}y + f_{33}z \end{pmatrix}$ is the RF of system (3). Besides this, if the system (3) is 2ω--periodic with respect to t, then its solution $(x(t), y(t), z(t))$ defined on the interval $[-\omega,\omega]$ with initial condition $(x(-\omega), y(-\omega), z(-\omega)) = \chi$ is 2ω--periodic if and only if $F(-\omega,\chi) = \chi$.

Proof. By checkout of the BR it is proved that the function $F = (x, f_{21} + f_{22}y + f_{23}z, f_{31} + f_{32}y + f_{33}z)^T$ is the RF of system (3). At this moment, the Poincare mapping of periodic system (3) is $T(x,y,z) = F(-\omega,x,y,z)$. By the previous introduction the assertions of the present theorem is hold. The proof is finished.

Under the hypotheses of Theorem 5, the first component of solution of system (3) is even function.

Example: Differential system

$$\begin{cases} x' = (1-e^{\sin t}(1-x^3\sin t))y - x^2 z\sin t, \\ y' = \frac{1}{2}y\cos t(x^6\sin^2 t + x^3 - 1) - \frac{1}{2}zx^2\cos te^{-\sin t}(1+\sin t + x^3\sin^2 t) \\ +y^2(3x^2\sin t - x^5\sin^2 t) + yze^{-\sin t}(x^4\sin^2 t - 2x\sin t), \\ z' = \frac{1}{2}yx^4\cos te^{\sin t}(1-\sin t + x^3\sin^2 t) + \frac{1}{2}z\cos t(1-x^3-x^6\sin^2 t) \\ -y^2(x^6\sin^2 t - 4x^3\sin t)e^{\sin t} - yz(3x^2\sin t - x^5\sin^2 t) \end{cases}$$

has RF

$$F(t,x,y,z) = \begin{pmatrix} x \\ e^{\sin t}(1-x^3\sin t)y + x^2 z\sin t \\ -x^4 y\sin t + e^{-\sin t}(1+x^3\sin t)z \end{pmatrix}.$$

Since this system is a 2π --periodic system, and $F(-\pi,x,y,z) \equiv (x,y,z)^T$, by Theorem 5, all the solutions of the considered system defined $[-\pi,\pi]$ are 2π --periodic.

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Scientific Research

Nonzero Solutions of Generalized Variational Inequalities*

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Abstract

The existence of nonzero solutions for a class of generalized variational inequalities is studied by fixed point index approach for multivalued mappings in finite dimensional spaces and reflexive Banach spaces. Some new existence theorems of nonzero solutions for this class of generalized variational inequalities are established.

Keywords: Variational Inequality, Fixed Point Index of Multivalued Mappings, Nonzero Solution

1. Introduction

Variational inequality theory with applications are an important part of nonlinear analysis and have been applied intensively to mechanics, differential equation, cybernetics, quantitative economics, optimization theory and nonlinear programming etc, [1-4].

Variational inequalities, generalized variational inequalities and generalized quasivariational inequalities were studied intensively in the last 30 years with topological method, variational method, semi-ordering method, fixed point method, minimax theorem of Ky Fan and KKM technique [1-4]. In 1998, motivated by the paper [5], Zhu [6] studied a system of variational inequalities involving the linear operators in reflexive Banach spaces by using the coincidence degree theory due to Mawhin [7]. Some existence results of positive solutions for this system of variational inequalities in reflexive Banach spaces were proved.

Let X be a real Banach space, X^* its dual and $(\cdot,\cdot)$ the pair between X^* and X. Suppose that K is a nonempty closed convex subset of X.

Find $u \in K$, $u \neq 0$, and $w \in g(u)$ such that

$$(Au, v-u) \geq (w, v-u), \ \forall v \in K \tag{1}$$

where mapping $A: K \to X^*$ is nonlinear and $g: K \to 2^{X^*}$ is a multi-valued mapping.

The existence of nonzero solutions for variational inequalities is an important topic of variational inequality theory. Y. Lai [8] discussed the variational inequality (1) when A is coercive or monotone and g is set-contractive or upper semi-continuous. K. Q. Wu *et al.*, [9] considered the variational inequality (1) when A is single-valued continuous and g is set-contractive.

On the other hand, recently, under some different conditions, [10,11] obtained some existence theorems of nonzero solutions for a class of generalized variational inequalities by fixed point index approach for multi-valued mappings in reflexive Banach space.

Based on the importance of studying the existence of nonzero solutions for variational inequalities, and motivated and inspired by recent research works in this field, in this paper, we discuss the existence of nonzero solutions for a class of generalized variational inequalities as follows:

Find $u \in K, u \neq 0$ such that

$$\begin{aligned}(Au, v-u) + j(v) - j(u) \\ \geq (g(u), v-u) + (f, v-u), \ \forall v \in K\end{aligned} \tag{2}$$

where $A, g: K \to X^*$ are two nonlinear mapping and $f \in X^*$.

A mapping $A: X \to X^*$ is called hemicontinuous at $x_0 \in X$ if for each $y \in X$, $A(x_0 + t_n y) \xrightarrow{w^*} Ax_0$ when $t_n \to +0$. A multivalued mapping $T: D(T) \subset X \to 2^{X^*}$ is said to be locally bounded in v if there exists a neighbourhood V of x for each $x \in X$ such that the set $T(V \cap D(T))$ is bounded in X^*. Suppose that K is a closed convex subset of X with $0 \in K$. For such K, the recession cone rcK of K is defined by $rcK = \{w \in X: v + w \in K, \forall v \in K\}$. It is easily seen that the recession cone is indeed a cone and we have that $rcK \neq \varnothing$. For a proper lower semicontinuous convex functional $j: X \to R \cup \{\infty\}$ with $j(0) = 0$ and $j(K) \subset R_+ = [0, +\infty)$, in the virtue of [12], the limit $\lim\limits_{t \to +\infty} \frac{1}{t} j(tw) =$

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$j_\infty(w)$ exists in $R\cup\{\infty\}$ for every $w\in X$ and j_∞ is also a lower semicontinuous convex functional with $j_\infty(0)=0$ and with the property that $j(u+v)\le j(u)+j_\infty(v), \forall u,v\in X$.

Suppose that K is a closed convex subset of X and U is an open subset of X with $U_K=U\cap K\ne\varnothing$. The closure and boundary of U_K relative to K are denoted by $\bar{U}_K$ and $\partial(U_K)$ respectively. Assume that $T:\bar{U}_K\to 2^K$ is an upper semicontinuous mapping with nonempty compact convex values and T is also condensing, *i.e.*, $\alpha(T(S))<\alpha(S)$ where α is the Kuratowski measure of noncompactness on X. If $x\notin T(x)$ for $x\in\partial(U_K)$, then the fixed point index, $i_K(T,U)$, is well defined(see[13]).

Proposition 1 [13] *Let K be a nonempty closed convex subset of real Banach space X and U be an open subset of X. Suppose that $T:\bar{U}_K\to 2^K$ is an upper semicontinuous mapping with nonempty compact convex values and $x\notin T(x)$ for $x\in\partial(U_K)$. Then the index, $i_K(T,U)$, has the following properties:*

1) *If $i_K(T,U)\ne 0$, then T has a fixed point;*

2) *For mapping $\widehat{X_0}$ with constant value $\{x_0\}$, if $x_0\in U_K$, then $i_K(\widehat{X_0},U)=1$;*

3) *Let U_1,U_2 be two open subsets of X with $U_1\cap U_2=\varnothing$.*

If $x\notin T(x)$ when $x\in\partial((U_1)_K)\cup\partial((U_2)_K)$, then $i_K(T,U_1\cup U_2)=i_K(T,U_1)+i_K(T,U_2)$;

4) *Let $H:[0,1]\times\bar{U}_K\to 2^K$ be an upper semicontinuous mapping with nonempty compact convex values and $\alpha(H([0,1]\times Q))<\alpha(Q)$ whenever $\alpha(Q)\ne 0, Q\subset\bar{U}_K$.*

If $x\notin H(t,x)$ for every $t\in[0,1], x\in\partial(U_K)$, then $i_K(H(1,\cdot),U)=i_K(H(0,\cdot),U)$.

For every $q\in X^*$, let $U(q)$ be the set of solutions in K of the following variational inequality

$$\begin{aligned}&(Au,v-u)+j(v)-j(u)\\&\quad\ge(q,v-u)+(f,v-u), \forall v\in K\end{aligned}\tag{3}$$

Define a mapping $K_A:X^*\to 2^K$ by

$K_A(q):=U(q),\quad q\in X^*.$

Obviously, $K_A(q)=\varnothing$ if and only if the variational inequality (3) has no solution in K.

2. Nonzero Solutions in R^n

Lemma 1 *Let X be a separable reflexive Banach space. Suppose that $A:X\to X^*$ is a bounded monotone hemicontinuous mapping (i.e., for every bounded subset D of X, $A(D)$ is bounded) and*

$j:K\to(-\infty,+\infty]$ is a proper convex lower semicontinuous functional. Assume that there exists $v_0\in K$ satisfying

$$\liminf_{\|u\|\to+\infty,u\in K}[(Au,u-v_0)+j(u)-j(v_0)]>0\tag{4}$$

Then for any given $f\in X^*$ there exists $u\in X$ such that

$$\begin{aligned}&(Au,v-u)+j(v)-j(u)\\&\quad\ge(f,v-u), \forall v\in X.\end{aligned}\tag{5}$$

Proof. Without loss of generality, assume that $f=0$, otherwise, set $\tilde{j}(v)=j(v)-(f,v)$. Let $K^r=\{x\in X:\|x\|\le r\}$. Because X is a separable reflexive Banach space, for given r, there exists a closed convex sets sequences $K_m, m=1,2,\ldots,$ satisfying the following conditions:

a) $K_m\subset K_{m+1}\subset K^r, m=1,2,\ldots;$

b) $K_m\subset X_m$, X_m is m-dimensional subspace of X;

c) $\bigcup_{m=1}^\infty K_m$ is dense in K^r.

First, we shall verify that for each m, there exists $u_m\in K_m$ such that

$$(Au_m,v-u_m)+j(v)-j(u_m)\ge 0, \forall v\in K_m.\tag{6}$$

Because X_m is a finite dimensional subspace (denoted its inner product by [.,.]), there exists a linear continuous mapping $\pi:X^*\to X_m$ such that $(g,\omega)=[\pi g,\omega]$ for all $\omega\in K_m$. Thus inequality (6) can be written

$$\begin{aligned}&[(-\pi Au+u)-u,v-u)]\\&\quad\le j(v)-j(u), \forall v\in K_m.\end{aligned}\tag{7}$$

Define a function $J_m(v):X_m\to(-\infty,+\infty]$ by

$$J_m(v)=\begin{cases}j(v), & v\in K_m\\ +\infty, & v\in X_m\setminus K_m.\end{cases}$$

Then inequality (7) can be written

$$\begin{aligned}&[(-\pi Au+u)-u,v-u)]\\&\quad\le J_m(v)-J_m(u), \forall v\in K_m\end{aligned}\tag{8}$$

which is equivalent to the equality

$$u=P_{J_m}(-\pi Au+u)\tag{9}$$

by [2,3], where P_{J_m} is an approximate mapping of J_m.

Obviously, $P_{J_m}(-\pi A+I):K_m\to K_m$ is continuous. According to Brouwer's fixed point theorem [2,3], there exists $u_m\in K_m$ satisfying the equality (9), that is, u_m is a solution of the variational inequality (6).

Second, we shall verify that for each r, there exists $u_r\in K^r$ such that

$$(Au_r, v-u_r)+j(v)-j(u_r)\ge 0, \forall v\in K^r. \qquad (10)$$

In fact, $K_m\subset K^r$ and A is a bounded mapping, which implies that there constant C such that $\|Au_m\|\le C$ for $m=1,2,\ldots$. Since X is a reflexive and K^r is weakly closed, there exists a subsequence $\{u_\mu\}\subset\{u_m\}$ such that $u_\mu\xrightarrow{w}u_r$ and $u_r\in K^r$. Because $\bigcup_{m=1}^{\infty}K_m$ is dense in K^r, for any given $\varepsilon>0$, there exists $u_0\in\bigcup_{m=1}^{\infty}K_m$ such that $\|u_r-u_0\|\le\varepsilon$. It then follows from (6) that

$$(Au_\mu, u_\mu-u_0)\le j(u_0)-j(u_\mu). \qquad (11)$$

when μ is sufficiently large. Thus we have

$$\begin{aligned}&\limsup_\mu(Au_\mu, u_\mu-u_r)\\ &\le\limsup_\mu(Au_\mu, u_\mu-u_0)+\limsup_\mu(Au_\mu, u_0-u_r)\\ &\le\limsup_\mu(j(u_0)-j(u_\mu))+C\cdot\varepsilon.\end{aligned}$$

Since j is a lower semicontinuous function and ε is an arbitrary positive number, we have

$$\limsup_\mu(Au_\mu, u_\mu-u_r)\le 0. \qquad (12)$$

This together with A being a monotone hemicontinuous mapping implies that

$$\begin{aligned}&\liminf_\mu(Au_\mu, u_\mu-v)\\ &\qquad\ge(Au_r, u_r-v), \forall v\in K^r.\end{aligned} \qquad (13)$$

If $v\in\bigcup_{m=1}^{\infty}K_m$, it then follows from (6) that

$$(Au_\mu, u_\mu-v)\le j(v)-j(u_\mu) \qquad (14)$$

when μ is sufficiently large. It thus follows from (13) that

$$\begin{aligned}(Au_r, u_r-v)&\le\liminf_\mu(Au_\mu, u_\mu-v)\\ &\le\liminf_\mu(j(v)-j(u_\mu))\\ &\le j(v)-j(u_r), \forall v\in\bigcup_{m=1}^{\infty}K_m.\end{aligned} \qquad (15)$$

Because $\bigcup_{m=1}^{\infty}K_m$ is dense in K^r, the above inequality holds for all $v\in K^r$. therefore u_r is a solution of the variational inequality (10).

New we shall verify that the variational inequality (5) has a solution. Taking $v=v_0$ in (10), we have

$$(Au_r, u_r-v_0)+j(u_r)-j(v_0)\le 0 \qquad (16)$$

and so it then follows from condition (4) that there exists constant $C>0$ such that $\|u_r\|\le C$. Taking $r>C$ then $\|u_r\|<r$ and so u_r is an inner point of B_r. Thus for any given $\omega\in X$, we have $(1-t)u_r+t\omega\in B_r$ by taking $t\in(0,1)$ small enough. Let $v=(1-t)u_r+t\omega$ in (10), then we obtain

$$t(Au_r, \omega-u_r)+t(j(\omega)-j(u_r))\ge 0$$

by j being a convex lower semicontinuous function. Thus

$$(Au_r, \omega-u_r)+j(\omega)-j(u_r)\ge 0, \forall\omega\in X.$$

Therefore u_r is a solution of the variational inequality (5).

Theorem 1 *Let K be a nonempty unbounded closed convex set in $X=R^n$ with $0\in K$. Suppose that $X\to X^*$ is a bounded monotone hemicontinuous mappingwith $(Au,u)\ge 0(\forall u\in K)$ and $j:K\to(-\infty,+\infty]$ is a bounded proper convex lowersemicontinuous functional with $j(0)=0$ (i.e., for every bounded subset D of K, $j(D)$ is bounded). Give a continuous mapping $g:K\to X^*$ and $f\in X^*$. Assume*

a) $\displaystyle\lim_{\|u\|\to 0}\frac{(Au,u)+j(u)}{\|u\|}=+\infty$ *;*

b) there exists constant $\alpha\ge 0$ such that

$$\liminf_{\|u\|\to+\infty}\frac{(Au,u)+j(u)}{\|u\|^{\alpha+1}}>\limsup_{\|u\|\to+\infty}\frac{\|g(u)\|}{\|u\|^{\alpha}}(u\in K);$$

c) there exists a point $u_0\in rcK\setminus\{0\}$ such that $(f,u_0)\ne 0$

Then (2) *has a nonzero solution.*

Proof. It is easy to see from condition (b) and Lemma 1 that the variational inequality (3) has a solution in K for every $q\in X^*$. Define a mapping $K_Ag:K\to 2^K$ by

$$(K_Ag)(u):=K_A(g(u)), u\in K$$

Then K_Ag is an upper semi-continuous mapping with nonempty compact convex values by [10, Lemma 1]. Let $K^R=\{x\in K:\|x\|\le R\}$. We shall verify that $i_K(K_Ag, K^R)=1$ for large enough R and $i_K(K_Ag, K^r)=0$ for small enough r.

Firstly, define a mapping by $H:[0,1]\times\overline{K^R}\to 2^K$, $H(t,u)=tK_A(g(u))$. It is easily seen that $H(t,u)$ is an upper semicontinuous mapping with nonempty compact convex values. We claim that there exists large enough R such that $u\notin H(t,u)$ for all $t\in(0,1)$, $u\in\partial(K^R)$. Otherwise, there exist two sequences $\{t_n\},\{u_n\},t_n\in[0,1]$, $t_n\neq 0,\|u_n\|\to+\infty$ such that

$$u_n\in H(t_n,u_n)=t_nK_A(g(u_n)) \quad\text{or}\quad \frac{u_n}{t_n}\in K_A(g(u_n)).$$

Thus

$$\begin{aligned}&(A(\frac{u_n}{t_n}),v-\frac{u_n}{t_n})+j(v)+j(\frac{u_n}{t_n})\\&\ge(g(u_n),v-\frac{u_n}{t_n})+(f,v-\frac{u_n}{t_n}),\ \forall u\in K\end{aligned}\tag{17}$$

Letting $v=0$ and denoting $z_n=\frac{u_n}{\|u_n\|}$ in (17), we obtain from (17) that

$$\begin{aligned}&(\frac{t_n}{\|u_n\|})^{\alpha+1}(A(\frac{u_n}{t_n}),\frac{u_n}{t_n})+(\frac{t_n}{\|u_n\|})^{\alpha+1}j(\frac{u_n}{t_n})\\&\le t_n^{\alpha}(\frac{g(u_n)}{\|u_n\|^{\alpha}},z_n)+(\frac{t_n}{\|u_n\|})^{\alpha}(f,z_n)\end{aligned}\tag{18}$$

Denote $y_n=\frac{u_n}{t_n}\in K$. Then $\|y_n\|\to+\infty$ .We can obtain from (18) that

$$\begin{aligned}\frac{(Ay_n,y_n)+j(y_n)}{\|y_n\|^{\alpha+1}}&\le t_n^{\alpha}\left\|\frac{g(u_n)}{\|u_n\|^{\alpha}}\right\|+\frac{\|f\|}{\|y_n\|^{\alpha}}\\&\le\left\|\frac{g(u_n)}{\|u_n\|^{\alpha}}\right\|+\frac{\|f\|}{\|y_n\|^{\alpha}}.\end{aligned}\tag{19}$$

Hence we have

$$\liminf_{\|u\|\to+\infty}\frac{(Au,u)+j(u)}{\|u\|^{\alpha+1}}\le\limsup_{\|u\|\to+\infty}\frac{\|g(u)\|}{\|u\|^{\alpha}}$$

which contradicts to condition (b). Therefore

$$\begin{aligned}i_K(K_Ag,K^R)&=i_K(H(1,\cdot),K^R)\\&=i_K(H(0,\cdot),K^R)\\&=i_K(\hat{0},K^R)=1\end{aligned}\tag{20}$$

by **Proposition 1**(4) and (2).

Secondly, we shall verify that $i_K(K_Ag,K^r)=0$ for small enough r ($r<1$). In fact, there exist constants $C_1,C_2,M>0$ from the boundedness of j, locally boundedness of A and condition (b) such that for all $u\in K^1$, we have

$$|j(u+u_0)-j(u)|\le C_1,\|g(u)\|\le C_2,$$

$$\begin{aligned}&|(g(u),u_0)|\le C_2\|u_0\|,\|Au\|\le M,\\&|(Au,u_0)|\le M\|u_0\|\end{aligned}\tag{21}$$

Since $(f,u_0)\neq 0$, let $(f,u_0)<0$. Take N large enough such that

$$(1-N)(f,u_0)>C_1+(C_2+M)\|u_0\|\tag{22}$$

Define a mapping by $H[0,1]\times\overline{K^r}\to 2^K,H(t,u)=K_A(g(u)-tNf)$. Then H is an upper semi- continuous mapping with nonempty compact convex values. We claim that there exists a small enough r such that $u\notin H(t,u)$ for all $u\in\partial(K^r),t\in[0,1]$. Otherwise, there exist sequences $\{t_n\},\{u_n\},t_n\in[0,1]$,

$u_n\in\partial(K^r),\|u_n\|\to 0$ such that $u_n\in H(t_n,u_n)=K_A(g(u_n)-t_nNf)$. Thus

$$\begin{aligned}&(Au_n,v-u_n)+j(v)-j(u_n)\\&\ge(g(u_n)-Nt_nf,v-u_n)+(f,v-u_n),\forall v\in K\end{aligned}$$

Taking $v=0,z_n=\frac{u_n}{\|u_n\|}$, we have

$$\begin{aligned}&\frac{1}{\|u_n\|}(Au_n,u_n)+\frac{j(u_n)}{\|u_n\|}\\&\le(g(u_n),z_n)+(1-t_nN)(f,z_n)\end{aligned}$$

Since $\frac{(Au_n,u_n)+j(u_n)}{\|u_n\|}\to+\infty$ and

$$\begin{aligned}&(g(u_n),z_n)+(1-t_nN)(f,z_n)\\&\le\|g(u_n)\|+(1+N)\|f\|\\&\le C_2+(1+N)\|f\|,\end{aligned}$$

we obtain a contradiction. Therefore $i_K(K_Ag,K^r)=i_K(H(0,\cdot),K^r)=i_K(H(1,\cdot),K^r)$ by Proposition 1 (4). If $i_K(H(1,\cdot),K^r)\neq 0$, then the mapping $H(1,\cdot):K\to 2^K$ has a fixed point u in K^r by Proposition 1(1), *i.e.*, $u\in H(1,u)=K_A(g(u)-Nf)$.

Thus

$$\begin{aligned}&(Au,v-u)+j(v)-j(u)\\&\ge(g(u)-Nf,v-u)+(f,v-u),\forall v\in K\end{aligned}$$

Taking $v=u+u_0$, we have

$$\begin{aligned}&(Au,u_0)+j(u+u_0)-j(u)\\&\ge(g(u),u_0)+(1-N)(f,u_0)\end{aligned}\tag{23}$$

Hence

$$\begin{aligned}&(1-N)(f,u_0)\\&\le(Au,u_0)+j(u+u_0)-j(u)-(g(u),u_0)\\&\le M\|u_0\|+C_1+C_2\|u_0\|=(C_2+M)\|u_0\|+C_1\end{aligned}$$

by (21) and (23). That contradicts to (22). Therefore, $i_K(H(1,\cdot),K^r)=0$ and then $i_K(K_A g,K^r)=0$.

It follows from **Proposition 1**(3) that $i_K(K_A g, K^R\setminus K^r)=1$. Therefore there exists a fixed point $u\in K^R\setminus\overline{K^r}$ which is a nonzero solution of (2).

3. Nonzero Solutions in Reflexive Banach Spaces

Theorem 2 *Let X be a reflexive Banach space and $K\subset X$ a nonempty unbounded closed convex set with $0\in K$. Suppose that $A:X\to X^*$ is a bounded monotone hemicontinuous mapping with $(Au,u)\ge 0$ for $u\in K$ and $j:K\to(-\infty,+\infty]$ is a bounded convex lower semicontinuous functional with $j(0)=0$. Assume that $g:K\to X^*$ is continuous from the weak topology on X to the strong topology on X^*. Give $f\in X^*$. The following conditions are assumed to be satisfied*

a) $(f,u_0)\ne 0$ for some $u_0\in rcK\setminus\{0\}$;

b) there constant $\alpha\ge 0$ such that $\liminf\limits_{\|u\|\to+\infty}\dfrac{(Au,u)+j(u)}{\|u\|^{\alpha+1}}>\limsup\limits_{\|u\|\to+\infty}\dfrac{\|g(u)\|}{\|u\|^{\alpha}}(u\in K)$;

c) $\liminf\limits_{u_s\xrightarrow{w}0} j(u_s)>0$.

Then (2) *has a nonzero solution.*

Proof. It is easily seen that $\lim\limits_{\|u\|\to 0}\dfrac{A(u,u)+j(u)}{\|u\|}=+\infty$ by the condition (c). Let $F\subset X$ be a finite dimensional subspace containing u_0. We shall show that all conditions in Theorem 1 are satisfied on space F.

Denote $K_F=K\cap F$ which is a nonempty unbounded closed convex set. Let $j_F:F\to X$ be an injective mapping and $j_F^*:X^*\to F^*$ its dual mapping. Denote $A_F=j_F^*(A|F):F\to F^*, g_F=j_F^*(g|K_F):K_F\to F^*$. We know that $A_F=j_F^*Aj_F$, $g_F=j_F^*gj_F$. Then, A_F,g_F are hemicontinuous and continuous respectively.

For $x_1,x_2\in K_F$, we have

$$\begin{aligned}&(A_F(x_1)-A_F(x_2),x_1-x_2)\\&=(j_F^*A(x_1)-j_F^*A(x_2),x_1-x_2)\\&=(Ax_1-Ax_2,j_F(x_1-x_2))\\&=(Ax_1-Ax_2,x_1-x_2)\ge 0\end{aligned}$$

by the monotony of A. This means that A_F is monotone. On the other hand, $j_F^*f\in F^*$ and $(j_F^*f,u_0)=(f,j_Fu_0)=(f,u_0)\ne 0$. Similarly, we have

$$\liminf_{\|u\|\to+\infty}\frac{(A_Fu,u)+j(u)}{\|u\|^{\alpha+1}}>\limsup_{\|u\|\to+\infty}\frac{\|g_F(u)\|}{\|u\|^{\alpha}}\quad(u\in K_F).$$

Therefore all conditions in Theorem 1 are satisfied on space F and so there exists $u_F\in K_F, u_F\ne 0$ such that

$$\begin{aligned}&(A_F(u_F),v-u_F)+j(v)-j(u_F)\\&\ge(g_F(u_F),v-u_F)+(j_F^*f,v-u_F),\forall v\in K_F\end{aligned}$$

It yields that

$$\begin{aligned}&(A(u_F),v-u_F)+j(v)-j(u_F)\\&\ge(g(u_F),v-u_F)+(f,v-u_F),\forall v\in K_F\end{aligned}$$

Taking $v=0$, we get

$(Au_F,u_F)+j(u_F)\le(g(u_F),u_F)+(f,u_F)$. Hence

$$\frac{(Au_F,u_F)+j(u_F)}{\|u_F\|^{\alpha+1}}\le\frac{\|g(u_F)\|}{\|u_F\|^{\alpha}}+\frac{\|f\|}{\|u_F\|^{\alpha}}.$$

This together with condition (b) implies that there exists a constant $M>0$ such that $\|u_F\|\le M$ for all finite dimensional subspace F containing u_0. Since X is reflexive and K is weakly closed, with a similar argument to that in the proof of Theorem 2 in [10] (also see [8]), we shall show that there exists $u'\in K$ such that for every finite dimensional subspace F containing u_0, u' is in the weak closure of the set $V_F=\bigcup\limits_{F\subset F_1}\{u_{F_1}\}$ where F_1 is a finite dimensional subspace in X.

In fact, since V_F is bounded, we know that $\overline{(V_F)}^w$ (the weak closure of the set V_F) is weakly compact. On the other hand, let $F^1,F^2,\cdots,F^m$ be finite dimensional subspaces containing u_0. Define $F^{(m)}:=span\{F^1,F^2,\cdots,F^m\}$. Then $F^{(m)}$ containing u_0 is a finite dimensional subspace. Hence,

$\bigcap\limits_{i=1}^m V_{F^i}=\bigcap\limits_{i=1}^m(\bigcup\limits_{F^i\subset F_1}\{u_{F_1}\})=\bigcup\limits_{F^{(m)}\subset F_1}\{u_{F_1}\}\ne\varnothing$, then $\bigcap\limits_F\overline{(V_F)}^w\ne\varnothing$. That is to say, there exists $u'\in K$ such that for every finite dimensional subspace F containing u_0, u' is in the weak closure of the set $V_F=\cup_{F\subset F_1}\{u_{F_1}\}$.

Now let $v\in K$ and F' a finite dimensional subspace of X which contains u_0 and v. Since u' belongs to the weak closure of the set $V_{F'} = \bigcup_{F' \subset F_1} \{u_{F_1}\}$. We may find a sequence $\{u_{F_\alpha}\}$ in $V_{F'}$ such that $u_{F_\alpha} \xrightarrow{w} u'$. However, u_{F_α} satisfies the following inequality

$$\begin{aligned}&(Au_{F_\alpha}, v - u_{F_\alpha}) + j(v) - j(u_{F_\alpha}) \\ &\quad \geq (g(u_{F_\alpha}), v - u_{F_\alpha}) + (f, v - u_{F_\alpha})\end{aligned} \tag{24}$$

The monotony of A implies that

$$\begin{aligned}&(Av, v - u_{F_\alpha}) + j(v) - j(u_{F_\alpha}) \\ &\quad \geq (g(u_{F_\alpha}), v - u_{F_\alpha}) + (f, v - u_{F_\alpha})\end{aligned}$$

Letting $u_{F_\alpha} \xrightarrow{w} u'$ yields that

$$\begin{aligned}&(Av, v - u') + j(v) - j(u') \\ &\quad \geq (g(u'), v - u') + (f, v - u'), \forall v \in K\end{aligned}$$

Thus

$$\begin{aligned}&(Au', v - u') + j(v) - j(u') \\ &\quad \geq (g(u'), v - u') + (f, v - u'), \forall v \in K\end{aligned}$$

by Minty's Theorem [2,3]. We claim that $u' \neq 0$. Otherwise, $u_{F_\alpha} \xrightarrow{w} 0$. Taking $v = 0$ in (24) yields that

$$\begin{aligned}j(u_{F_\alpha}) &\leq -(Au_{F_\alpha}, u_{F_\alpha}) + (g(u_{F_\alpha}), u_{F_\alpha}) + (f, u_{F_\alpha}) \\ &\leq (g(u_{F_\alpha}), u_{F_\alpha}) + (f, u_{F_\alpha})\end{aligned}$$

The right side of the above inequality tends to 0, which contradicts to the condition (c). Therefore u' is a nonzero solution of (2).

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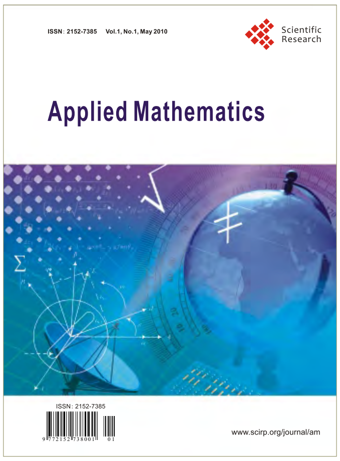

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