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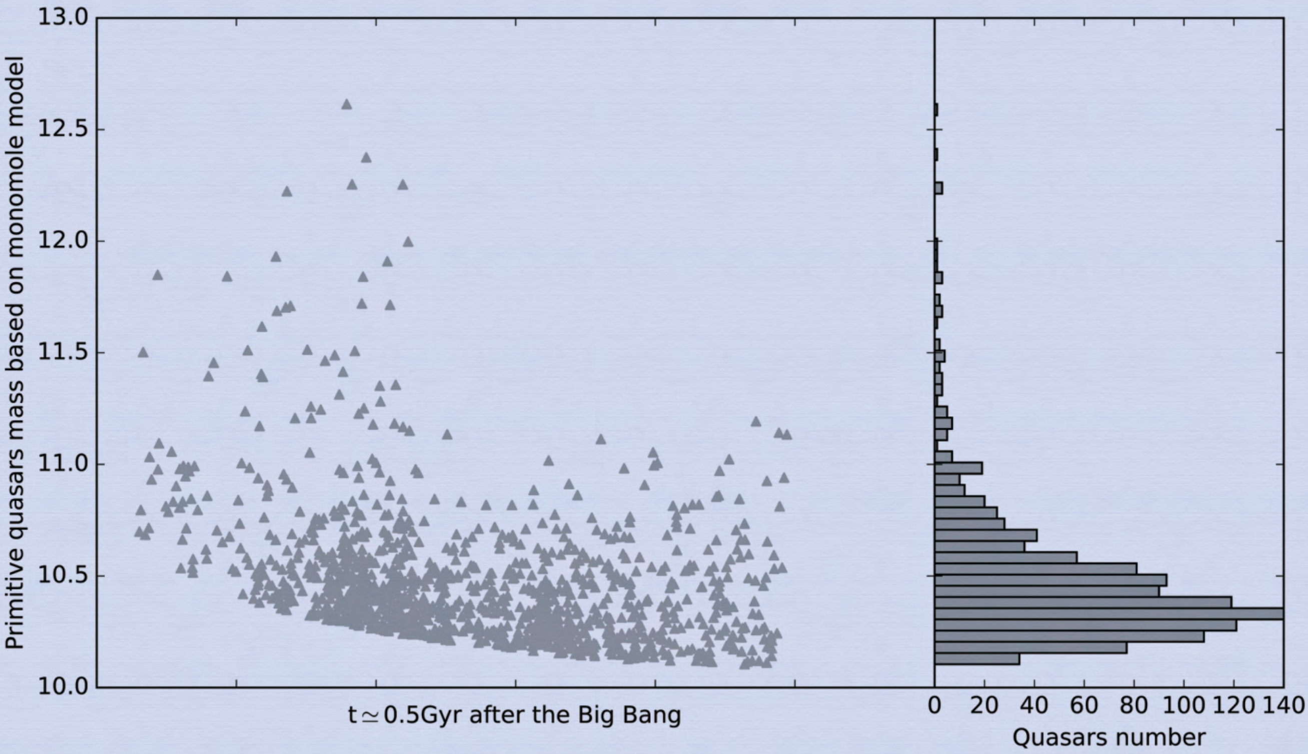


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# Table of Contents

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# To the Complete Set of Equations for a Static Problem of General Relativity

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## Abstract

The paper is concerned with the formulation of the static problem of general relativity. As known, this problem is reduced to ten equations for the components of the Einstein tensor and the solution of these equations is associated with two principal problems. First, since the components of the Einstein tensor identically satisfy four conservation equations, only six of these equations are mutually independent. So, the set of the Einstein equations actually contains six independent equations for ten components of the metric tensor and should be supplemented with four additional equations which are missing in the original theory. Second, for a deformable solid the Einstein tensor is associated with the energy tensor which is expressed in terms of six stresses induced by gravitation. These stresses are not known and the relativity theory does not propose any equations for them. Thus, the static problem of general relativity cannot be properly formulated because the set of governing equations is not complete. In the paper, the problem of completeness of the general relativity governing set of equations is analyzed in application to the spherically symmetric static problem and the proposed approach is further described for the general case. As an example, linearized axisymmetric problem is considered.

## Keywords

General Relativity, Coordinate Conditions, Compatibility Stress Equations, Spherically Symmetric Problem

## 1. Introduction. General Relativity Equations

The Einstein equation which specifies the Einstein tensor has the following form:

$$E_i^j = R_i^j - \frac{1}{2}\delta_i^j R, \quad (i, j = 1,2,3,4) \tag{1}$$

in which $R_i^j$ ($R = R_i^i$) are the components of the Ricci curvature tensor (we use mixed components because for the spherically symmetric problem considered further they coincide with the physical components). The Einstein tensor is associated with the energy tensor as

$$E_i^j = \chi T_i^j \tag{2}$$

where

$$\chi = 8\pi G/c^4 \tag{3}$$

is the relativity gravitational constant expressed in terms of the classical constant $G$ and the velocity of light $c$. The energy tensor expressed with the aid of Equation (1) and Equation (2) identically satisfies the conservation equations

$$T_{i,k}^k = 0 \tag{4}$$

For the static problem,

$$T_i^j = \sigma_i^j, \quad T_i^4 = 0 \quad (i, j = 1,2,3), \quad T_4^4 = \mu c^2 \tag{5}$$

where $\sigma_i^j$ is the stress tensor and $\mu$ is the density.

Consider two problems associated with the formulation of the general relativity static problem. First, substituting Equations (2) in Equation (1), we arrive at ten equations for ten components of the metric tensor in four-dimensional Riemannian space. However, the right parts of these equations identically satisfy Equations (4) which means that only six of ten Equation (1) are mutually independent. Thus, we have six equations for ten unknown functions. The additional equations which are usually referred to as coordinate conditions should be imposed on the metric tensor. As known, the metric tensor of the Euclidean space must satisfy the Lame equations. Such equations do not exist for the Riemannian space. However, we can suppose that the Riemannian space induced by gravitation cannot be arbitrary and must be somehow restricted, e.g., by coordinate conditions. The necessity of such conditions was first mentioned by D. Hilbert [1]. By now the widely recognized general form of these conditions has not been proposed. Existing particular conditions are discussed further in application to the spherically symmetric problem.

Second, changing $E$ to $T$ and then to $\sigma$ in Equations (1), we arrive at the set of equations containing stresses on the left sides. These stresses are not known. Traditionally, Equation (1) is used to study the gravitation in the empty space for which $T_i^j = 0$ and Equation (1) are homogeneous. For solids the stresses are not zero and the equations cannot be solved. The solution can be obtained if the solid is simulated with perfect fluid. In this case the nonzero stresses $\sigma_1^1 = \sigma_2^2 = \sigma_3^3 = -p$, where $p$ is the pressure that can be found from the corresponding equation of Equation (4). However, in equilibrium state, the perfect fluid can have only spherical shape and the solution can be obtained for this particular case only. For the general case, consider the analogy with the theory of elasticity. Conservation equations, Equation (4), correspond to equilibrium equations of this theory, whereas Equation (1) is similar to the equations which allow us to express the stresses in terms of stress functions and to satisfy the equilibrium equations. Thus, the metric tensor of general relativity is analogous to the tensor of stress functions in theory of elasticity. In this theory, the stress functions are found from the compatibility equations which postulate that the geometry of the stressed solid is Euclidean. In general relativity, the geometry is Riemannian and the compatibility equations of the theory of elasticity cannot be directly applied.

Thus, the traditional set of the general relativity equations is not complete and should be supplemented with some additional equations that are discussed further. The proposed approach is demonstrated in application to the spherically symmetric problem which has the exact solution.

## 2. Spherically Symmetric Static Problem

### 2.1. Classical Linear Solution

For comparison with the general relativity solutions that are discussed further, consider the problem of the theory of elasticity for a linear elastic isotropic solid sphere loaded with gravitation forces following from the Newton theory. For a sphere with constant density $\mu$, the gravitational potential $\varphi$ is the solution of the Poisson equation

$$\Delta\varphi = \varphi'' + \frac{2}{r}\varphi' = 4\pi G\mu \tag{6}$$

Here, $(\cdots)' = \mathrm{d}(\cdots)/\mathrm{d}r$ and $r$ is the radial coordinate ($0 \le r \le R$). For the external space ($r \ge R$, index "$e$"), $\mu = 0$ and the solution of Equation (6) is $\varphi_e = -Gm/r$ in which

$$m = \frac{4}{3}\pi R^3 \mu \tag{7}$$

is the sphere mass. Introduce the so-called gravitational radius

$$r_g = \frac{2Gm}{c^2} \tag{8}$$

Then, $\varphi_e = -r_g c^2 / 2r$. For the internal space ($0 \le r \le R$, index "$i$"), the regular solution of Equation (6) is

$$\varphi_i = \frac{2}{3}\pi G\mu r^2 + C$$

Determining constant $C$ from the boundary condition $\varphi_i(R) = \varphi_e(R)$ and using Equation (8), we get

$$\varphi_i = -\frac{r_g c^2}{4R}\left(3 - \frac{r^2}{R^2}\right) \tag{9}$$

The equilibrium equation for the sphere under the action of gravitational body forces $f_g = -\mu\varphi_i'$ is

$$\sigma_r' + \frac{2}{r}(\sigma_r - \sigma_\theta) - \frac{\mu r_g c^2}{2R^3} r = 0 \tag{10}$$

where $\sigma_r$ and $\sigma_\theta$ are the radial and the circumferential stresses. The second equation for the stresses follows from the compatibility equation. There are two ways to derive this equation. First, introduce the corresponding strains expressed in terms of the radial displacement $u$ as

$$\varepsilon_r = u', \quad \varepsilon_\theta = u/r \tag{11}$$

The compatibility equation follows from these equations and has the form

$$\left(r\varepsilon_\theta\right)' = \varepsilon_r \tag{12}$$

Express the strains in terms of stresses with the aid of Hooke's law

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - 2\nu\sigma_\theta\right), \quad \varepsilon_\theta = \frac{1}{E}\left[\left(1-\nu\right)\sigma_\theta - \nu\sigma_r\right] \tag{13}$$

in which $E$ is the elastic modulus and $\nu$ is the Poisson's ratio. Substituting the strains in Equation (12), we finally get

$$r\left[\left(1-\nu\right)\sigma'_\theta - \nu\sigma_r\right] + \left(1+\nu\right)\left(\sigma_\theta - \sigma_r\right) = 0 \tag{14}$$

Equation (12) means that the geometry of the deformed sphere is Euclidean. In general relativity, the geometry is Riemannian, the displacement $u$ and Equations (11) do not exist. However, there is the second way to obtain Equation (14) not attracting Equations (11). This approach is based on the principle of minimum of the complementary energy

$$U = \frac{2\pi}{E}\int_0^R \left[\sigma_r^2 + 2\left(1-\nu\right)\sigma_\theta^2 - 4\nu\sigma_r\sigma_\theta\right] r^2 \mathrm{d}r$$

under the condition that the stresses satisfy the equilibrium equation, Equation (10). Introducing this equation with the aid of the Lagrange multiplier $\lambda$, construct the augmented functional

$$U = \frac{2\pi}{E}\int_0^R F \mathrm{d}r,$$

$$F = \left[\sigma_r^2 + 2\left(1-\nu\right)\sigma_\theta^2 - 4\nu\sigma_r\sigma_\theta\right] r^2 + \lambda\left(r\right)\left[\sigma'_r + \frac{2}{r}\left(\sigma_r - \sigma_\theta\right) - \frac{\mu r_g c^2}{2R^3} r\right] \tag{15}$$

The Euler equations providing $\delta U = 0$

$$\frac{\partial F}{\partial \sigma_r} - \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\partial F}{\partial \sigma'_r}\right) = 0, \quad \frac{\partial F}{\partial \sigma_\theta} = 0 \tag{16}$$

take the form

$$2\left(\sigma_r - 2\nu\sigma_\theta\right) r^2 + \frac{2}{r}\lambda - \lambda' = 0, \quad 2\left(1-\nu\right)\sigma_\theta - 2\nu\sigma_r - \frac{\lambda}{r^3} = 0 \tag{17}$$

Expressing $\lambda$ from the second equation and substituting in the first equation, we arrive at the compatibility Equation (14).

Thus, we get two equations, Equation (10) and Equation (14) for two stresses. The final solution which satisfies the boundary condition $\sigma_r\left(R\right) = 0$ is

$$\bar{\sigma}_r = -k\bar{r}_g\left(1-\bar{r}^2\right), \quad \bar{\sigma}_\theta = -k\bar{r}_g\left(1+\frac{1+\nu}{3-\nu}\bar{r}^2\right), \quad k = \frac{3-\nu}{20\left(1-\nu\right)} \tag{18}$$

Here,

$$\bar{\sigma}_r = \frac{\sigma_r}{\mu c^2}, \quad \bar{\sigma}_\theta = \frac{\sigma_\theta}{\mu c^2}, \quad \bar{r}_g = \frac{r_g}{R}, \quad \bar{r} = \frac{r}{R} \tag{19}$$

For a sphere of perfect fluid, $\sigma_r = \sigma_\theta = -p$ and the pressure $p$ can be found from the equilibrium Equation (10) not attracting the compatibility Equation (14). The result is

$$\bar{p} = \frac{\bar{r}_g}{4}\left(1 - \bar{r}^2\right) \tag{20}$$

In general relativity, the space geometry is Riemannian and the line element in spherical coordinates $r, \theta, \varphi$ is taken in the form

$$\mathrm{d}s^2 = g_{11}\mathrm{d}r^2 + g_{22}\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) - g_{44}c^2\mathrm{d}t^2 \tag{21}$$

The components of the metric tensor depend on the radial coordinate only. For the foregoing linear solution, these components are [2]

$$g_{11} = 1 + \frac{\bar{r}_g}{\bar{r}}, \quad g_{22} = r^2 \tag{22}$$

In case $\bar{r}_g = 0$, the space is Euclidean and gravitation vanishes. For real objects, the ratio $\bar{r}_g$, as a rule, is extremely small. For example, for Earth $\bar{r}_g = 1.4 \times 10^{-6}$, for Sun $\bar{r}_g = 4.3 \times 10^{-6}$, for the largest of the observed visible stars—red supergiant UI Scutti ( $R = 11.9 \times 10^{11}\,\mathrm{m}$, $m = 64 \times 10^{30}\,\mathrm{kg}$ [3]) $\bar{r}_g = 8 \times 10^{-9}$.

### 2.2. General Relativity Solution

For a spherically symmetric problem, the field equations following from Equation (1), Equation (2) and Equation (5) reduce to

$$\chi\sigma_r = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 + \frac{g'_{22}g'_{44}}{2g_{22}g_{44}}\right] \tag{23}$$

$$\begin{aligned}\chi\sigma_\theta = -\frac{1}{2g_{11}}\Bigg[&\frac{g''_{44}}{g_{44}} - \frac{1}{2}\left(\frac{g'_{44}}{g_{44}}\right)^2 + \frac{g''_{22}}{g_{22}} - \frac{1}{2}\left(\frac{g'_{22}}{g_{22}}\right)^2 \\ &+ \frac{g'_{22}}{2g_{22}}\left(\frac{g'_{44}}{g_{44}} - \frac{g'_{11}}{g_{11}}\right) - \frac{g'_{11}g'_{44}}{2g_{11}g_{44}}\Bigg]\end{aligned} \tag{24}$$

$$\chi\mu c^2 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g''_{22}}{g_{22}} - \frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 - \frac{g'_{11}g'_{22}}{2g_{11}g_{22}}\right] \tag{25}$$

The only one conservation equation, Equation (4), becomes

$$\sigma'_r + \frac{g'_{22}}{g_{22}}\left(\sigma_r - \sigma_\theta\right) + \frac{g'_{44}}{2g_{44}}\left(\sigma_r - \mu c^2\right) = 0 \tag{26}$$

The solution of the external ( $r \ge R$ ) problem must satisfy the asymptotic conditions and to reduce to Equation (22) for $r \to \infty$. The solution for the internal ( $0 \le r \le R$ ) problem must satisfy the symmetry condition at the sphere center according to which $g_{11}(0) = 1$, $g_{22}(0) = 0$. Both solutions must satisfy the boundary conditions on the sphere surface, *i.e.*

$$g_{11}^{e}(R)=g_{11}^{i}(r),\quad g_{22}^{e}(R)=g_{22}^{i}(R),\quad g_{44}^{e}(r)=g_{44}^{i}(R) \tag{27}$$

As in the general case (Section 1), substitution of the left parts of Equations (23)-(25) in Equation (26) identically satisfies this equation. So, only three of four Equations (23)-(26) are mutually independent. Traditionally [4], the simplest set of equations including Equation (23), Equation (25) and Equation (26) is used. The obtained solution identically satisfies Equation (24).

To solve the problem, we should supplement Equation (23), Equation (25) and Equation (26) which include three components of the metric tensor and two stresses with one coordinate condition for the metric tensor and one equation for the stresses. The first coordinate condition was proposed by K. Schwarzchild [5] who changed the spherical coordinates to $x_1=r^3/3$, $x_2=-\cos\theta$, $x_3=\varphi$, $x_4=t$ and applied the condition $g=1$, where $g$ is the determinant of the metric tensor components in coordinates $x_i$. This condition is equivalent to $g_{22}=r^2$ [6] and reduces the order of Equation (25). As a result, the solution does not contain the proper number of integration constants and the first boundary condition in Equation (27) cannot be satisfied [6]. The internal problem was solved for a sphere of perfect fluid [7] and did not require the additional equation. The other way involves the application of the so-called harmonic coordinate conditions which in the general case have the following form [8]

$$\frac{\partial}{\partial x^{i}}\left(\sqrt{g}\,g^{ij}\right)=0$$

External spherically symmetric problem was solved with the harmonic coordinate condition by V. Fock [9]. Internal problem and boundary conditions were not considered.

To obtain the general solution of the spherically symmetric static problem, apply the set of Equation (23), Equation (25) and Equation (26). To simplify these equations, introduce new notations for the components of the metric tensor, *i.e.*, put $g_{11}=q^2$, $g_{22}=\rho^2$, $g_{44}=h^2$. Then, Equation (23), Equation (25) and Equation (26) reduce to

$$\chi\sigma_r=\frac{1}{\rho^2}-\frac{\rho'}{q^2\rho}\left(\frac{\rho'}{\rho}+\frac{2h'}{h}\right),\quad \chi\mu c^2=\frac{1}{\rho^2}-\frac{1}{q^2}\left[\left(\frac{\rho'}{\rho}\right)^2+\frac{2\rho''}{\rho}-\frac{2q'\rho'}{q\rho}\right] \tag{28}$$

$$\sigma_r'+\frac{2\rho'}{\rho}\left(\sigma_r-\sigma_\theta\right)+\frac{h'}{h}\left(\sigma_r-\mu c^2\right)=0 \tag{29}$$

For the external space ($\sigma_r=\sigma_\theta=0,\mu=0$), the solution of Equation (28) which satisfies the asymptotic conditions for $r\to\infty$ is [10]

$$q_e^2=\frac{\rho_e\left(\rho_e'\right)^2}{\rho_e-r_g},\quad h_e^2=1-\frac{r_g}{\rho_e} \tag{30}$$

To determine $\rho_e(r)$, we need to add a coordinate condition. Introduce a new interpretation of the Riemannian space [6] [11] according to which it is a mathematical model of the actual nonhomogeneous Euclidean space characterized with the space density $d=\sqrt{g_R/g_E}$ in which $g_R$ and $g_E$ are the determinants of the metric tensors in Riemannian and Euclidean three-dimensional spaces in the same coordinates. Assume that in space coordinates $x^1,x^2,x^3$ the space density satisfies the following variational equation:

$$\delta D=0\,,\quad D=\iiint d\sqrt{g_E}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3=\iiint\sqrt{g_R}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 \tag{31}$$

Equation (31), written for a four-dimensional space, is known in general relativity as a possible way to derive the field equations [12]. However, if the variations $\delta g_{ij}$ are mutually independent, $\delta D\neq 0$. The situation becomes different if these variations are not independent. For spherically symmetric problem,

$$d_e=\frac{q_e\rho_e^2}{r^2}=\frac{\rho_e'\rho_e^2}{r^2\sqrt{1-r_g/\rho_e}} \tag{32}$$

The Euler equation similar to Equation (16) is satisfied identically which means that $\delta D=0$ for any function $\rho_e(r)$. To specify this function, we can minimize $d_e$ taking $d_e=1$. Thus, it is assumed that gravitation transforming Euclidean space into Riemannian does not change the density of the metric tensor. For $d_e=1$, Equation (32) yields the following equation for $\rho_e(r)$:

$$\rho_e'\rho_e^2=r^2\sqrt{1-r_g/\rho_e} \tag{33}$$

For the internal space ( $\mu=const$ ), the solution of the second equation in Equation (28) which satisfies the regularity condition at the sphere center is [10]

$$q_i^2=\frac{(\rho_i')^2}{1-u\rho_i^2}\,,\quad u=\frac{1}{3}\chi\mu c^2 \tag{34}$$

As for the external space, take Equation (31) as the coordinate condition in which

$$d_i=\frac{q_i\rho_i^2}{r^2}=\frac{\rho_i'\rho_i^2}{r^2\sqrt{1-u\rho_i^2}} \tag{35}$$

where $d_i$ is the space density. The Euler equation is satisfied identically and we can take $d_i=1$. This means that that the sphere mass is not affected by gravitation and is specified by Equation (7). Using this equation in conjunction with Equation (3) and Equation (8) for $\chi$ and $r_g$, we get $u=r_g/R^3$. Then, Equation (34) and Equation (35) take the following final forms:

$$q_i^2=\frac{(\rho_i')^2}{1-r_g\rho_i^2/R^3}\,,\quad d_i=\frac{\rho_i'\rho_i^2}{r^2\sqrt{1-r_g\rho_i^2/R^3}}$$

Putting $d_i=1$, we arrive at the following equation for $\rho_i(r)$:

$$\rho_i'\rho_i^2=r^2\sqrt{1-r_g\rho_i^2/R^3}$$

The solution of this equation which satisfies the boundary condition $\rho_i(r=0)=0$ is [10]

$$\frac{1}{\sqrt{\bar{r}_g}}\sin^{-1}\left(\bar{\rho}_i\sqrt{\bar{r}_g}\right)-\bar{\rho}_i\sqrt{1-\bar{r}_g\bar{\rho}_i^2}=\frac{2}{3}\bar{r}_g\bar{r}^3 \tag{36}$$

where in addition to Equation (19) $\bar{\rho} = \rho/R$. For the sphere surface $\bar{r} = 1$, $\bar{\rho}_i = \bar{\rho}_1$, and Equation (36) yields

$$\frac{1}{\sqrt{\bar{r}_g}} \sin^{-1}\left(\bar{\rho}_1 \sqrt{\bar{r}_g}\right) - \bar{\rho}_1 \sqrt{1 - \bar{r}_g \bar{\rho}_1^2} = \frac{2}{3} \bar{r}_g \tag{37}$$

The general solution of Equation (33) is [10]

$$\left(\frac{1}{3}\bar{\rho}_e^2 + \frac{5}{12}\bar{r}_g \bar{\rho}_e + \frac{5}{8}\bar{r}_g^2\right)\sqrt{\bar{\rho}_e\left(\bar{\rho}_e - \bar{r}_g\right)} + \frac{5}{8}\bar{r}_g^3 \ln\left(\sqrt{\bar{\rho}_e} + \sqrt{\bar{\rho}_e - \bar{r}_g}\right) = \frac{1}{3}\bar{r}^3 + C \tag{38}$$

The integration constant can be found from the boundary condition on the sphere surface according to which $\bar{\rho}_e\left(\bar{r} = 1\right) = \bar{\rho}_1$. Then,

$$C = \left(\frac{1}{3}\bar{\rho}_1^2 + \frac{5}{12}\bar{r}_g \bar{\rho}_1 + \frac{5}{8}\bar{r}_g^2\right)\sqrt{\bar{\rho}_1\left(\bar{\rho}_1 - \bar{r}_g\right)} + \frac{5}{8}\bar{r}_g^3 \ln\left(\sqrt{\bar{\rho}_1} + \sqrt{\bar{\rho}_1 - \bar{r}_g}\right) - \frac{1}{3} \tag{39}$$

For $\bar{r}_g \ll 1$, Equation (36) and Equation (38) reduce to $\bar{\rho}_i = \bar{\rho}_e = \bar{r}$.

Thus, the functions $\rho_i(r)$ and $\rho_e(r)$ are specified by Equation (36) and Equation (38). The obtained solution satisfies the asymptotic and the boundary conditions in Equation (27) [10]. As follows from Equation (39), the solution exists if $\bar{\rho}_1 \ge \bar{r}_g$. Otherwise, the solution becomes imaginary. The minimum possible value of $\bar{\rho}_1$ is $\bar{r}_g$. Assume that this minimum value correspond to the sphere radius $R_g$. Then, substituting $\bar{\rho}_1 = \rho_1/R_g = r_g/R_g$ in Equation (37), we get

$$\sqrt{\frac{R_g}{r_g}} \sin^{-1}\left(\frac{r_g}{R_g}\sqrt{\frac{r_g}{R_g}}\right) - \frac{r_g}{R_g}\sqrt{1 - \left(\frac{r_g}{R_g}\right)^3} = \frac{2r_g}{3R_g}$$

The solution of this equation is $R_g = 1.115 r_g$. Thus, the obtained solution gives the critical radius which is larger than the gravitational radius. In contrast to the Schwarzchild solution, for the sphere with the critical radius $R_g$ the solution is not singular and gives finite values for the metric coefficients. Particularly, for $\rho = \rho_1$ we get $q_e = q_i = 1.243$ and $\rho_e = \rho_i = 0.8968R$. For $R < R_g$, the solution becomes imaginary which means that the general relativity is not valid for such high levels of gravitation. Dependences of the space metric coefficients on the radial coordinate for the sphere with the critical radius $R_g$ is shown in **Figure 1**.

As can be seen, $g(r \to \infty) = 1$ and $\rho(r \to \infty) = r$ (dashed line in **Figure 1**).

Consider the propagation of light from the sphere surface. The trajectory of light in the equatorial ($\theta = \pi/2$) plane is specified by the following equations [13]:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\frac{h_e}{q_e}\sqrt{1 - \left(\frac{sh_e}{\rho_e}\right)^2}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = cs\left(\frac{h_e}{\rho_e}\right)^2 \tag{40}$$

in which $q_e$ and $h_e$ are given by Equation (30), $\rho_e(r)$ is the solution of Equation (38) and $s$ is the integration constant that can be found from the initial condition. Physical components of the velocity are

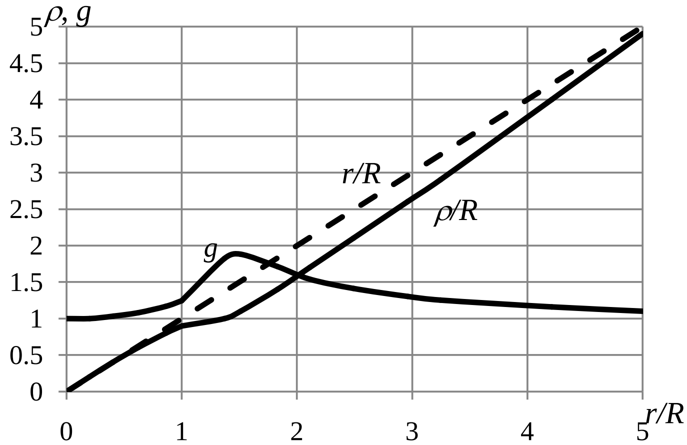


**Figure 1.** Dependences of the space metric coefficients on the radial coordinate for the sphere with the critical radius.

$$v_r = \frac{g_e}{h_e}\frac{\mathrm{d}r}{\mathrm{d}t} = c\sqrt{1-\left(\frac{sh_e}{\rho_e}\right)^2}, \quad v_\varphi = c\frac{sh_e}{\rho_e} \tag{41}$$

so that $v_r^2 + v_\varphi^2 = c^2$. Assume that light propagates from point A on the sphere surface ($\bar{\rho} = \bar{\rho}_1$, $\bar{r} = 1$) at angle $\alpha$ with respect to the radius (**Figure 2**).

The initial conditions are $v_r = c\cos\alpha$, $v_\varphi = c\sin\alpha$ and Equations (41) yield $s = \rho_1 \sin\alpha / h_1$, where $h_1 = h_e(\rho_e = \rho_1)$. Consider the case $\alpha = 0$ for which $s = 0$ and $v_r = c, v_\varphi = 0$. This result allows us to conclude that in the radial direction light propagates for any spherical object with velocity *c*. However, if $\alpha \neq 0$, the situation can be different. Using Equation (30) and Equation (40), we can obtain the following equation for the light trajectory:

$$\frac{\mathrm{d}\rho_e}{\mathrm{d}\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\rho_e}{\mathrm{d}r} = \rho_e^2\sqrt{\frac{1}{\rho_1^2\sin^2\alpha}\left(1-\frac{r_g}{\rho_1}\right) - \frac{1}{\rho_e^2}\left(1-\frac{r_g}{\rho_e}\right)} \tag{42}$$

Numerical integration of this equation allows us to plot $\rho_e(\varphi)$. Using further Equation (38), we change $\rho_e$ to *r*. The resulting trajectory $\bar{r}(\varphi)$ for $\alpha = \pi/2$ and $\bar{r}_g = 0.5$ is shown in **Figure 2** with the dashed line. As follows from Equation (42), for the sphere with the critical radius, $\rho_1 = r_g$ the trajectory becomes imaginary and light does not propagate.

Return to the internal problem and determine the stresses. Consider the first equation in Equation (28). Substitute $q_i$ from Equation (34) and express

$$\frac{h_i'}{h_i} = \frac{\rho_i \rho_i'(u - \chi\sigma_r)}{2(1-u\rho_i^2)} \tag{43}$$

Here, $u = \chi\mu c^2/3 = r_g/R^3$. Substitute Equation (43) in Equation (29) to get

$$\sigma_r' - \frac{2\rho_i'}{\rho_i}(\sigma_r - \sigma_\theta) + \frac{\rho_i\rho_i'}{2(1-u\rho_i^2)}(u-\chi\sigma_r)(\sigma_r - \mu c^2) = 0 \tag{44}$$

The first equation for the stresses follows from this equation if we change variable *r* to $\rho$, use notations (19), and Equation (3), Equation (7) for $\chi$ and $r_g$, *i.e.*,

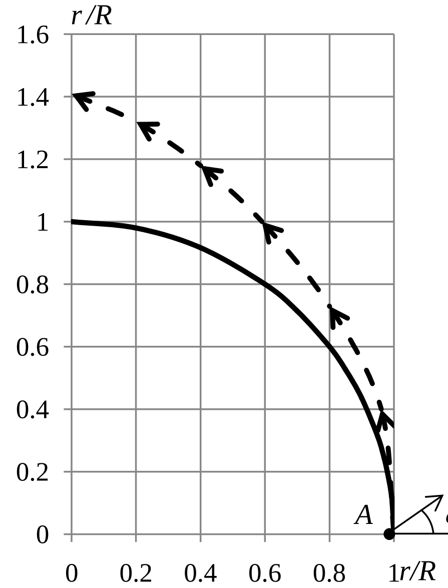


**Figure 2.** Propagation of light from the sphere surface.

$$\frac{d\bar{\sigma}_r}{d\bar{\rho}_i}+\frac{2}{\bar{\rho}_i}(\bar{\sigma}_r-\bar{\sigma}_\theta)-\frac{\bar{r}_g\bar{\rho}_i}{2(1-\bar{r}_g\bar{\rho}_i^2)}(1-3\bar{\sigma}_r)(1-\bar{\sigma}_r)=0 \tag{45}$$

For a sphere of perfect fluid, $\sigma_r=\sigma_\theta=-p$ and Equation (45) reduces to

$$\frac{d\bar{p}}{d\bar{\rho}_i}+\frac{\bar{r}_g\bar{\rho}_i}{2(1-\bar{r}_g\bar{\rho}_i^2)}(1+3\bar{p})(1+\bar{p})=0$$

The solution of this equation which satisfies the boundary condition $\bar{p}(\bar{\rho}_i=\bar{\rho}_1)=0$ is [10]

$$\bar{p}=\frac{\sqrt{1-\bar{r}_g\bar{\rho}_i^2}-\sqrt{1-\bar{r}_g\bar{\rho}_1^2}}{\sqrt{1-\bar{r}_g\bar{\rho}_i^2}-3\sqrt{1-\bar{r}_g\bar{\rho}_1^2}} \tag{46}$$

In contrast to the Schwarzchild solution, the pressure is not singular. For $\bar{r}_g \ll 1$, Equation (46) degenerates into Equation (20). For the fluid sphere, Equation (43), being transformed to variables $\bar{p},\bar{\rho}_i,\bar{r}_g$, becomes

$$\frac{1}{h_i}\frac{dh_i}{d\bar{\rho}_i}=\frac{\bar{r}_g\bar{\rho}_i(1+3\bar{p})}{2(1-\bar{r}_g\bar{\rho}_i^2)}$$

Integrating, we can find $h_i$ for the fluid sphere. The integration constant allows us to satisfy the last boundary condition in Equation (27).

To obtain the stresses, we need to supplement Equation (45) with an additional equation. To derive this equation, we minimize the functional in Equation (15), where with regard to Equation (44)

$$F=\left[\sigma_r^2+2(1-\nu)\sigma_\theta^2-4\nu\sigma_r\sigma_\theta\right]q_i\rho_i^2$$
$$+\lambda\left[\sigma_r'+\frac{2\rho_i'}{\rho_i}(\sigma_r-\sigma_\theta)+\frac{\rho_i\rho_i'(u-\chi\sigma_r)}{2(1-u\rho_i^2)}(\sigma_r-\mu c^2)\right]$$

Using Equation (34) for $q_i$, we can present the Euler equations, Equation (16), as

$$2\left(\sigma_r - 2\nu\sigma_\theta\right)\frac{\rho_i^2\rho_i'}{\sqrt{1-u\rho_i^2}} + \lambda\frac{\rho_i'}{\rho_i}\left[2 + \frac{\rho_i^2}{2\left(1-u\rho_i^2\right)}\left(u - 2\chi\sigma_r + \chi\mu c^2\right)\right] - \lambda' = 0$$

$$2\left[\left(1-\nu\right)\sigma_\theta - \nu\sigma_r\right]\frac{\rho_i^2}{\sqrt{1-u\rho_i^2}} - \frac{\lambda}{\rho_i} = 0$$

Expressing $\lambda$ from the second equation, substituting in the first equation and reducing the resulting equation to the form similar to Equation (45), we arrive at

$$\bar{\rho}_i\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{\rho}_i} + \bar{\sigma}_r - 2\nu\bar{\sigma}_\theta + \bar{\sigma}\left\{2 + \frac{1}{1-\bar{r}_g\bar{\rho}_i^2}\left[\bar{r}_g\rho_i^2\left(4-3\bar{\sigma}_r\right) - 3\right]\right\} = 0 \tag{47}$$

where $\bar{\sigma} = \left(1-\nu\right)\bar{\sigma}_\theta - \nu\bar{\sigma}_r$. For $r_g = 0$ and $\rho = r$, this equation coincides with Equation (14). Numerical integration of Equation (45) and Equation (47) under the boundary conditions $\sigma_r\left(\rho_i = 0\right) = \sigma_\theta\left(\rho_i = 0\right)$ and $\sigma_r\left(\rho_i = \rho_1\right) = 0$ allows us to obtain the dependences of stresses on $\bar{\rho}_i$ which can be changed to $\bar{r}$ with the aid of Equation (38). The dependences $\bar{\sigma}_r\left(\bar{r}\right)$ and $\bar{\sigma}_\theta\left(\bar{r}\right)$ corresponding to $\bar{r}_g = 0.25$ and $\nu = 0$ are shown in **Figure 3** with solid lines. Dashed lines correspond to the linear classical solution in Equation (18).

## 3. The General Theory

Return to Section 1 and consider the general case. Ten Einstein's equations

$$R^{ij} - \frac{1}{2}g^{ij}R = \chi T^{ij} \tag{48}$$

in which the energy tensor

$$T^{ij} = \sigma^{ij} \quad \left(i, j = 1, 2, 3\right), \quad T^{i4} = 0 \quad \left(i = 1, 2, 3\right), \quad T^{44} = \mu c^2$$

satisfies the conservation equations

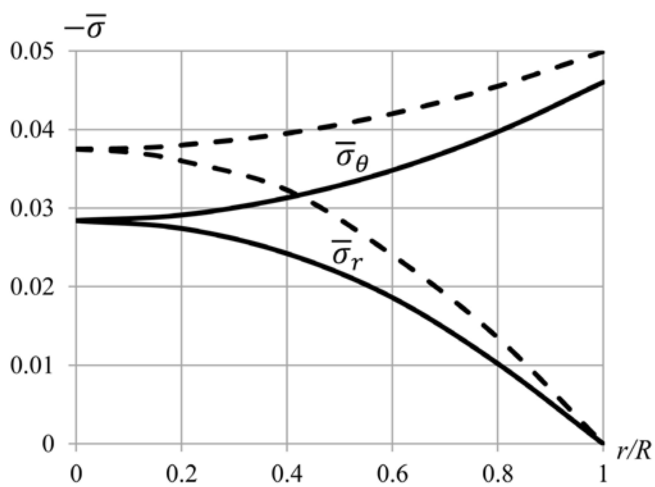


**Figure 3.** Dependences of the normalized stresses on the radial coordinate corresponding to the obtained solution (——) and the classical linear solution (- - - -).

$$\partial_k \sigma^{ik} + \Gamma^i_{mk}\sigma^{mk} + \Gamma^n_{kn}\sigma^{ik} + \Gamma^i_{44}\mu c^2 = 0 \tag{49}$$

and includes 10 components of the metric tensor $g^{ij}$. Because of Equation (49), only six of Equation (48) are mutually independent and we have six equations for 16 functions, *i.e.*, 10 coefficients $g^{ij}$ and six stresses $\sigma^{ij}$. Assume that we use six independent equations of Equation (48) to express six metric coefficients in terms of four. To derive four additional equations, we propose to use the variational equation, Equation (31), in which six metric coefficients are expressed in terms of four. Variation with respect to these coefficients allows us to write four Euler's equations and to obtain the set of 10 equations for the metric tensor.

To derive the equations for stresses, introduce the strain energy

$$U = \frac{1}{2}\iiint \sigma^{ij}\varepsilon_{ij}\sqrt{g_R}\,\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$$

in which $g_R$ is the determinant of the metric tensor in the Riemannian three-dimensional space. Expressing strains in terms of stresses through Hooke's la

$$\varepsilon_{mn} = c_{mnij}\sigma^{ij}$$

in which $c_{mnij}$ is the compliance tensor and introducing Equation (49) with aid of the Lagrange multipliers, construct the augmented functional

$$U = \frac{1}{2}\iiint F\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3,$$

$$F = c_{mnij}\sigma^{mn}\sigma^{ij}\sqrt{g_R} + \lambda_i\left(\partial_k\sigma^{ik} + \Gamma^i_{mk}\sigma^{mk} + \Gamma^n_{kn}\sigma^{ik} + \Gamma^i_{44}\mu c^2\right)$$

Minimization with respect to the stresses and $\lambda$-multipliers yields 10 equations for six stresses and four multipliers [14]. Thus, we have arrived at the complete set of 20 equations for 10 metric coefficients 6 stresses and 4 multipliers.

## 4. Linearized Axisymmetric Problem

Spherically symmetric problem discussed above requires only one coordinate condition. To demonstrate a more complicated case, consider an axisymmetric problem for which we need two conditions. Since the general problem can hardly be solved because the equations are too complicated, obtain the linearized solution for the external space. The line element in cylindrical coordinates $r, \varphi, z$ can be presented as

$$\mathrm{d}s^2 = (1+f_{11})\mathrm{d}r^2 + r^2(1+f_{22})\mathrm{d}\varphi^2 + (1+f_{33})\mathrm{d}z^2 + f_{13}\mathrm{d}r\mathrm{d}z - (1+f_{44})c^2\mathrm{d}t^2$$

Assume that functions $f_{mn}(r,z)$ are small in comparison with unity. For the external space with zero stresses and density, Equations (1), *i.e.*
$E_{11} = E_{13} = E_{22} = E_{33} = E_{44} = 0$, reduce to

$$\begin{aligned} & r\frac{\partial^2}{\partial z^2}(f_{22}+f_{44}) + \frac{\partial}{\partial r}(f_{33}+f_{44}) - 2\frac{\partial f_{13}}{\partial z} = 0 \\ & r\frac{\partial^2}{\partial r\partial z}(f_{22}+f_{44}) - \frac{\partial}{\partial z}(f_{11}-f_{22}) = 0 \end{aligned} \tag{50}$$

$$2\frac{\partial^2 f_{13}}{\partial r\partial z}-\frac{\partial^2}{\partial r^2}\left(f_{33}+f_{44}\right)-\frac{\partial^2}{\partial z^2}\left(f_{11}+f_{44}\right)=0$$
$$r\frac{\partial^2}{\partial r^2}\left(f_{22}+f_{44}\right)-\frac{\partial}{\partial r}\left(f_{11}-2f_{22}-f_{44}\right)=0 \tag{51}$$

$$\frac{\partial^2}{\partial r^2}\left(f_{22}+f_{33}\right)+\frac{\partial^2}{\partial z^2}\left(f_{11}+f_{22}\right)-2\frac{\partial^2 f_{13}}{\partial r\partial z}$$
$$-\frac{\partial}{\partial r}\left(f_{11}-2f_{22}-f_{33}\right)-2\frac{\partial f_{13}}{\partial z}=0 \tag{52}$$

For the axially symmetric problem, we have two conservation equations, so only three of five Equations (50)-(52) are mutually independent. Consider Equation (52) and subtract from it the first equation in Equation (50) and the first equation in Equation (51). The resulting equation

$$\Delta f_{44}=\frac{\partial^2 f_{44}}{\partial r^2}+\frac{1}{r}\frac{\partial f_{44}}{\partial r}+\frac{\partial^2 f_{44}}{\partial z^2}=0 \tag{53}$$

allows us to conclude that $f_{44}(r,z)$ is the classical gravitational potential which can be found in terms of exponential functions with respect to $z$ and Bessel functions with respect to $r$ [15]. For the external problem, the solution must satisfy the asymptotic conditions according to which $f_{44}\to 0$ for $r\to\infty$ and $z\to\infty$. Proceeding, express $f_{13}$ from the first equation in Equations (50), *i.e.*,

$$\frac{\partial f_{13}}{\partial z}=\frac{1}{2}\frac{\partial}{\partial r}\left(f_{33}+f_{44}\right)+\frac{r}{2}\frac{\partial^2}{\partial z^2}\left(f_{22}+f_{44}\right) \tag{54}$$

and substitute this result in the first equation in Equation (51) to get

$$r\frac{\partial^3}{\partial r\partial z^2}\left(f_{22}+f_{44}\right)-\frac{\partial^2}{\partial z^2}\left(f_{11}-f_{22}\right)=0$$

This equation can be ignored because it follows from the second equation in Equation (50). Integration of Equation (50) yields

$$f_{11}=f_{22}+r\frac{\partial}{\partial r}\left(f_{22}-f_{44}\right)+\varphi_1(r) \tag{55}$$

where $\varphi_1(r)$ is the integration function. Substituting Equation (55) in the second equation of Equation (51), we get $\varphi_1'=0$, so $\varphi_1=C$. Thus, Equation (50) and Equation (51) allow us to express $f_{13}$ and $f_{11}$ in terms of two unknown functions— $f_{22}$ and $f_{33}$. To proceed, we need to introduce two coordinate conditions. As earlier, apply Equation (31), *i.e.*, $\delta D=0$. The linearized space density is $d=1+f_{11}+f_{22}+f_{33}$. Using Equation (55), we can construct the following functional:

$$D=\iiint F\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\,,\quad F=r\left[1+2f_{22}+r\frac{\partial}{\partial r}\left(f_{22}+f_{44}\right)+f_{33}+C\right]$$

The condition $\delta D=0$ is satisfied if $f_{33}=0$ and $f_{22}$ is an arbitrary function. To identify this function, we use, as earlier, the minimum condition $d=1$. Then, $f_{22}=-f_{11}$ and Equation (55) yields

$$r\frac{\partial f_{22}}{\partial r}+2f_{22}=r\frac{\partial f_{44}}{\partial r}-C$$

The solution of this equation is

$$f_{22} = -f_{11} = \frac{1}{r^2}\left[ f_{44} - C\ln r + \varphi_2(z) \right]$$

Integration of Equation (54) allows us to find the last metric coefficient, *i.e.*,

$$f_{13} = r\frac{\partial f_{44}}{\partial z} + \int \frac{\partial f_{44}}{\partial r}\mathrm{d}z + \frac{1}{r^3}\varphi_2'(z) + \varphi_3(r)$$

Using the asymptotic conditions, we can conclude that constant $C$ and functions $\varphi_2, \varphi_3$ are zero. Thus, the solution is

$$f_{11} = -f_{22} = -\frac{f_{44}}{r^2}, \quad f_{13} = r\frac{\partial f_{44}}{\partial z} + \int \frac{\partial f_{44}}{\partial r}\mathrm{d}z, \quad f_{33} = 0$$

where $f_{44}$ is the solution of Equation (53). As follows from the foregoing derivation, the space density is uniform in the external space.

## 5. Gravitation and Space Density

The space density introduced in Section 2.2 allows us to propose the new interpretation of gravitation. As follows from the foregoing discussion, the isolated object in space can be in equilibrium under the action of gravitation and stresses induced by gravitation. It is important that the gradient of the space density outside the object is zero. Two objects in space cannot be in equilibrium and it is natural to suppose that the space density between them is not uniform. To take the equilibrium state and to reduce the gradient of the space density between the objects, they should move towards each other. Two situations are possible resulting in stable equilibrium or stable motion. First, the collision and the merge into one object can take place resulting in the equilibrium of the new object and zero gradient of the space density. Second case can take place if the trajectories of the moving objects are affected by perturbations induced by other objects in space. In this case, the collision does not occur and the objects orbit in elliptical paths.

## 6. Conclusion

The general relativity equations are supplemented with the coordinate conditions following from the stationarity condition of the three-dimensional metric tensor density and equations for the stresses similar to compatibility equations of the theory of elasticity. The solution of the obtained complete set of equations is demonstrated for linearized and general spherically symmetric problems and linearized axially symmetric problem. The space density which is the ratio of the three-dimensional metric tensor densities in Riemannian and Euclidean spaces in the same coordinates is introduced and used to explain the attraction of objects under the action of gravitation.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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# Implications for Discovery of Strong Radial Magnetic Field at the Galactic Center—Challenge to Black Hole Models

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## Abstract

The black hole model will be excluded by a very strong radial magnetic field near the Galactic Center which has been detected in 2013. Following it, the explosion mechanism, for both supernova and the hot big bang of the Universe, driven by magnetic monopoles is proposed in this paper.

## Keywords

Black Hole Model, Strong Radial Magnetic Field, Magnetic Monopoles

## 1. Two Dilemmas

The radial magnetic field near the Galactic Center (GC) has been detected in 2013 [1].

$$B \geq 8\left(\frac{n_e}{26\ \text{cm}^{-3}}\right)^{-1} \text{mG} \quad (\text{at } r \approx 0.12\ \text{pc}). \tag{1}$$

where $n_e$ is the number density of electrons. The first question is what is the role of the strong radial magnetic field? It is well known that a charged particle is hard to transfer cross the magnetic field line by the Lorentz force. According to the principle of the Alphen magneto-hydrodynamic freezing effect, the accretion (plasma) disk will be prevented from approaching to the GC by the strong radial magnetic field when the magnetic field is stronger more than the Alphen critical magnetic field

$$B > B_{\text{Alphen}} = 1.3\left(\frac{n_e}{26\ \text{cm}^{-3}}\right)^{1/2} \text{mG} \quad (\text{at } r \approx 0.12\ \text{pc}) \tag{2}$$

(The mass of the central supermassive object is taken as $4.6\times10^{6} M_{\odot}$ ).

And the accretion (plasma) disk may not enter in the neighborhood of the GC. Thus the radiations observed [2] from the GC direction are hardly emitted by the gas of accretion disk around the black hole (BH) of the GC which is the popular idea. This is a dilemma of the standard accretion disk model of BH in the GC [2] [3] [4].

The second question is what is the origin of the strong magnetic field near the GC?

It is well known that the most efficiency dynamo of producing magnetic field up to now is the advanced $\alpha$-turbulence dynamo in the sunspot (proposed by Parker) [5].

But the resulting magnetic field produced by the similar $\alpha$-turbulence dynamo in the interstellar gas is only $B \le 0.1\,\mu\text{G}$ at $r \approx 0.12\ \text{pc}$, which is five-order of magnitude less than the observed lower limit for the field strength, 8 mG. The observed magnetic field with a lower limit of 8 mG near the GC is hardly produced by so far known generator mechanism. This is the second dilemma.

To sum up, we conclude that the discovery of abnormally strong radial magnetic fields at the center of the GC presents the two dilemmas above for astrophysics.

However, these two dilemmas may simultaneously be solved by our model "Super-massive star with magnetic monopoles" [6] for quasars and active galactic nuclei (AGNs) (including the super-massive object at the GC) proposed since 1985. The discovered strong radial magnetic field near the GC is just consistent with our prediction on quantity.

## 2. A Supermassive Star Model with Magnetic Monopoles (SMSMM)

Taking the RC effect, that the MM may catalyze a nucleon decay, as an energy source, however, this dilemma in the GC may be naturally solved by our SMSMM at the GC (non BH). Three predictions, at least, in our model [6] are quantitatively confirmed by astronomical observations afterwards: 1) The radial magnetic field produced by the MMs condensed in the core of the super-massive object is about $B \approx (10 \sim 50)\,\text{mG}$ at $r = 0.12\ \text{pc}$ which is consistent with the lower limit of the observed magnetic field. This is a key prediction. All other models will be excluded when the key prediction is confirmed by the astronomical observation.

2) Plenty of positrons are emitted from the GC, the production rate is $10^{43}$ $e^{+}$/sec or so. This prediction is consistent with high-energy astrophysical observations [7] in 2003.

3) We predicted the surface temperature of the super-massive stellar object at the galactic center to be 123 K and the corresponding $10^{13}$ Hz (at the sub-millimeter range), and this is quite close to the observed value of $10^{12}$ Hz [8].

These observations may show the signals for existence of MMs such as the supermassive object at the GC.

To sum up, we conclude three important significances of our predictions above have been confirmed by the astronomical observations:

1) The Black Hole model in the Galactic Center is nonphysical.

2) These are the astrophysical evidences for the existence of MMs.

I hereby declare that astronomical observations are actually physical experiments in space.

3) Our model is reasonable. Especially the RC effect may be real and we may use it as an energy source for the supernova explosion and for the Earth core.

## 3. Query on the BH Models for Quasars and AGNs

It is now generally believed by most astronomers that bright quasars observed at large redshift (for example, $z > 1$ or even $z > 6$) are supermassive black holes formed in the primordial universe.

The spectacularly huge luminosity is supplied by the matter of a rotating accretion disk around the BH. We point out that the most serious contradiction in the BH model comes from the mass of the quasars original black hole.

We asked to compare birth in the early universe. In order to find the original mass of a quasar when it was born in the early universe, we need to subtract the mass added by accretion from the mass of the quasar determined today from the time of its birth ( $t \approx 0$ ) to the time of $t(z)$. However, according to various possible accretion theories in the current research, taking the data of 105,783 quasars from [9] which are based by SLOAN Digital sky survey (SDSS), we find that the mass of these black holes (quasars) with medium and low redshift is mostly negative or very small after deduction (see **Figure 1**). This is totally ridiculous.

However, the situation is totally different when we take our model (SMSMM). The mass of the supermassive object must decrease gradually due to the baryons decaying catalyzed by the MMs. And the nucleon decaying products (including the pions, muons, positrons and the radiation) would go out of the object continuously.

In order to compare the mass of the quasars (active galactic nuclei) observed at different redshift in the primordial era, the mass lost during their evolution due to radiation by the nucleon decay catalyzed by MMs. should be added into the mass of the quasars (AGNs) observed at different redshift now.

Adding the lost mass during their evolution, the statistic distribution of number of the quasars (AGNs) with their primordial mass of shows a Gauss distribution (see **Figure 2**). It is reasonable.

## 4. Supernova Explosion Driven by MMs

What role played by the MM in supernova (SN) explosion?

The MMs in the interior of stars and planets are almost by captured from space during their life time after their formation. The total number of MMs captured from space by the progenitors of SN after their formation may be estimated to be

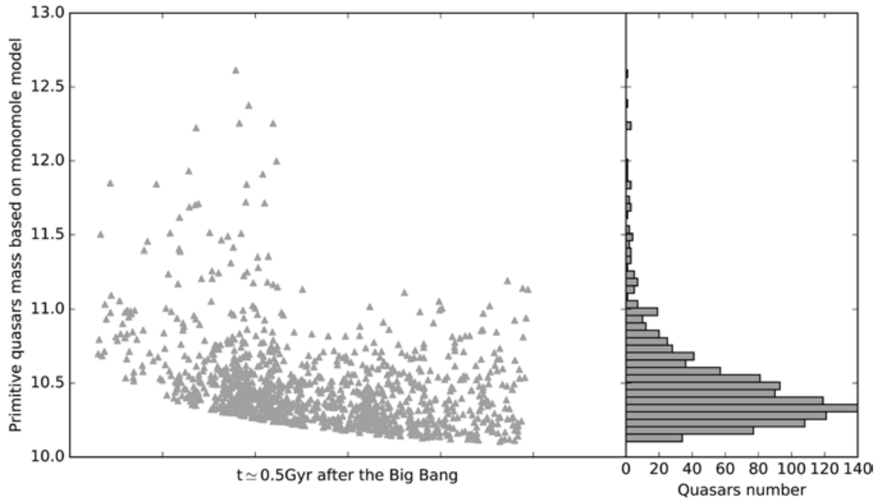


**Figure 1.** Distribution of the mass of primordial mass of quasars in black hole model, the solid line represents the quasar mass error not considered in the statistics, dash-dot line and dotted line represent the upper and lower limits of the quasar mass error are considered separately in the statistics (Data for 105,783 quasars are from [9]), the result is from Zheng Li and Ming Zhang's work (in preparing).

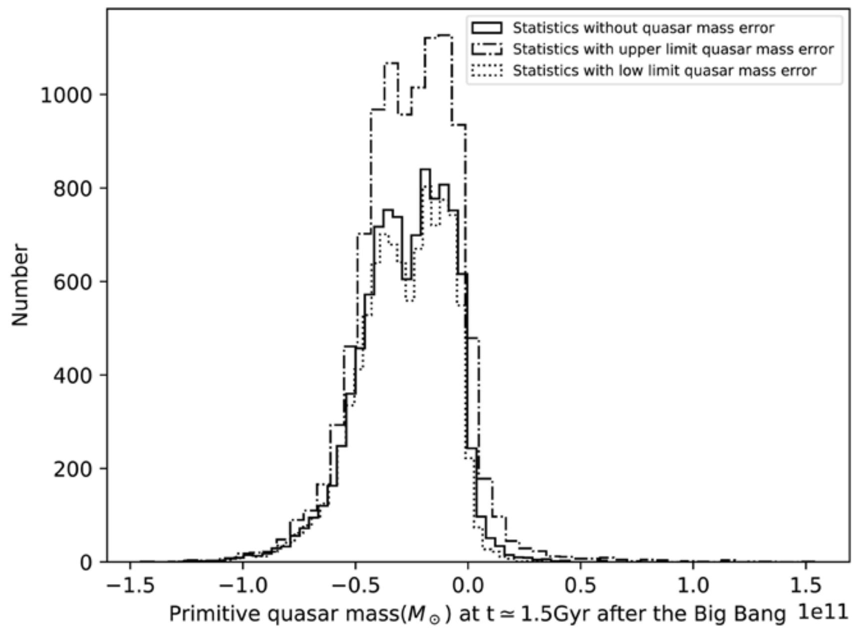


**Figure 2.** Distribution of the primordial mass of quasars in the model of supermassive star with magnetic monopoles, the quasar masses we take log10 (Data for 105,783 quasars are from [9]), the result is from Zheng Li and Ming Zhang's work (in preparing).

$$N_m = 4\pi R^2 \Phi_m t \approx 1.0\times 10^{31}\left(\frac{\Phi_m^{(0)}}{10^{-2}\Phi_m^{(up)}}\right)\left(\frac{R_{RG}}{10^3 R_\odot}\right)^2\left(\frac{t_{RG}}{10^6\,\mathrm{Yr}}\right). \tag{3}$$

where $R$ denotes the radius of the star (or planet) and $t$ is the life time of the star (or planet), $\Phi_m$ is the flux of the MM flight in the space. The superscript (0) is the sign of its value in the space. The number of MMs captured from space by the progenitor of SN is mainly obtained in its red giant stage due to the radius of corresponding red giant being hundreds of times larger than the radius of the corresponding main sequence star, although the life time of its red giant stage is about 1/10 of times shorter than the life time of its corresponding main sequence star. $\Phi_m^{(up)}$ is the up limit of the flux for the MM flight in the space (Parker, 1970) [10].

$$\Phi_m^{(up)} \approx \left(10^{-12} \sim 10^{-16}\right) \mathrm{cm}^{-2} \cdot \mathrm{s}^{-1} \cdot \mathrm{sr}^{-1}. \tag{4}$$

Taking the RC effect as the energy source, the luminosity of the supernova is.

$$L_m \approx 2.5\times 10^{43} a \left(\frac{\xi}{100}\right)\left(\frac{n_B^{(c)}}{n_{nuc}}\right)\left(\frac{T_c}{10^{11}\,\mathrm{K}}\right)^{1/2} \mathrm{erg/s}. \tag{5}$$

$$a = \left(\frac{R_{RG}}{10^3 R_\odot}\right)^2 \left(\frac{t_{RG}}{1\times 10^6\,\mathrm{Yr}}\right). \tag{6}$$

$$\xi \equiv \left(\frac{\sigma}{10^{-30}\,\mathrm{cm}^2}\right)\left(\frac{\Phi_m^{(0)}}{10^{-2}\Phi_m^{(up)}}\right). \tag{7}$$

where $n_B^{(c)}$ is the number density of the baryons in the center for the core of the SN, $n_{nuc}$ is the nuclear (number) density, $\sigma$ is the cross section of the RC effect. For collapsed supernova, its central density increase with the core mass of the collapsing supernova we use some approximate (reasonable) estimates:

$$t_{RG} = 10^6 \left(\frac{M}{20M_\odot}\right)^{-1} \mathrm{Yr} \tag{8}$$

$$\frac{R_{RG}}{R_\odot} = 10^3 \left(\frac{M}{20M_\odot}\right)^{\beta}, \quad \beta \approx (1.0 \sim 1.5). \tag{9}$$

Hence

$$a \approx \left(\frac{M}{20M_\odot}\right)^{(2\beta-1)}. \tag{10}$$

$$n_B^{(c)} / n_{nuc} \propto \left(M_c / 20M_\odot\right)^2. \tag{11}$$

The relationship between the peak luminosity of a supernova and the mass of its progenitor may be written as

$$L_m^{(peak)} \approx 5.0\times 10^{43} \left(\frac{M}{20M_\odot}\right)^{2\beta+1} \left(\frac{\xi}{100}\right)\left(\frac{T_c}{10^{11}\,\mathrm{K}}\right)^{1/2} \mathrm{ergs/s}. \tag{12}$$

The supernova will explode when the peak luminosity of the supernova is higher than the Eddingtons luminosity which is the critical luminosity of a stable star:

$$L_{peak} > L_{Edd}. \tag{13}$$

$$L_{Edd} = \frac{4\pi c G M}{\kappa} \approx 1.3\times 10^{38} \left(\frac{\kappa}{0.4}\right)^{-1} \left(\frac{M}{M_{sun}}\right) \mathrm{ergs/s}. \tag{14}$$

We can use the ratio, $b$ to measure the magnitude of the supernova explosion.

$$b \equiv L_{peak} / L_{Edd} . \tag{15}$$

$$b \approx 2\times10^{4} \left(\frac{M}{20M_{\odot}}\right)^{2\beta} . \tag{16}$$

For the super luminosity supernova ASASSN-15lh (Sep. /2015), $L = 2.2\times10^{49}$ ergs/s at the 15th day after the peak, it is easily explained by an assumption: 1) Its progenitor with the initial mass more than $10^4 M_{\odot}$, and its radius $R \approx 10^{4-5} R_{\odot}$, $t \approx 10^4$ Yr. 2) The number density of the baryons in the center for the core of the SN during its collapse is reach to $n_B^{(c)} \approx 10^3 n_{nuc}$.

The SN may be very weak without strong explosion when its RC luminosity is not much higher than the Eddington's luminosity, $L_{Edd} \approx 1.3\times10^{38} (M/M_{\odot})$.

(Such as Cas A and the G1.9 + 0.3). The Equation (12) is the relationship of the peak luminosity of SN with the mass of their progenitors.

Most of the MMs will be rapid thrown outwards with the plasma by the strong Coulomb electromagnetic interaction, and they will go away from the stars. However, trace MMs will return back to the core of the star. The residual MM still continue to catalyze the nucleons decaying. The corresponding radiation pressure is outward against material collapse.

When the RC luminosity catalyzed by the MMs in the core is less than the Eddington's luminosity of the remnant, they will reach at the dynamic equilibrium. Neither material is pushed outward by the radiation pressure, nor will the core collapse to the center. The central density cannot tend to infinite and the SN remnant is not a black hole. It means that no black holes with stellar mass formed through supernova explosion of massive stars.

Using the same idea we may also explain naturally following two mysteries: 1) why the Earth's core is in a melting state. The parameter value in the Equation (5) and (12), Equation $\xi \approx 100$, is estimated based on the measured outward heat flow from the earth's core and on the energy yield rate of nucleon decay catalyzed by magnetic monopoles (RC effect); 2) why no white dwarf with surface temperature lower than $10^3$ K has not been observed up to now?

## 5. Physics on the Hot Big Bang of the Universe Driven by MMs

On the standard model of the hot big bang cosmology, the early universe is depicted by extrapolating back to a hot and dense initial state of Planck length and Planck time derived with the help of the uncertainty principle. However, the formation of the big bang itself has not been investigated, *i.e.* nobody gives answer to the question what is the physical reason of Hot Big Bang of the Universe?

Using the same idea that nucleons may decay catalyzed by the MM with strong interaction cross section (*i.e.* RC effect), we may also explain naturally the physical reason of Hot Big Bang of the Universe.

It is generally estimated and believed that there are $2.0 \times 10^{11}$ galaxies. Every galaxy is roughly the size of over Milky galaxy with $10^{11}$ stars, then the total number of stars in the Universe is about $10^{23}$. The mass of the sun is $2.0 \times 10^{33}$ gram, then the total mass of the Universe of the baryons is $2.0 \times 10^{56}$ gram and the total number of the baryons is $1.0 \times 10^{80}$. If the content of the magnetic monopoles of the same polarity contained in the Universe is $\xi = N_m / N_B = 10^{-20}\left(\xi/\xi^{(up)}\right)$, here $N_m$ and $N_B$ are the number of magnetic monopole and baryons, respectively. $\xi^{(up)}$ is the Parker up limit (Parker, 1970), and $\xi^{(up)} \approx 10 \sim 20$. So the total number of the magnetic monopoles of the same polarity contained in the universe may be estimated to be $N_m = 10^{60}\left(\xi/\xi^{(up)}\right)$ (Peng and Chou, 2001; Peng *et al.*, 2017). The magnetic monopoles in the high temperature baryon plasma are strongly compressed and moving very fast toward the center via electromagnetic interaction. The RC luminosity produced by nucleon decay which is catalyzed by the magnetic monopoles is given by

$$L_m = N_m n_B^{(c)} \langle \sigma v_T \rangle m_B c^2 \approx 10^{75}\left(\frac{n_B}{n_{nuc}}\right)\left(\frac{\xi}{\xi^{(up)}}\right)\left(\frac{\sigma_{(RC)}}{10^{-80}\,\mathrm{cm}^2}\right)\mathrm{ergs/sec}. \quad (17)$$

when the total mass in the universe is compressed to become super-massive body, the corresponding Eddington luminosity is given by

$$L_{Edd} = 10^{38}\left(\frac{M}{M_\odot}\right)\mathrm{ergs/sec} \approx 10^{61}\,\mathrm{ergs/sec}. \quad (18)$$

If the whole universe is compressed such that

$$\left(\frac{n_B^{(c)}}{n_{nuc}}\right) > 10^{-10}\left[\left(\frac{\xi}{\xi^{(up)}}\right)\left(\frac{\sigma_{RC}}{10^{-80}\,\mathrm{cm}^2}\right)\right]^{-1}. \quad (19)$$

then $L_m > 10^4 L_{Edd}$ and the whole Universe must violently explode outward leading naturally to the hot big bang. This is just the physical mechanism for Hot Big Bang of the Universe.

In view of this, we may propose an oscillating model of the Universe between the expansion phase of the big bang and then the contracting phase by the gravitational attraction.

## 6. Conclusion

In the traditional standard hot big bang cosmology, it is extrapolated back to the initial singularity of the universe. This is done purely by theoretical speculation. Our model of the hot big bang is obtained in terms of the Rubakov-Callan luminosity and no other theoretical arguments or anticipation is required. In our model, the expression phase may finally be ended and followed by the contraction phase due to gravitational attraction.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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# Probabilistic Cosmology

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## Abstract

In this paper, an alternative approach to cosmology is discussed. Rather than starting from the field equations of general relativity, one can investigate the probability space of all possible universes and try to decide what kind of universe is the most probable one. Here two quite different models for this probability space are presented: the combinatorial model and the random curvature model. In addition, it is briefly discussed how these models could be applied to explain two fundamental problems of cosmology: Time's Arrow and the accelerating expansion.

## Keywords

Cosmology, Ensemble, Time's Arrow, Accelerating Expansion, Multiverse

## 1. Introduction

What is the most common way of solving a problem in physics? In the tradition which goes back at least to Newton, the dominating answer tends to be: Find the equations of motion (and solve them).

This answer has been enormously successful in the history of modern physics. This was also the way modern cosmology started a century ago. To illustrate this approach, let us consider a closed universe of the simplest possible topological type, *i.e.* a 4-sphere. And let us in addition also assume it to be homogeneous and isotropic at every moment of time. Can we then determine how the radius $a(t)$ of the universe varies with time? The pioneering work for answering this question was based on the so called Friedmann equation (see Friedmann [1]):

$$\frac{6}{a(t)^2}\left(\frac{\mathrm{d}a}{\mathrm{d}t}(t)\right)^2+\frac{6}{a(t)^2}=\frac{6a_0}{a(t)^3}. \tag{1.1}$$

This is essentially just the time-time-component of Einstein's general field equations

$$G_{ij} = 8\pi T_{ij}, \tag{1.2}$$

where $G_{ij}$ is the Einstein tensor and $T_{ij}$ is the stress-energy tensor (for further details, see e.g. Misner, Thorne, Wheeler [2] or Wald [3]). Multiplying by the factor $a(t)^2/6$, (1.1) can be rewritten as

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}(t)\right)^2 = \frac{a_0}{a(t)} - 1 \quad \text{or} \quad \frac{\mathrm{d}a}{\mathrm{d}t}(t) = \pm\sqrt{\frac{a_0}{a(t)} - 1}, \tag{1.3}$$

from which it is easy to see that a solution, starting from $a = 0$ at the Big Bang, will grow until it reaches its maximal value $a_0$ and then, by symmetric considerations, decrease until it again becomes zero (at the Big Crunch). In fact, it is not difficult to see that the solutions of (1.3) are cycloidal curves, so the Friedmann equation made it possible to predict the time-development of the universe from the beginning to the end (see **Figure 1**). Nowadays however, this is neither the best nor the most popular model for cosmology. In particular, it fails to explain the accelerating expansion (see Adam, Riess *et al.* [4] and Perlmutter *et al.* [5]).

But, if we now return to the original question, there are also other answers to how problems in physics can be solved. For example, in statistical mechanics a competing answer could be: Consider all possibilities and find the most probable kind. As a trivial illustration, consider the situation where a gas is initially confined by a wall to a part of an otherwise empty container as in **Figure 2** (left). What happens if the wall is removed? Clearly, the gas will immediately start to spread out (middle) to fill up the whole container, and at the end of this process (right), the gas will be very evenly distributed over the interior. How should we explain this phenomenon? It is of course in principle possible to try to use the

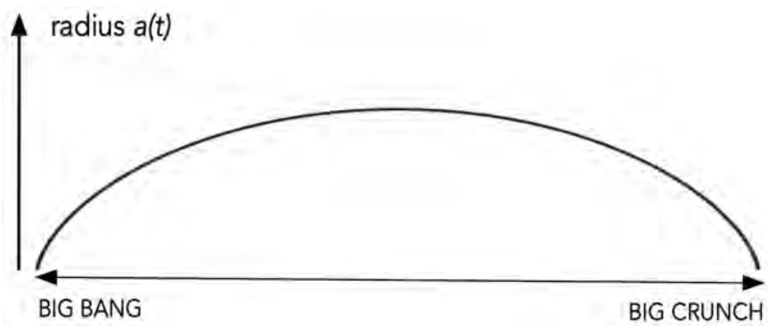


**Figure 1.** The time development of the closed Friedmann model.

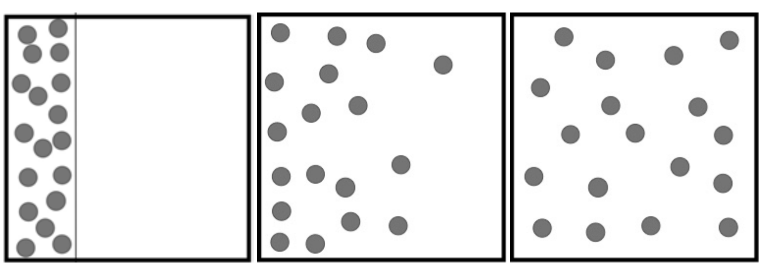

**Figure 2.** Three stages of a gas filling up a container.

equations of motion for some set of initial conditions. But this would lead to extremely complicated computational problems, even for a moderate number of particles. A much simpler method, and in a sense a more convincing one, would be to note that the number of micro-states of the gas which correspond to the evenly distributed macro-state is enormously much larger than the number of micro-states which correspond to the un-evenly distributed macro-state we started with, and in fact much larger than the number of micro-states corresponding to any other macro-state as well. Thus, neglecting almost everything about the microscopic processes that underlies the development and instead just considering it to be more or less a random process, we can, using little more than high school mathematics, convincingly argue that the gas will end up in a state of even distribution.

The purpose of the present paper is to try to show how this way of reasoning could be used in modern cosmology, and also that it may have the potential to change our views on cosmology in general.

How should this way of thinking be implemented? The idea is to consider the set of all possible universes as a huge probability space, and then try to find out what types of universes dominate in this probability space.

It goes without saying that it is impossible to construct a model for the probability space of all universes in any kind of detail. Rather, the strategy in this paper will be the opposite one; to find extremely simple models. Still, the hope is that these models will somehow reflect fundamental properties that are not so easy to spot within are more traditional framework.

Is this a multiverse theory? Answering this question will mainly be left to the reader, since it is more a question about philosophical interpretation than about science. From the specific point of view of the author however, the answer may be said to be yes: to consider all universes to have the same ontological status does seem to be the most natural interpretation. But on the other hand, it should be kept in mind that we are here concerned with multiverses of a very restricted kind, which can be viewed as a natural outflow of Feynmann's "Democracy of all Histories" approach to physics. Thus, this is (regardless of the interpretation) essentially based on ordinary quantum physics and has no relation to the more speculative multiverse theories that have been discussed in recent years.

The main examples of such simplified multiverse models which I want to discuss in this paper are the following:

- The combinatorial multiverse.
- The random curvature multiverse.

From a certain point of view, these examples are both extreme in the sense that it is difficult to imagine how the simplification could be driven much further. Still, it is part of the ambition in this paper to show that even such simplified models have the potential to explain fundamental properties of our universe. The examples that will be taken to illustrate this point have been discussed separately before, so I will simply sketch what can be done in these two cases to illustrate the general method. For more detailed calculations, see Tamm

[6], Tamm [7], Tamm [8], Tamm [9]. The examples are as follows:

- The asymmetry of time.
- The accelerating expansion.

It should also be said that the purpose here is not to make firm predictions to be compared to observations. Such predictions may very well be made at a later stage, but so far the ambition has rather been to make the models as simple as possible, and also they may contain various parameters which may be difficult to determine.

I will be mainly concerned with closed universes. Part of the reason for this will become clearer as we proceed. But it is of course possible to try to apply the same way of thinking to open universes as well. It is just harder to treat them in the context of probability spaces, something which is well known from ordinary quantum physics.

The underlying physics will mainly be treated in a kind of semi-classical setting. This is in a way very natural since the problems are macroscopic, but still in the end it may be argued that a quantum mechanical treatment would be preferable. As has already been said, in this paper I consider simplicity in the presentation to be the most important thing. However, I will briefly come back to this question in Section 7.

## 2. The Combinatorial Multiverse

In this section we will study the perhaps simplest of all models, namely the combinatorial one. Thus, for each moment of time between the endpoints $-T_0$ and $T_0$ (corresponding to the Big Bang and the Big Crunch), consider the set of all possible configurations or "states" that a possible universe could be in. To simplify still further, let us assume time to be discrete (and integer valued). Thus, we have moments of time

$$-T_0, -T_0+1, -T_0+2, \cdots, T_0-2, T_0-1, T_0, \tag{2.1}$$

and for each such moment of time we have a certain number of possible states. At times $-T_0$ and $T_0$, we assume that there is only one unique state, but for each moment of time between the endpoints, there are many different states. All these states are then the nodes of a huge graph, and a universe is a path in this graph where the edges are specified by the dynamics of the model: between each pair of adjacent moments of time, say $t$ and $t+1$, there will be a certain number of edges between the corresponding states, indicating those time-developments which are possible, and the totality of all such edges defines what we mean by the dynamics of the model. A very schematic picture is shown in **Figure 3**.

***Remark*** **1.** The word "state" here should be interpreted with some caution. It should not be interpreted as representing ordinary quantum states in the usual sense. Rather, states here may be thought of as "distinguishable configurations", which is clearly a kind of semi-classical approximation (see Tamm [6]).

In particular, it is important to note that a state can lead to different states in the future, *i.e.* each state may be thought of as a fork in the road of history. For

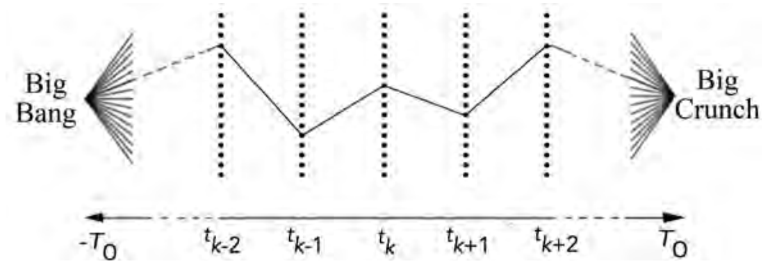


**Figure 3.** One universe in the combinatorial multiverse [7].

example, the decay or non-decay of a certain particle may lead to completely different futures within a reasonably short time, in spite of the fact that the development of the underlying wave-function is supposed to be unique.

As it stands however, this model is too simple to generate any results. In fact, there are no observable differences at all between the states, which means that there are no measurable variables which could be related to the (so far non-specified) dynamics. In the next section, which is devoted to the second law of thermodynamics, we will therefore consider one additional variable: the entropy.

## 3. Time's Arrow

The term "Time's Arrow" was coined by Eddington [10] and refers to the fact that macroscopic time is directed; there is an arrow pointing from the past towards the future. For some reason we can remember yesterday but we cannot remember tomorrow. Another formulation, which perhaps lends itself better to physical reasoning, is to say that entropy grows in the direction towards the future.

The problem with Time's Arrow is that the underlying equations of motion, which are supposed to be responsible for the macroscopic behavior, are essentially invariant under reversal of the direction of time. This can also be expressed by saying that on the microscopic level there is no arrow. So where does the macroscopic arrow come from?

There seems to be no question in physics where the tentative answers have been so diverse (see e.g. Barbour [11], Halliwell, Perez-Mercander, Zurek [12], Zeh [13]). One way to resolve this problem could be to simply just state that the boundary conditions of the universe are very different in the future and in the past. If we assume that the universe starts from a very improbable state of very low entropy immediately after the Big Bang, and then develops towards more and more probable states in the future very much like the gas in **Figure 2**, then the growth of entropy in between may appear to be perfectly natural, something which was in essence clear already to Ludwig Boltzmann. But assuming such differences in the boundary conditions would amount to little more than assuming an arrow of time from the start.

Probabilistic cosmology however, offers a different view-point. We can consider the probability space of all possible universes with a fixed four-volume, and this probability space may very well be perfectly time-symmetric, *i.e.* it would look exactly the same if we would reverse the direction of time. However, this would not at all imply that the time in each single universe would share this property. In fact, it could very well be that the symmetry would be broken so that the overwhelming majority of all universes would have a directed time, in the sense that the entropy would be monotonic. To put it shortly, all these universes would have the same endpoints, but only half of them would have the same Big Bang as we have. In the other half, our Big Bang would instead be the Big Crunch.

To model this in a way which is sufficiently simple to allow for computations, we will make use of the combinatorial multiverse in the previous section, but with the concept of entropy added to it.

Thus, let us assume that to every state we can assign a certain number $S$ which we call the entropy of the state. To make the model as simple as possible, let us also assume that $S$ only takes integer values.

How many states correspond to a given value of $S$? According to Boltzmann, we have that

$$S = k_B \log \Omega \Leftrightarrow \Omega = W^S, W = e^{1/k_B}. \tag{3.1}$$

Although this formula was derived under special circumstances, it does represent a generally excepted truth in statistical mechanics: the number of states grows exponentially with the entropy. In the following, this will be taken to hold true at every given moment of time for the universe as a whole. A schematic modification of Figure 3 for a very small multiverse is shown in Figure 4. In this picture, one possible path (universe) is shown, in this case with monotonically increasing entropy. However, before the model can be put into use we still need to specify which paths are allowed, *i.e.* specify the (time-symmetric) dynamics of the model. In other words, we need to agree on some rule for deciding which states are accessible from a given state.

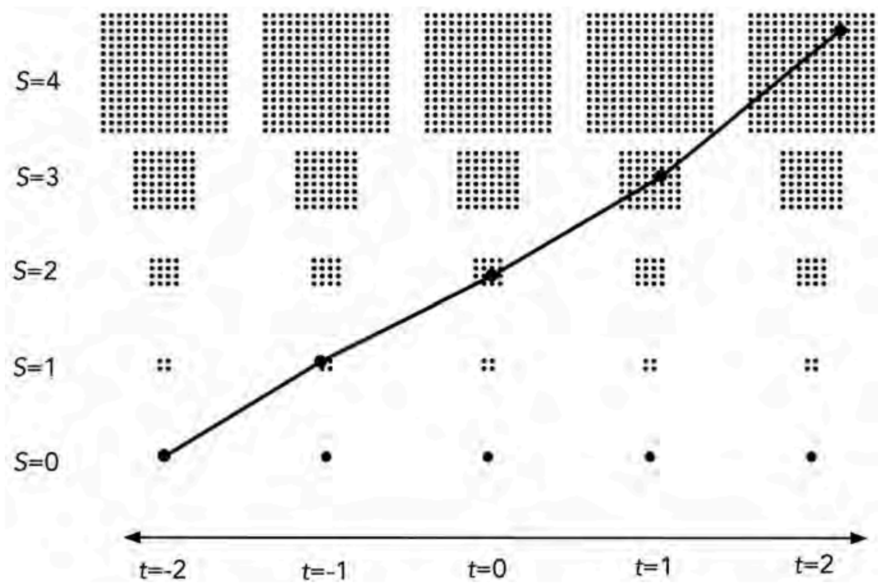


Figure 4. A very small combinatorial multiverse with entropy, where $W = 4$. A particular universe with monotonically increasing entropy is also shown [8].

To this end, simplifying still further, we assume that the entropy can only change by ±1 during each unit of time. The idea is then to make use of Boltzmanns intuition that the universe with time moves from less probable states to more probable ones. In its original form however, this idea has a definite direction of time built in to it, which obviously makes it unsuitable in the present context. Therefore, we will instead make use of the following probabilistic time-symmetric version:

**Principle 1.** (*The Time Symmetric Boltzmann Principle*) *For every state at time $t$ with entropy $S$, the dynamics allows for a very large number $K$ of "accessible states" with entropy $S+1$ at times $t-1$ and $t+1$. But on the other hand, the chance $\delta$ for finding an edge leading to a state with entropy $S-1$ (at time $t-1$ or $t+1$) is very small.*

Note that with this simplified dynamics, we do not compare the differences in probability between different paths in any detail. Rather, we just classify transitions as possible or not possible.

***Remark* 2.** For the conditions in the symmetric Boltzmann principle to be compatible, it is necessary that $K \ll W$, where $W$ is the constant in (3.1). In fact, in this case it is easy to see that only a fraction $K/W$ of states can be reached from states with lower entropy at the previous (or next) moment of time, so in this case, $\delta = K/W \ll 1$.

In addition to this, we also need some assumptions at the ends (BB and BC). In this case, let us assume that the entropy is zero, but that during the very first and last units of time, "everything is possible", *i.e.* that there is a positive probability for a transition to any of the states at the next (previous) moment. However, the probability does not necessarily have to be the same for all states. Rather, it seems very natural to assume that the probability for such a transition decreases rapidly with the entropy of the state, *i.e.* the by far most probable transitions lead to states with very low entropy. This is of course just a coarse way of modeling the very extreme situation just after the Big Bang or just before the Big Crunch.

Summing up the discussion, we can now define the combinatorial multiverse with entropy added in the following way:

**Definition 1.** A *universe* $U$ is a chain of states, one state $\Sigma_t$ at time $t$ for each $t$, with the property that the transition between $\Sigma_t$ and $\Sigma_{t'}$ is always possible according to the dynamical laws, where $t' = t \pm 1$.

**Definition 2.** The *combinatorial multiverse* (*with entropy*) $M$ is the set of all possible universes $U$ in the sense of Definition 1.

Note that with the above definitions, the probability weight of a certain universe only depends on the weights of the first and last steps, since for all other steps we have simply put the weights equal to one.

## 4. The Broken Symmetry

I will in this paper not go into the technical details for computing the number of universes with different kinds of behavior of the entropy, for this I simply refer to Tamm [6], Tamm [7], Tamm [8]. But it may still be worthwhile to briefly discuss how the combinatorial multiverse can be used to explain time asymmetry.

With suitable choices for the parameters of the model, it is easy to convince oneself that the probability for a universe with a monotonic behavior of the entropy is enormously much larger than, e.g. the probability for a universe with low entropy at both ends. In fact, if $\delta$ in the time symmetric Boltzmann principle is small enough, then the probability for such a behavior will be so small that it is almost neglectable in comparison with the probability for a monotonic behavior, even if in the monotonic case the last (or first) step will be very improbable.

Thus, since it seems to be an experimental fact that we live in a universe with low entropy at least one end, we have in a sense arrived at an explanation for the fact that an observer who is confined to such a universe will, with overwhelming probability, experience a directed time: there are simply so many more universes of this kind.

Is this a sufficient explanation for the arrow of time? From the point of view of the author, this model should rather be considered as a first step towards such an explanation, and more refined models should be designed. Certainly, there are many simplifications in the above model, and some of them may even appear to be rather extreme. But on the other hand, most of them can be said to be quite harmless for explaining the underlying mechanism, e.g. discrete time and integer-valued entropy.

But there is one assumption which is somewhat problematic in the Symmetric Boltzmann Principle above: if we apply it probabilistically, then it leads to a kind of Markov property in the sense that the probability for the entropy to go up or down at a certain step is completely independent of the pre-history. This is quite in contrast to our own universe, where an event (e.g. a supernova), can leave traces that can still be seen billions of years later.

However, one can attempt to construct slightly more complicated models which do not have this behavior. For example, a kind of assumption which would not have this Markov property would be to assume that the probability for the increase/decrease of the entropy at a certain step (forwards or backwards in time) should depend on the $n$ previous (following) steps. If we for instance let $n = 2$, this would mean that an increase (or decrease) of the entropy from time $t_k$ to $t_{k+1}$ will be more likely if we already know that at the previous step from $t_{k-1}$ to $t_k$ the entropy has increased (decreased).

In fact, it can also be argued that such a modified model would not only be more realistic, but would also in a sense give clearer results than the above model. For instance, one can attempt to prove that in such a model, the total probability mass of all universes with directed time (in anyone of the two directions) must be very close to 1. And certainly, there are other ways to improve further.

Still, the gap to a more realistic dynamics based on, say, ordinary Newtonian or quantum mechanical mechanics is of course large. This is, for better or for worse, both the strength and the weakness of probabilistic cosmology as it is presented here: extreme simplification may be the price we have to pay in order to see the forest in spite of all the trees.

## 5. The Random Curvature Multiverse

In this section, we will briefly discuss another kind of simple model for a multiverse, which is however quite different from the combinatorial multiverse in Sections 2, 3 and 4. Here it will not be the entropy but rather the scalar curvature which will be the central concept. Nevertheless, the basic approach is the same: we start from very general statistical assumptions and try to determine the most probable type of universe.

Thus, let us consider the probability space of all possible metrics on a certain space-time manifold, only subject to the condition that the total 4-volume is a fixed number. Scalar curvature is essentially additive in separate regions, so what can we say about the probability for a certain value of the total scalar curvature in a region $D$ which is a union of many smaller regions?

***Remark* 3.** It is generally believed that the fluctuations in $R$ become more and more violent when we move towards shorter and shorter length scales. From this point of view, one can wonder if it makes sense to consider the total scalar curvature in a region at all?

The easiest way to get around this difficulty is to simply consider the mean scalar curvature at some (short) length scale. As it turns out, everything to come is essentially independent of the choice of this length scale, so I will not comment further on this here.

To each such smaller region we assume that there is a certain probability distribution for the different possible metrics. Exactly what this probability distribution actually looks like on the microlevel is of course difficult to know, but the point is that under quite general assumptions this will not be important. Let us just suppose that it depends only on the scalar curvature. This is in fact very much in the spirit of the early theory of general relativity, where $R$ plays a central role (compare e.g. the deduction of the field equations from the Hilbert Palatini principle in Misner, Thorne, Wheeler [2]). We also suppose, starting from the idea that zero curvature is the most natural state, that the mean value of this distribution is zero. This assumption may be non-obvious, but nevertheless serves as a good starting point.

If we now consider the total curvature $R$ in $D$ to be the sum of the contributions from all the smaller subregions, and if we (roughly) treat these contributions as independent variables, then the central limit theorem (see Fischer [14]) says that the probability for a certain value of $R$ is

$$\sim \exp\left\{-\mu_{\Delta} R^2\right\}, \tag{5.1}$$

where $\Delta$ is the volume of $D$.

In the following, we will simply take this as the natural probability weight for the metric $g$ in $D$ on a macroscopic scale where we do not observe any fluctuations of the curvature. In other words, the factor in (5.1) can be considered as a kind of measure of the resistance of space-time against bending.

What about the probability weight of a larger set $\Omega = \bigcup_\alpha D_\alpha$ with metric $g$? Assuming multiplicativity (which essentially means that different regions are treated as independent of each other), and that all the regions have roughly the same volume $\Delta$, we get the (un-normalized) probability

$$P(\Omega) \sim \Pi_\alpha \exp\left\{-\mu_\Delta R_{g_\alpha}^2\right\} = \exp\left\{-\mu_\Delta \sum_\alpha R_{g_\alpha}^2\right\} \approx \exp\left\{-\mu \int_\Omega R_g^2 \mathrm{d}V\right\}. \quad (5.2)$$

where $\mu$ is a fixed constant. (Here we have, in the transition from sum to integral, tacitly made use of the additive property of the variance in normal distributions.) So what we get is a kind of Ensemble of all possible metrics in $\Omega$, where each metric gets a probability weight as above.

In classical general relativity, $R$ is usually assumed to be zero everywhere, as long as there is no mass present. This is of course very well compatible with the present Ensemble, since $R = 0$ will obviously maximize the exponential in (5.2). However, when it comes to cosmology things become more complicated, and this kind of Ensemble may lead to non-trivial consequences.

The word Ensemble originally steams from statistical mechanics. So the idea is now to apply methods from classical statistical mechanics to the whole multiverse (see e.g. Huang [15] for some background about Ensembles). First compute the "state sum": $\Xi = \sum_g \exp\left\{-\int_\Omega \mu R_g^2 \mathrm{d}V\right\}$. Minus the logarithm of the state sum, $\mathcal{F} = -\log \Xi$, is what is usually refered to as the "Helmholtz Free Energy". According to standard wisdom in statistical mechanics, the macrostates which minimize $\mathcal{F}$ (among all states with a given volume), are the by far most probable ones, *i.e.* the ones which may be realized.

In general, finding these macrostates can be difficult, since they are determined by a sensitive interplay between the size of the terms in the state sum and their corresponding "densities of state". However, in the case of interest here, corresponding to low curvature, it can be a reasonable first order approximation to assume that the density of states is the same for all competing states. In this case, $\mathcal{F}$ can essentially be computed as minus the logarithm of the largest term in $\Xi$:

$$\mathcal{F} \sim \mu \int_\Omega R_g^2 \mathrm{d}V. \quad (5.3)$$

***Remark* 4.** Note that terms like the "Helmholtz Free Energy" are used here to associate to a fundamental statistical principle. But it should of course be kept in mind that in this situation we deal with 4-dimensional states, and that this is not directly related to ordinary 3-dimensional energy.

On the other hand, there *is* an analogy between the integral in (5.3) and the concept of action (which in a certain sense can be thought of as a kind of 4-dimensional energy). From this point of view, the principle of minimizing the free energy as above also becomes analogous to the usual principle of least action.

What happens if we minimize the action/free energy $\mathcal{F}$ in the case of a closed, homogeneous, isotropic universe? (Compare with the closed Friedmann model in Section 1). As long as we consider an empty universe without mass, the answer will be just a four-sphere. In fact, it turns out that in Lorentz geometry, such a sphere has $R = 0$ everywhere, which obviously makes it minimizing. Figure 5 looks rather similar to the closed Friedmann universe in Figure 1, but it is not exactly the same.

Just as in the case with the combinatorial multiverse without entropy, the model is so far too simple to be able to generate any interesting results. To make it more interesting, we need to also include matter. This will be initiated in the next section.

## 6. A Geometric Model for the Accelerating Expansion

Can probabilistic cosmology explain the accelerating expansion? (or more generally, determine the scale factor, explain inflation etc.).

A commonly made implication of the accelerating expansion is that the universe must be open. On the other hand, as has been pointed out in Section 1, probabilistic cosmology is most easily applied to closed universes, since there are problems with making the set of all open universes into a probability space. As it turns out however, this is not an issue in the present situation, since one of the conclusions is in fact that accelerating expansion may be a very natural phenomenon also in closed universes. In this section, I will sketch a very simple model based on the random curvature multiverse of the previous section.

Let us now once more return to the closed, homogeneous, isotropic universe we started with in Section 1. If we accept the accelerating expansion as a reality, then the most common way of explaining it is to reinterpret the field equation, which leads to the idea of dark energy. But is it evident that the field equations are the right starting point?

An alternative approach is offered by probabilistic cosmology. As we saw in the previous section, the cosmology of an empty random curvature multiverse is rather simple. But if we also take into account matter, the situation becomes much more interesting. So how should the gravitational forces be included?

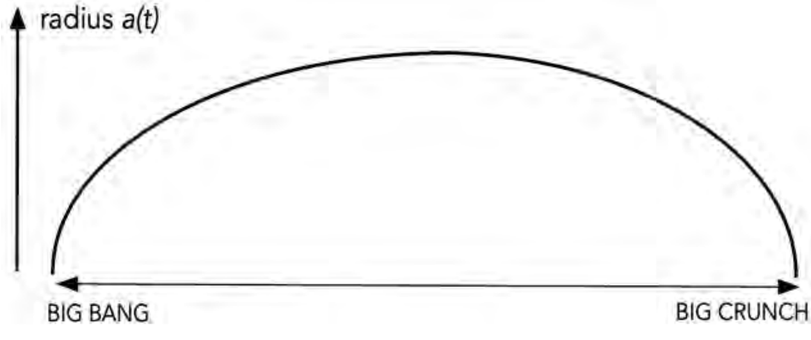


Figure 5. The form of an empty universe.

The easiest way, and also the most traditional one (although perhaps not the most fundamental one), is to actually continue the analogy between minimizing $\mathcal{F}$ and the principle of least action. In this case, we can simply add the ordinary action associated with gravitation to $\mathcal{F}$ to obtain the total action.

To make everything as easy as possible, let us just make use of the usual classical concept of potential energy. In this case it is easy to see that the total gravitational energy at a certain moment of time $t$ should be of the form

$$const \cdot \frac{1}{a(t)}, \tag{6.1}$$

which then leads to a contribution to the total action:

$$-\beta \int_{-T_0}^{T_0} \frac{1}{a(t)} \mathrm{d}t, \tag{6.2}$$

for some constant $\beta$.

***Remark* 5.** The form of the expression in (6.1) of course just expresses the fact that the (negative) potential energy between two bodies is inversely proportional to their distance. From this it follows easily e.g. that an expanding homogeneous gas will behave exactly in this way.

However, our universe as we know it does not expand as a homogeneous gas. This may have been a reasonable picture during the very first part of our history. But for the present expansion, it is much better to imagine the expansion as taking place in between galaxies of more or less fixed size and mass. This will still give rise to an expression like in (6.2) for the action, but possibly with quite a different value of $\beta$. The distribution of galaxies is by the way also an interesting field for probabilistic cosmology, but it would lead too far to go into this here.

This is one reason why the present model should not be expected to give accurate results near the endpoints. Another reason is that in this case, gravitational physics alone may not be enough to explain the expansion rate.

Summing up, the problems becomes to minimize

$$\Phi = \int_{\Omega} R^2 \mathrm{d}V - \beta \int_{-T_0}^{T_0} \frac{1}{a(t)} \mathrm{d}t \tag{6.3}$$

where $\Omega$ stands for the whole universe, under the condition that

$$V = \int_{\Omega} \mathrm{d}V = \frac{\pi^2}{2} \int_{-T_0}^{T_0} a(t)^3 \, \mathrm{d}t \tag{6.4}$$

is a fixed number, corresponding to the total 4-volume of $\Omega$. What do the solutions to this minimizing problem look like?

A traditional method of attack is to look at the Euler-Lagrange equation for the functional

$$\Phi_{\lambda} = \int_{\Omega} R^2 \mathrm{d}V - \beta \int_{-T_0}^{T_0} \frac{1}{a(t)} \mathrm{d}t + \lambda \left( V - \int_{\Omega} \mathrm{d}V \right). \tag{6.5}$$

It should be noted that a solution to this equation is in general not the same as a global minimum of $\Phi$, even if condition (6.4) is satisfied, since there could also be other stationary solutions. As it turns out, there are strong indications that the solution in this case is unique, which would then imply that finding the global minimum is in this case equivalent to solving the Euler-Lagrange equation. This is simply because the solutions to the equations in this paper tend to be uniquely determined (at least in the time-symmetric, homogeneous and isotropic case). But a rigorous treatment of this question leads to difficult and unsolved problems, which also require a much heavier mathematical machinery than I can go into here.

Having said this, we can still study the solutions to the Euler-Lagrange equation on a time-interval corresponding to the main part of the time-span of each universe. An example is plotted in **Figure 6**. If we compare this plot with the one in **Figure 5**, we note that here there is an interval of time in the beginning of the development where the function $a(t)$ is convex (and a similar interval towards the end). This corresponds exactly to a phase of accelerating expansion.

***Remark* 6.** It is quite a mathematical task to give a complete treatment of the minimizing problem in this section. Some more details are given in Tamm [9], but still a lot of work remains to be done.

However, it may be worthwhile to comment on the difference in underlying intuitive perspective between the classical theory and the present one.

In the classical closed Friedmann universe in Section 1, matter gives rise to an attractive force which makes the universe re-contract into a Big Crunch. From the intuitive point of view of a classical initial value problem, this makes the behavior in **Figure 1** very natural.

In the present context however, the perspective is somewhat different. Here the total volume is given from the start. Perhaps we may think of the universe as built up from a fixed number of elementary constituents of some kind, each with a fixed "elementary" volume. So how will the empty universe in **Figure 5** react when we add mass? Clearly, the influence should still be contractive, and the contractive force should be strongest close to the endpoints. But since the volume is fixed, contraction near the ends must imply expansion somewhere in between. From this point of view, the behavior in **Figure 6** becomes very natural.

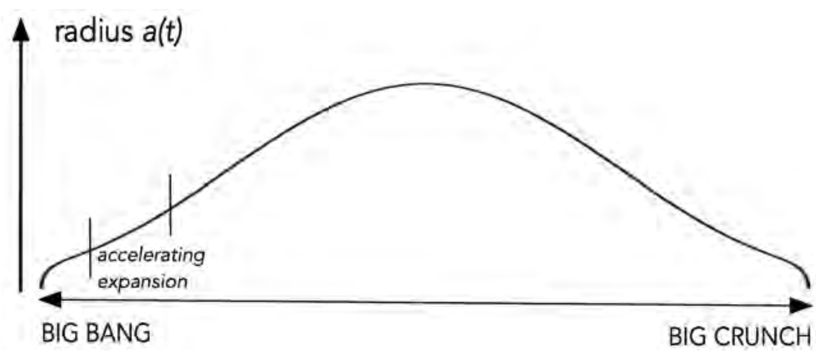


**Figure 6.** An example of a solution of the Euler-Lagrange equation [9].

***Remark* 7.** As has already been stated, the above Lagrangian approach may be the easiest way to include mass in the model. But there are bolder alternatives. One can for instance conversely attempt to interpret gravitational action in terms of curvature instead. In fact, it can be argued that in general, the presence of mass implies non-zero scalar curvature, and thus that mass in itself will contribute to the scalar curvature. Moreover, it can also be seen that two interacting bodies will give rise to less curvature than the sum of their separate contributions, in fact in a way similar to (6.2). This way of viewing the problem has the interesting property that in a sense it puts mass and the curvature of space-time on an equal footing: in both cases their influence on the physical development comes from their contributions to the integral

$$\int_{\Omega} R^2 \mathrm{d}V. \tag{6.6}$$

However, it would lead too far to go into all this here, so this discussion will have to be continued elsewhere.

## 7. Conclusions

The two examples in this paper both represent extremely simplified models for the multiverse, but they also represent two very different kinds of simplification. This is in fact one of the main reasons for choosing them as examples in this paper; to show that there are very different ways to implement probabilistic cosmology.

But would it not be better to try to create a common model which could include all aspects of probabilistic cosmology in a unified way? This would very much be like wishing for a grand unifying theory for all of fundamental physics: it would be wonderful to have one, but it is not obvious that we do ourselves a favor by advocating such a theory if the time is not ripe for it. From this point of view, the only reasonable way forward would seem to be to use different kinds of simplifications in different contexts, and only in the end we may hope that all these different aspects will unite into a more complete and unified picture.

Having said this, it is still worth pointing out that the best (at least in the opinion of the author) proxy to such a united approach that we have is Feynmann's democracy of all histories approach to physics. And, at least from an abstract point of view, this approach seems to be very well suited for the use of probabilistic cosmology. From a more applied point of view however, there may still be a long way to go before e.g. both the Combinatorial Multiverse and the Random Curvature Multiverse can be treated within such a common framework.

Summing up, it has been my purpose in this paper to show that a probabilistic approach could be a powerful tool for producing new answers in cosmology. But will the answers be the right ones? This is of course just the usual problem in science: new ideas and perspectives can be fascinating and interesting. But that does not necessarily make them correct. What is right and what is wrong can only be answered after the long and tedious process of comparing with observations and alternative explanations. Still, the more instruments we have in our tool box, the better are the perspectives for a success.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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# Local and Global Flatness in Cosmology

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## Abstract

We raise the question of how the curvature parameter $k$ is related to the curvature of the universe. We also show that, for a cosmological model that can be interpreted geometrically as a pseudo-hypersphere with a time-dependent radius, the Einstein field equations are not sufficient to fully describe the model. In addition, the differential equation system of Bianchi identities is required to describe the temporal evolution of the universe. We discuss the facts using the example of the de Sitter universe, the subluminal universe and the $R_h = ct$ model by Melia. In particular, we discuss the formal differences between the two latter models and claim that both models are identical. We also examine the possibility of introducing non-comoving coordinates.

## Keywords

Curvature Parameter, Bianchi Identities, de Sitter Cosmos, Subluminal Cosmos, $R_h = ct$ Cosmos, Geometric Horizon

## 1. Introduction

In many papers on expanding cosmological models, the topic is introduced with findings on the curvature parameter $k$. An expanding model is based on the metric in the *canonical form*

$$\mathrm{d}s^2 = \frac{1}{1-k\dfrac{r'^2}{\mathcal{R}^2}}\mathrm{d}r'^2 + r'^2\,\mathrm{d}\Omega^2 - \mathrm{d}t'^2 . \tag{1.1}$$

Here, $r'$ is the comoving radial coordinate of an observer participating in an expanding motion and $\Omega$ the solid angle. $t'$ is the cosmic time, which applies equally to all comoving observers and, at the same time, is the proper time of these observers. For $k = 1$ the underlying space should be positively curved and closed. For $k = 0$ the space is described as flat and $k = -1$ negatively curved. The two latter universes are open, they exhibit infinite extension.

In an earlier paper [1], we showed that $k=0$ does not necessarily mean that the universe described by the line element (1.1) is flat. We discuss this problem once again in Sec. 2. Sec. 3 deals extensively with the two versions of the de Sitter universe and its inconsistencies. In Sec. 4, we extend the considerations to the subluminal universe and to Melia's model in Sec. 5. We also discuss the 3-dimensional Ricci scalar and how meaningful the relation ${}^3R=0$ is for $k=0$. In Sec. 6, we explore the possibilities of finding coordinate systems for non-comoving systems.

Furthermore, we will use the following variables: $\mathcal{R}$ radius of the universe, $\mathcal{K}$ scale factor, $H$ Hubble parameter, $B,C,U$ curvature quantities, $D$ tidal forces.

## 2. The Curvature Parameter

In our paper [1], we examined in detail the free fall in the Schwarzschild field, with the intention of extending the associated methods to expanding cosmological models. With the transformation

$$\sqrt{\frac{2M}{r}}=\frac{r}{\mathcal{R}} \tag{2.1}$$

the Schwarzschild metric

$$\mathrm{d}s^2=\frac{1}{1-\dfrac{2M}{r}}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 \tag{2.2}$$

can be converted into the canonical form

$$\mathrm{d}s^2=\frac{1}{1-\dfrac{r^2}{\mathcal{R}^2}}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2-\left(1-\frac{r^2}{\mathcal{R}^2}\right)\mathrm{d}t^2. \tag{2.3}$$

Here, $\mathcal{R}=\mathcal{R}(r)$ is half the radius of curvature of the Schwarzschild parabola and according to (2.1) has the validity range $\mathcal{R}=[2M,\infty]$. At the waist of Flamm's paraboloid one has $\mathcal{R}=2M$ and this marks the event horizon.

Comparison with (1.1) shows that the curvature parameter of the metric is $k=1$, Schwarzschild geometry thus builds on a positively curved space. Furthermore, (2.3) formally corresponds to the line element of the de Sitter universe. We will build on this.

Lemaître used a coordinate transformation to transform the Schwarzschild metric into the form

$$\mathrm{d}s^2=\mathcal{K}^2\left[\mathrm{d}r'^2+\mathcal{R}^2\mathrm{d}\Omega^2\right]-\mathrm{d}t'^2,\quad \mathcal{K}=\frac{r}{\mathcal{R}}. \tag{2.4}$$

As in the cosmological models, $\mathcal{K}$ is referred to here as a *scale factor*. The line element is of type $k=0$. The new coordinate system ($i'$) accompanies a free-falling observer family. $t'$ is the common time for all observers and $r'$ the comoving radial coordinate. From the metric (2.4), we learn that $g_{4'4'}=1$. This means that there is no gravity present in this system. To get more insight into the problem, we should remember the following: Observers hover in a closed elevator. Since they are not familiar with their environment, they consider themselves motionless in a flat gravitation-free space. Such considerations have been discussed in the literature under the term "Einstein's elevator".

There is no doubt that there has been no change in the curvature of space due to the motion of the free-falling observers. $k=0$ does not mean that the underlying space is *globally flat*, but rather that it is only *locally flat* for the free-falling observers. This consideration is missing in papers which deal with cosmological models that expand in free fall.

## 3. The Two Versions of the de Sitter Universe

De Sitter designed a static cosmological model with a metric in the form (2.3). Its metric is of type $k=1$ and can be interpreted as a metric on a 4-dimensional pseudo-hypersphere embedded in a 5-dimensional flat space. The pseudo-hypersphere has the time-independent radius $\mathcal{R}$. A transformation given by Lemaître [2] [3] transforms this metric into the form (2.4) with the scale factor $\mathcal{K}=e^{\psi'}$ via:

$$\mathrm{d}s^2=\mathcal{K}^2\left[\mathrm{d}r'^2+r'^2\,\mathrm{d}\vartheta^2+r'^2\sin^2\vartheta\mathrm{d}\varphi^2\right]-\mathrm{d}t'^2. \tag{3.1}$$

It is of type $k=0$. Other models, the anti-de-Sitter model, the Lanczos and the Lanczos-like model have similar characteristics. These models are grouped into the de Sitter family. The behavior of these models in transformations from comoving to non-comoving coordinates has been extensively studied by Florides [4]. We [5] [6] [7] have complemented the Lemaître coordinate transformations using Lorentz transformations.

Since the scale factor over $t'=\mathcal{R}\psi'$ is time-dependent, the dS metric is considered in the form (2.4) as the metric of an expanding universe. However, this interpretation leads to contradictions. First of all, this view violates the principles of the general theory of relativity: A coordinate transformation cannot change the physical content of a theory. All possible coordinate systems are equal, and the choice of a particular coordinate system is usually a matter of utility.

The conservation law leads to another discrepancy. If one has redefined the cosmological constant that is unpopular with many authors using $\lambda=3/\mathcal{R}^2$, $\kappa p=-3/\mathcal{R}^2$, $\kappa\mu_0=3/\mathcal{R}^2$, one realizes that the mass density $\mu_0$ of the universe is constant, despite assumed expansion. Some authors have therefore tried to explain the constant mass density by producing new mass. However, this approach has proved unsatisfactory. Furthermore, Mitra [8] pointed out that, due to the equation of state $p+\mu_0=0$, no matter flow and no energy transport can be detected, even for the non-comoving observer.

The possibility of assigning a Lorentz transformation to the Lemaître coordinate transformation sheds some light on the problem [5] [6]. A Lorentz transformation transforms the static observer system into an accelerated one. In the static system, there are forces at every point in the cosmos that want to move the observers apart in all directions. The comoving observer system now follows these forces in free fall. According to Einstein's elevator principle [1], these forces are no longer perceptible in the comoving system. Instead of said forces, tidal forces [9] [10] occur. Mathematically, this process is carried out via the inhomogeneous transformation law of the Ricci-rotation coefficients.

This also makes it clear that neither a coordinate transformation nor a Lorentz transformation can change the geometric base structure of the space. $k=0$ in (3.1) thus results from Einstein's elevator principle and cannot be considered as a criterion for the flatness of the space. The question of how the de Sitter model is to be understood in its two versions was widely discussed among German physicists at the time. Finally, they turned to the great mathematician Klein [11]. His detailed answer ended the discussion. It is not known whether Klein's authority or the argumentative content of his work was the decisive factor. However, we cannot find any link between the geometries of the hyperspheres or their space-time slices and Klein's statements. We have not found any work that responds to Klein's publication.

The geometric structure of the pseudo-hypersphere may best expressed with the metric in the form of

$$\begin{aligned} \mathrm{d}s^2 &= \mathcal{R}^2\mathrm{d}\eta^2 + \mathcal{R}^2\sin^2\eta\,\mathrm{d}\vartheta^2 + \mathcal{R}^2\sin^2\eta\sin^2\vartheta\,\mathrm{d}\varphi^2 + \mathcal{R}^2\cos^2\eta\,\mathrm{d}i\psi^2 \\ r &= \mathcal{R}\sin\eta, \quad \mathcal{R}\mathrm{d}i\psi = i\mathrm{d}t. \end{aligned} \tag{3.2}$$

From it one takes the differential of the proper time

$$i\mathrm{d}T = \mathcal{R}\cos\eta\,\mathrm{d}i\psi, \quad \cos\eta = \sqrt{1-r^2/\mathcal{R}^2}\,. \tag{3.3}$$

Parallel slices through the pseudo-hypersphere at an arbitrary position $r$ generate pseudo-circles (*hyperbolae of constant curvature*) with the $r$-dependent radii $\mathcal{R}\cos\eta$ and the imaginary angle $i\psi$. The pseudo-circles are open, they range from $-\infty$ to $+\infty$ and have the same curvature everywhere, even at infinity. The pseudo-circle is drawn in the literature as a hyperbola in pseudo-real representation, which visualizes the dS cosmos as a one-shell hyperboloid. This has the advantage that the infinity of the timelines is recognizable. However, taking this representation literally can lead to errors. No hyperbolic property is recognizable in the dS model, no slice through the pseudo-hypersphere leads to a hyperbola.

At the point $r=\mathcal{R}$, the equator of the pseudo-hypersphere, $\mathcal{R}\cos\eta=0$ and the pseudo-circle degenerates to a point in the pseudo-real representation. No time passes there, just as time stops at the event horizon of the Schwarzschild field. As can be seen from (3.3), this point lies at $r=\mathcal{R}$, *i.e.* at the equatorial spherical surface of the 3-dimensional hypersphere, which is assigned to an observer at an arbitrarily chosen pole. We call this area the *geometric horizon*.

The above-mentioned Lorentz transformation is associated with the Lemaître transformation. From it, the relative speed of the observers, which are driven apart by the forces of the universe, can be read from

$$v = \sin\eta = \frac{r}{\mathcal{R}}\,. \tag{3.4}$$

Thus, the relative velocity is *geometrically* determined. At a pole arbitrarily fixed with $r=0$, it has the value $v=0$ and, on the equatorial spherical surface, the value $v=1$, which is the value of the speed of light in the natural system of measurement. Thus, this horizon is also a *cosmic horizon*. In [12], we have shown that the observers' recession velocity can only reach the speed of light asymptotically. This means that in the dS universe, the basic laws of special relativity are not violated.

## 4. The Subluminal Model

The dS universe dealt with in the last section is not particularly suitable for the adaptation of astrophysical data. Nevertheless, it is significant for historical reasons. It has been instrumental in driving research into expanding cosmological models and is still the starting model for new expanding approaches. It has also been criticized that in expanding universes whose metric is known in comoving systems and to which a mass distribution can be assigned, no forces are acting on the masses. The expansion in free fall is responsible for the missing forces and consequently the common cosmic time for all observers. In [12], we envisaged an extended dS model in which the observers drift apart more slowly than in free fall and recognized forces acting on such observers. This model is only of mathematical importance, but the presented technique may be useful for building more sophisticated models.

Another, rather promising attempt was a model [13] that builds on the dS universe, but drops the condition $\mathcal{R}=const.$. We have called it a subluminal model because it definitively rules out that the recession velocity of the galaxies exceeds the speed of light. The subluminal universe is positively curved and closed. It has the position-independent pressure $\kappa p=-1/\mathcal{R}^2$ and the time-dependent mass density $\kappa\mu_0=3/\mathcal{R}^2$ with the equation of state $\mu_0+3p=0$. Pressure and mass density result from the exact solutions of Einstein's field equations. The subluminal model therefore differs significantly from the FRW standard model in which the pressure is inserted by hand and is therefore not an exact solution to Einstein's field equations. Since the Einstein field equations do not fully determine FRW models, it is necessary to introduce numerous parameters, namely, the Ωs and the deceleration parameter. These quantities must then be filled using astrophysical data. The subluminal model needs only one parameter, the radius of curvature of the universe, or the scale factor. The Friedman equation takes the simple form $\mathcal{R}^{\bullet}=1$, $\mathcal{R}^{\bullet\bullet}=0$. The expansion rate of the model is constant.

For models that build on a pseudo-hypersphere with a non-constant radius, Einstein's field equations are insufficient to determine all the quantities of the model. The metric on a surface will determine the properties of that surface, but it will not be able to predict the change in the curvature of that surface. This is what the contracted Bianchi identities $R_{[m\,n\cdot r\cdot}{}^{s}{}_{\|s]}=0$ provide. They describe possible changes of the Riemann curvature tensor. For a genuine expanding cosmological model, two differential equation systems are needed

$$\begin{aligned} &\text{(I)} \quad R_{mn} - \frac{1}{2} g_{mn} R = -\kappa T_{mn} \\ &\text{(II)} \quad R_{m\ \ \|n}^{\ n} - \frac{1}{2} R_{\|m} = 0 \end{aligned} . \tag{4.1}$$

The system (II) leads to the conservation law $T_{m\ \ \|n}^{\ n} = 0$. This is often used in the literature to establish an outstanding relation to variables. However, little reference is made to the above considerations. For models with constant $\mathcal{R}$, the conservation law is trivial. Therefore, there is no need to use the system (II) to complement such a model.

The subluminal model has a geometric horizon, namely the equatorial surface of the hypersphere. As with the dS universe, it is determined by the relation (3.4) and, at the same time, it is the cosmic horizon. No galaxy can exceed the speed of light; it can only reach it asymptotically. Therefore, a galactic island formation is excluded. The possibility that superluminal speeds can occur has been deduced from Hubble's law. Using the redshift, it describes a linear relation between the recession velocity of the galaxies and the distance to a dislocated observer. However, this relation only allows for superluminal speeds if one assumes that in the Hubble equation $v = Hr$ the variable $r$ is unbounded. Arbitrary distances are only possible in open infinite universes. In a closed universe with a geometrical horizon, the radial variable can only take the amount $\mathcal{R}$, the radius of the universe, as the highest value.

We favor the view that infinite universes, be they flat or open, negatively curved ones, are ruled out as a way of describing Nature, this is because, on the one hand, infinities are hard to imagine, and on the other hand because we want to avoid conclusions from Hubble's law, which lead to acausalities and contradictions to the special theory of relativity.

Attempts have also been made to avoid the disagreeable implications of Hubble's law by arguing that the Hubble velocity is a coordinate speed that does not make reliable predictions. This problem does not apply to the subluminal model. If one differentiates $r = \mathcal{R}\sin\eta$ according to cosmic time, one first obtains the non-invariant expression $r^{\bullet} = \frac{\mathcal{R}^{\bullet}}{\mathcal{R}} r$, which reduces to $\mathcal{R}^{\bullet} = 1$ due to $v = r^{\bullet} = r/\mathcal{R} = \sin\eta$. However, we have shown in [13] that this expression can be translated to $v = \mathrm{d}x^1/\mathrm{d}T$. Here, $\mathrm{d}x^1$ and $\mathrm{d}T$ are the proper length and proper time of a non-comoving observer. Thus, the recession velocity is defined independently of the coordinates and is also the velocity used in the Lorentz transformation, which transforms the non-comoving system into the comoving system.

In the introduction we explained, with the aid of the well-known Schwarzschild model, why gravity cannot be experienced in a free-falling elevator; we then transferred the problem to cosmic free-falling observers. We now want to address the problem in greater mathematical depth by borrowing a quantity from the Ricci-rotation coefficients that is closely related to the curvature of the space-like greater circles of the pseudo-hypersphere.

The static dS metric is of type (2.3) and is the seed metric for the subluminal model. From this metric, using the standard technique of the tetrad representation, we obtain the above-considered quantity

$$B_m = \left\{ \frac{1}{r}\cos\eta, 0, 0, 0 \right\}, \quad m = 1,2,3,4. \tag{4.2}$$

Here $\eta$ is the polar angle of the pseudo-hypersphere and $\cos\eta = \sqrt{1 - r^2/\mathrm{R}^2}$. After a Lorentz transformation from the static system into the comoving system, this variable takes the form

$$B_{m'} = \left\{ \alpha\frac{1}{r}\cos\eta, 0, 0, -i\alpha v\frac{1}{r}\cos\eta \right\}. \tag{4.3}$$

Here, according to (3.4), $v = \sin\eta$ is the relative velocity between the two systems and $\alpha = 1\big/\sqrt{1-\sin^2\eta} = 1/\cos\eta$ is the assigned Lorentz factor. Finally, we have

$$B_{m'} = \left\{ \frac{1}{r}, 0, 0, -\frac{i}{\mathrm{R}} \right\}. \tag{4.4}$$

The spatial part of the quantity $B$ is $B_{\alpha'} = \left\{ \frac{1}{r}, 0, 0 \right\}, \alpha' = 1', 2', 3'$ and corresponds to the expression of a flat geometric form. The same[1] applies to the curvature of the space-like parallels of the pseudo-hypersphere

$$C_{m'} = \left\{ \frac{1}{r}, \frac{1}{r}\cot\vartheta, 0, -\frac{i}{\mathrm{R}} \right\} \tag{4.5}$$

and for a time-like slice on the pseudo-hypersphere, so for a pseudo-circle

$$U_m = \{U_1, 0, 0, 0\} \to {}'U_{m'} = \left\{ 0, 0, 0, -\frac{i}{\mathrm{R}} \right\}. \tag{4.6}$$

From (4.3) it can be seen that even in the free-falling system, the space curvature is still present via the *geometric* term $\cos\eta$, but is compensated by the *kinematic* term $\alpha$. If one writes all components of the quantity $B$ in the 5-dimensional embedding space of the pseudo-hypersphere, one has with the local extra dimension $0'$

$$B_{a'} = \left\{ \frac{1}{r}\sin\eta, \alpha\frac{1}{r}\cos\eta, 0, 0, -i\alpha v\frac{1}{r}\cos\eta \right\}, \quad a' = 0', 1', 2', 3', 4'. \tag{4.7}$$

This quantity can hardly be assigned to a flat space. Since all of the above expressions can be deduced directly from the type $k = 0$ metric,

$$\mathrm{d}s^2 = \mathrm{K}^2\left(\mathrm{d}r'^2 + r'^2\,\mathrm{d}\Omega^2\right) - \mathrm{d}t'^2 \tag{4.8}$$

one will not be able to assume that $k = 0$ inevitably leads to a flat space.

It should not be overlooked that after the Lorentz transformation, or, if one wishes, after the Lemaître transformation, fourth components appear in the three basic quantities of the model, the sum of which produces the expansion scalar. To understand the meaning of these quantities, let us return to the

[1]Details on the calculation of Ricci-rotation coefficients can be found in our monographs [9] [10].

Schwarzschild model.

The new forces are the tidal forces, which act on the observer in Einstein's elevator but are too weak in Earth's proximity to be perceived by observers. Misner, Thorne, and Wheeler [14] derived these forces in their textbook with the aid of the geodesic deviation from the Riemann curvature tensor and Sharan [15] also illustrated them in his textbook. However, in an early article [16], we deduced the tidal forces directly from Einstein's field equations. We now want to transfer the process to the cosmological problem.

We summarize the three new components in (4.4), (4.5) and (4.6) to a quantity[2] $D_{\alpha\beta}$

$$D_{11} = D_{22} = D_{33} = -\frac{i}{\mathcal{R}}, \quad D_{[\alpha\beta]} = 0 . \tag{4.9}$$

These are the gravitational forces that act on a freely expanding observer. As the space expands, they uniformly enlarge a volume around the observer in all three spatial directions. The Ricci-rotation coefficients are decomposed according to

$$A_{mn}{}^{s} = {}^*B_{mn}{}^{s} + {}^*C_{mn}{}^{s} + D_{mn}{}^{s} . \tag{4.10}$$

where ${}^*B$ and ${}^*C$ are the spatial parts of $B$ and $C$. They appear to be flat, as stated above, and the subequations of Einstein's field equations drop out of Einstein's field equations with these quantities. As such, we only need to consider

$$D_{mn}{}^{s} = u_n D_m{}^{s} - u^s D_{mn}, \quad D_{sn}{}^{s} = u_n D_s{}^{s}, \quad u_n = \{0,0,0,1\} . \tag{4.11}$$

The Ricci only contains relations with tidal forces:

$$R_{mn} = -\left[ D_{mn\wedge s} u^s + D_{mn} D_s{}^{s} \right] - 2u_n D_{[s}{}^{s}{}_{\wedge \underline{m}]} - u_m u_n \left[ D_s{}^{s}{}_{\wedge r} u^r + D_{sr} D^{sr} \right] . \tag{4.12}$$

Here $\underline{m} = 1,2,3$ and $\Phi_{m\wedge n} = \Phi_{m|n} - {}^*B_{nm}{}^{s}\Phi_s - {}^*C_{nm}{}^{s}\Phi_s$ defines the space-like covariant derivative. The relation

$$D_{[s}{}^{s}{}_{\wedge \underline{s}]} = 0 \tag{4.13}$$

indicates that the curvature of the model is location-independent and considerably simplifies the Ricci. The two remaining brackets in (4.12) lead to the results obtained using the Friedman equation. Of the Einstein field equations, only the relations

$$\begin{aligned} R_{\alpha\beta} &= -\left[ D_{\alpha\beta\wedge s} u^s + D_{\alpha\beta} D_\gamma{}^{\gamma} \right], \quad {}^3R = -\left[ D_\gamma{}^{\gamma}{}_{\wedge s} u^s + D_\delta{}^{\delta} D_\gamma{}^{\gamma} \right] \\ R_{44} &= -\left[ D_\gamma{}^{\gamma}{}_{\wedge s} u^s + D_{\alpha\beta} D^{\alpha\beta} \right], \quad {}^4R = -\left[ D_\gamma{}^{\gamma}{}_{\wedge s} u^s + D_{\alpha\beta} D^{\alpha\beta} \right] \end{aligned} \tag{4.14}$$

remain. After a short calculation, one obtains

$$R_{\alpha\beta} = 2g_{\alpha\beta}\frac{1}{\mathcal{R}^2}, \quad {}^3R = \frac{6}{\mathcal{R}^2}, \quad R_{44} = 0, \quad {}^4R = 0 . \tag{4.15}$$

We note that the 3-dimensional Ricci scalar ${}^3R$ does not vanish. Finally, for the Einstein tensor one has

$$G_{\alpha\beta} = -g_{\alpha\beta}\frac{1}{\mathcal{R}^2}, \quad G_{44} = -\frac{3}{\mathcal{R}^2} \tag{4.16}$$

[2]Since we handle everything in the comoving system, we now omit the primes on the indices.

and from that one obtains $\kappa p = -1/\mathcal{R}^2, \kappa\mu_0 = 3/\mathcal{R}^2$ and $\mu_0 + 3p = 0$, the results which are already known.

It can be seen from the above system of equations that curvature effects can also be described in the freely expanding system with the Einstein field equations.

## 5. The Model of Melia

In numerous papers[3], Melia has proposed a cosmological model that is flat and infinite and thus also contains an infinite amount of matter. Matter, space, time, and infinity were thus created at the Big Bang. Melia has called his model the $R_h = ct$ model, where $R_h$ is the non-comoving radial coordinate at the cosmic horizon of the expanding model but $t$ is the cosmic time, *i.e.* the time in the system that comoves with the expansion.

A flat infinite model has no geometric horizon that defines the cosmic horizon. Melia, building on a flat universe, creates an *event horizon* by comparing it with the Schwarzschild theory. An enclosed mass $M = M(r_h)$ of a certain volume in the universe determines the Hubble radius[4] $r_h = 2GM/c^2$ and leads to the relation $r_h = ct'$, $(R_h = ct)$. The Hubble radius is the distance light has traveled since the Big Bang and $t'$ the age of the universe. $r_h$ is the point at which the rate of expansion has reached the speed of light. From a point beyond $r_h$, there is no connection to an observer within $r_h$. Therefore, all considerations about whether Olbers' paradox can significantly influence the brightness of the sky of fixed stars are ruled out.

Since we believe that, despite the different view of the curvature parameter $k$, Melia's model is identical to our subluminal model, we have to show how Melia's quantities relate to our quantities. On the pseudo-hypersphere, $r = \mathcal{R}\sin\eta$ applies with $\mathcal{R}$ as the time-dependent radius of curvature and $\eta$ as the polar angle. At the equatorial surface, $\sin\eta = 1$, thus

$$r_h = \mathcal{R}. \tag{5.1}$$

This is the basic relation which connects the two models under discussion. The geometric horizon corresponds to the Hubble horizon and thus to Melia's event horizon. From $r_h = ct'$ one immediately gains $\dot{\mathcal{R}} = c$ or in the natural measuring system

$$\dot{\mathcal{R}} = 1, \tag{5.2}$$

a relation that we obtained with the equation system II from (4.1) and which leads to the solution of the Friedman equation. Equation (5.2) accounts for the simplicity of the subluminal model. This model also has a *geometric speed*, like all models based on a concrete geometric form. It is, as already addressed in Sec. 3,

$$v = \sin\eta = \frac{r}{\mathcal{R}} \tag{5.3}$$

[3]Most papers by Melia and colleagues are listed in [13].

[4]Melia's variables $R, t$ correspond to our variables $r, t'$, as we have used in earlier papers.

and is the recession velocity of galaxies. For $r = r_h$, one has $v_h = 1$. The recession velocity has reached the speed of light (asymptotically) at the equator. Higher speeds than the speed of light do not occur in either model, so the fundamental laws of special relativity remain.

To examine the relationship between the two models in greater depth, let us return to the above-mentioned definition of Melia's cosmic horizon. With respect to the cosmological principle of Weyl and the Birkhoff theorem, Melia determines the Hubble radius with

$$r_h = \frac{2GM\left(r_h\right)}{c^2}.$$

Here,

$$M\left(r_h\right) = \frac{4\pi}{3}\frac{r_h^3}{c^2}\mu_0$$

is the mass enclosed by the sphere with the radius $r_h$ and $\mu_0$ is the assigned mass density. Thus, with the aid of (5.1)

$$r_h = \sqrt{\frac{3c^4}{8\pi G\mu_0}} = \sqrt{\frac{3}{\kappa\mu_0}} = \mathfrak{R}. \tag{5.4}$$

This immediately results in

$$\kappa\mu_0 = \frac{3}{\mathfrak{R}^2}. \tag{5.5}$$

The mass density decreases as the universe increases in the radius $\mathfrak{R}$. These and similar relations can also be found in the Einstein universe, Friedman universe and the models of the dS family. However, this relation is not evident in the $R_h = ct$ model.

In addition, the discussion of whether the velocity defined by (5.3) is an invariant expression or a coordinate velocity remains brief. As was already explained in Sec. 4, (5.3) can be reduced to the invariant relation $v = \mathrm{d}x^1/\mathrm{d}T$, with the de Sitter proper length $\mathrm{d}x^1$ and the proper time $\mathrm{d}T$ in the non-comoving system.

Both models describe the relation between the non-comoving radial coordinate $r$ and the comoving $r'$ with

$$r = \mathfrak{K}\left(t'\right)r', \tag{5.6}$$

where $\mathfrak{K}$ is the time-dependent scale factor. We still have to show that the $R_h = ct$ model is compatible with the curvature of the pseudo-hypersphere. With

$$r = \mathfrak{R}\sin\eta, \quad r' = \mathfrak{R}_0\sin\eta, \quad \mathfrak{R} = \mathfrak{K}\mathfrak{R}_0, \quad \mathfrak{R}_0 = const. \tag{5.7}$$

we can write the Hubble parameter with both the scale factor and the pseudo-hypersphere's radius of curvature

$$H = \frac{\mathfrak{R}^{\bullet}}{\mathfrak{R}} = \frac{\mathfrak{K}^{\bullet}}{\mathfrak{K}}. \tag{5.8}$$

$\mathcal{R}_0$ is the radius of curvature of the pseudo-hypersphere, if it is measured with the aid of comoving, *i.e.* expanding rods and therefore appears to be a constant quantity for the comoving observer.

Lastly, it would still be necessary to investigate whether the evaluation of Einstein's field equations results in different criteria for space curvature. Melia essentially relies on the Friedman equation; however, this is only one part of Einstein's field equations, namely the 44-components of Einstein's field equations. We did not find the remaining three subequations of Einstein's field equations in his papers. But they are precisely what is needed to provide information about the curvature structure of space. Two methods are available for working through the problem: the coordinate method with the Christoffel symbols $\Gamma$ as connexion coefficients and the tetrad formalism with the Ricci-rotation coefficients. Both types of coefficients depend on the relation

$$A_{nm}{}^{s} = \underset{n}{e}{}^{k} \underset{m}{e}{}^{i} \overset{s}{e}_{j} \Gamma^{j}_{ki} + \overset{s}{e}_{j} \underset{m|n}{e}{}^{j} .$$

The Ricci-rotation coefficients yield three quantities, the curvatures of the greater circles, parallels, and pseudo-circles on the pseudo-hypersphere. The Christoffel symbols provide a larger number of components, most of which contain a collection of trigonometric functions that can only be assigned very indirectly or perhaps not at all to geometric objects. The procedure is therefore not particularly suited for clarifying the question of whether $k = 0$ must lead to a flat space.

Unless new arguments are submitted later that indicate a *global* flat space of the $R_h = ct$ model, the current situation is to be interpreted in such a way that both models, the $R_h = ct$ model and the subluminal model, are positively curved and therefore identical.

Melia has an extensive set of astrophysical data and has shown in some articles that this data can be best adapted to the $R_h = ct$ model, much better than to other FRW models. Thus, our subluminal model is well supported by Melia's data and analyses.

When developing our model, we did not envisage finding a model that closely relates to astrophysical data. Our goal was to provide mathematical foundations for a model that

1) is an exact solution to Einstein's field equations,

2) involves pressure, which is a result of this exact solution and is not inserted by hand, as is the case with numerous models,

3) does not allow superluminal speed and,

4) can be fully described geometrically.

The fact that this model is supported by astrophysical data was initially surprising to us, but it justifies our efforts. However, Melia's model also has an additional mathematical profile due to the subluminal model.

## 6. Coordinate Systems

Most cosmological models assume a metric written in comoving coordinates.

This metric is also the natural framework for a model, because the rods and clocks associated with such a system are the ones we currently have available. Nevertheless, there is a need to present the obtained model in non-comoving coordinates as well. Of course, the question remains as to which new insights can be gained when looking for new representations. If one processes a model in tetrad calculus, a single coordinate system is sufficient to carry out operations such as differentiation and integration. Different coordinate systems are generally useful, but are of essentially equal value for certain problems. Comoving observers are characterized by $r' = const.$, non-comoving observers by $r = const.$. The question is how to realize the latter in practice. The position of such an observer must be continuously recalculated and a fixation to the calculated point in space requires significant technical effort.

It is preferable to search for non-comoving coordinates if one does not have a static *reference system*, because one does not know the Lorentz transformation which converts expanding systems into static ones. This is the case if the model does not provide geometric velocities or if one has not fully exploited the geometry.

If one has successfully set up an expanding model and knows the metric in comoving coordinates, one also has the corresponding tetrads $\overset{m'}{e}_{i'}$. If one also knows the geometrical speed or has determined the recession velocity in another way, one can also adjust the Lorentz transformation into a non-comoving system. With this and the inhomogeneous law of transformation of the Ricci-rotation coefficients, all field quantities can be calculated in the non-comoving system. With these, one can set up the stress-energy-momentum tensor and the conservation law and recalculate the field equations. These operations can all be done without the explicit use of a coordinate system.

However, if one wants to immediately start with a static system, the following possibility is a viable option: First of all, one transforms the expanding 4-bein ($m'$) with a Lorentz transformation into a static ($m$)

$$\overset{m}{e}_{i'} = L^{m}_{m'} \overset{m'}{e}_{i'}.$$

This would be enough to calculate the Ricci-rotation coefficients for the non-comoving system. However, it is also quite inconvenient, since the new tetrads are still indicated in the comoving coordinate system ($i'$). Now the question arises as to whether the new 4-bein system can be diagonalized with a coordinate transformation $\Lambda$ with

$$\overset{m}{e}_{i} = L^{m}_{m'} \overset{m'}{e}_{i'} \Lambda^{i'}_{i}. \tag{6.1}$$

If, with a little intuition, one has found $\Lambda$, it must be ascertained whether this transformation is holonomic, *i.e.* whether it relates to coordinate lines. Thus, the relations

$$\Lambda^{i'}_{[i|k]} = 0 \Rightarrow \Lambda^{i'}_{i} = x^{i'}{}_{|i} \tag{6.2}$$

must apply. That this need not always be the case has been demonstrated by a generalized dS model [12]. As can indeed be found for this model. However, these do not fulfil the relations (6.2). Therefore, the coordinates are anholonomic, meaning that there are no coordinate lines. The Ricci-rotation coefficients can therefore not be calculated with the 4-bein alone, but must be complemented by the object of the anholonomity

$$'A_{m'n'}{}^{s'} = A_{m'n'}{}^{s'} + \Lambda_{m'n'}{}^{s'} + \Lambda^{s'}{}_{m'n'} + \Lambda^{s'}{}_{n'm'}, \quad \Lambda_{m'n'}{}^{s'} = \underset{m'}{e}^{i'} \underset{n'}{e}^{k'} \overset{s'}{e}_{j'} \Lambda^{j'}_{j} \Lambda^{j}_{[k'|i']} . \tag{6.3}$$

This of course questions the usefulness of the method.

The subluminal model provides a geometric velocity, with which the Lorentz matrix can be formed. With it, all field quantities can be transformed into the non-comoving system [13]. In particular, with the aid of the transformation analogous to (4.6) one has derived the radial force

$$\begin{gathered} U_m = \hat{U}_m + f_m, \quad \hat{U}_m = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, 0\right\}, \quad f_m = \left\{i\alpha^2 v \digamma_4, 0, 0, -i\alpha^2 v \digamma_1\right\} \\ \digamma_{m'} = \left\{0, 0, 0, -\frac{i}{\mathcal{R}}\right\}, \quad \digamma_m = L^{m'}_m \digamma_{m'} = \left\{-\alpha v \frac{1}{\mathcal{R}}, 0, 0, -\alpha \frac{i}{\mathcal{R}}\right\} \end{gathered} \tag{6.4}$$

which one is accustomed to derive from $g_{44}$ or $\overset{4}{e}_4$. Since $U_1$ is not a gradient, it is not possible to go in the opposite direction and derive the metric coefficient $g_{44}$ from (6.4). The quantity $\digamma_m$ prevents this from being possible, wherein said quantity was obtained from the expansion of the universe. It is only if one switches off the expansion $(\digamma = 0)$ that the whole expression is reduced to the known dS quantity $\hat{U}$, which can be derived from $g_{44}$. Thus, to a non-comoving observer cannot be assigned a fully non-comoving coordinate system. It should also be remembered that in (6.4) we are looking for the quantity $g_{44}$, which is a solution to the differential equation system I. However, the expression containing the quantity $\digamma$ is a solution to the differential equation system II.

The search for the lapse function $g_{44}$ is probably historical. Even in the Schwarzschild model, the metric coefficients $g_{44}$ were used to calculate the gravitational redshift and/or time dilation. Recalling our discussion of free fall in the Schwarzschild field, we find that the ratio of the proper time of the free-falling observer and that of the static observer

$$\frac{\mathrm{d}T'}{\mathrm{d}T} = \frac{1}{\alpha} = \sqrt{1 - \frac{2M}{r}}$$

is identical to $\sqrt{g_{44}}$. The time dilation can thus be deduced from the transformation behavior of the two observer systems. This applies equally to cosmological problems. For the subluminal model, one has

$$\frac{\mathrm{d}T'}{\mathrm{d}T} = \frac{1}{\alpha} = \sqrt{1 - \frac{r^2}{\mathcal{R}^2}} .$$

In this model, there is no need to resort to a possibly existing metric component $g_{44}$. If a model has a geometric velocity, the Lorentz factor can be created for observer systems moving relatively to each other and thus determine the redshift as a function of *r*.

No general method is known from the literature with which one could determine for which model static coordinates are possible. Investigations in this direction have been undertaken by Mitra [17] and Gautreau [18] [19], among others. In his papers, Melia has also tried to bring FRW metrics into the Schwarzschild form.

Apart from some marginal notes, we cannot contribute anything to this. It could be that Florides [4], with his six models, has already exhausted all the possibilities.

The subluminal model was developed by the simple generalization $\mathbb{R} = \mathbb{R}(t')$ from the dS model. The subluminal model therefore consists of a set of self-similar dS universes dislocated in the 5-dimensional space. The question thus arises as to whether these universes, together with time, can be covered by a single coordinate system.

On the other hand, one tries to set up a metric for a surface in non-comoving coordinates which describes not only the properties of the surface but also the temporal change of this surface. This attempt is reminiscent of the German story of Baron Münchhausen, who pulls himself out of the swamp by his own braid. The properties of the surface would have to be separated here, distinguishing between those belonging to system I and those belonging to system II.

## 7. Conclusion

In this paper, we have tried to establish a connection between our subluminal model and Melia's $R_h = ct$ model. We have argued that a cosmological metric with the curvature parameter $k = 0$ does not necessarily require global flatness of the universe, but rather a local flatness due to the free fall of the expanding universe. We have confirmed our point of view by gradually introducing curvature variables into the $R_h = ct$ model, bringing the $R_h = ct$ model into the formal vicinity of the subluminal model. The identity of both models is thus ensured.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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# Differential Homological Algebra and General Relativity

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## Abstract

In 1916, F.S. Macaulay developed specific localization techniques for dealing with "unmixed polynomial ideals" in commutative algebra, transforming them into what he called "inverse systems" of partial differential equations. In 1970, D.C. Spencer and coworkers studied the formal theory of such systems, using methods of homological algebra that were giving rise to "differential homological algebra", replacing unmixed polynomial ideals by "pure differential modules". The use of "differential extension modules" and "differential double duality" is essential for such a purpose. In particular, 0-pure differential modules are torsion-free and admit an "absolute parametrization" by means of arbitrary potential like functions. In 2012, we have been able to extend this result to arbitrary pure differential modules, introducing a "relative parametrization" where the potentials should satisfy compatible "differential constraints". We recently noticed that General Relativity is just a way to parametrize the Cauchy stress equations by means of the formal adjoint of the Ricci operator in order to obtain a "minimum parametrization" by adding sufficiently many compatible differential constraints, exactly like the Lorenz condition in electromagnetism. In order to make this difficult paper rather self-contained, these unusual purely mathematical results are illustrated by many explicit examples, two of them dealing with contact transformations, and even strengthening the comments we recently provided on the mathematical foundations of General Relativity and Gauge Theory. They also bring additional doubts on the origin and existence of gravitational waves.

## Keywords

Homological Algebra, Extension Module, Torsion-Free Module, Pure Differential Module, Purity Filtration, Involution, Electromagnetism, General Relativity

## 1. Introduction

The main purpose of this paper is to prove how apparently totally abstract mathematical tools, ranging among the most difficult ones existing in differential geometry and homological algebra today, can also become useful and enlighten many engineering or physical concepts (see the review Zbl 1079.93001 for the only application to control theory).

In the second section, we first sketch and then recall, with more specific references, the main (difficult) mathematical results on *differential extension modules* and *differential double duality* that are absolutely needed in order to understand the *purity* concept and, in particular, the so-called *purity filtration* of a differential module ([1] [2] [3] [4]). We also explain the unexpected link existing between *involutivity* and *purity* allowing to exhibit a *relative parametrization* of a pure differential module, even defined by a system of linear PD equations with coefficients in a non-constant differential field *K*. It is important to notice that the *reduced Spencer form*, which is used for such a purpose, generalizes the *Kalman form* existing for an OD classical control system and we shall illustrate this fact.

The third section will present for the first time a few explicit motivating academic examples in order to illustrate the above mathematical results, in particular the unexpected striking situations met in the study of contact and unimodular contact structures.

In the fourth section, we finally provide examples of applications, studying the mathematical foundations of OD/PD control theory (CT) ([3] [5]), electromagnetism (EM) ([6] [7]) and general relativity (GR) ([8] [9] [10]). Most of these examples can be now used as test examples for certain computer algebra packages recently developed for such a purpose ([11] [12]).

## 2. Mathematical Tools

Let $D=K[d_1,\cdots,d_n]=K[d]$ be the ring of differential operators with coefficients in a differential field $K$ of characteristic zero, that is such that $\mathbb{Q}\subset K$, with $n$ commuting derivations $\partial_1,\cdots,\partial_n$ and commutation relations $d_i a = a d_i + \partial_i a, \forall a\in K$. If $y^1,\cdots,y^m$ are $m$ differential indeterminates, we may identify $Dy^1+\cdots+Dy^m=Dy$ with $D^m$ and consider the finitely presented left differential module $M={}_DM$ with presentation $D^p\to D^m\to M\to 0$ determined by a given linear multidimensional system with $n$ independent variables, $m$ unknowns and $p$ equations. Applying the functor $hom_D(\bullet,D)$, we get the exact sequence $0\to hom_D(M,D)\to D^m\to D^p\to N_D\to 0$ of *right differential modules* that can be transformed by a side-changing functor to an exact sequence of finitely generated *left differential modules*. This new presentation corresponds to the *formal adjoint* $ad(\mathcal{D})$ of the linear differential operator $\mathcal{D}$ determined by the initial presentation but now with $p$ unknowns and $m$ equations, obtaining therefore a new finitely generated *left differential module* $N={}_DN$ and we may consider $hom_D(M,D)$ as the *module of equations* of the *compatibility*

*conditions* (CC) of $ad(\mathcal{D})$, a result which is not evident at first sight (see [3] [13]). Using now a maximum free submodule $0 \to D^l \to hom_D(M,D)$ or, equivalently, a maximum number of differentially linearly independent CC, and repeating this standard procedure while using the well known facts that the co-kernel of this monomorphism is a torsion module and $ad(ad(\mathcal{D})) = \mathcal{D}$, we obtain therefore an embedding $0 \to hom_D(hom_D(M,D),D) \to D^l$ of left differential modules for a certain integer $1 \le l < m$ because $K$ is a field and thus $D$ is a Noetherian bimodule over itself, a result leading to
$l = rk_D(hom_D(M,D)) = rk_D(M) < m$ as in ([14], p. 341, [15] p. 1228) (see Section 3 for the definition of the *differential rank* $rk_D$). Now, the kernel of the map $\epsilon : M \to hom_D(hom_D(M,D),D) : m \to \epsilon(m)(f) = f(m)$, $\forall f \in hom_D(M,D)$ is the torsion submodule $t(M) = \{m \in M \mid \exists 0 \ne P \in D, Pm = 0\}$ and $\epsilon$ is injective if and only if $M$ is torsion-free, that is $t(M) = 0$. In that case, we obtain by composition an embedding $0 \to M \to D^l$ of $M$ into a free module (that can also be obtained by localization if we introduce the ring of fractions $S^{-1}D = DS^{-1}$ when $S = D - \{0\}$). This result is quite important for applications as it provides a (minimal) parametrization of the linear differential operator $D$ and amounts to the controllability of a classical control system when $n = 1$ ([3] [16]). This parametrization will be called an "*absolute parametrization*" as it only involves arbitrary "*potential-like*" functions (see [4] [8] [9] [15] [17] [18] [19] [20] for more details and examples, in particular the fact that the Einstein equations cannot be parametrized).

It is however essential to notice that such an approach is leading to a "*vicious circle*" because the only constructive way to check whether $M$ is torsion-free or not is to use the differential double duality. For this reason, we briefly recall the five steps of the corresponding test allowing to know whether a given differential system or operator can be parametrized or not:

- STEP 1: Write down the system in the form of a differential operator $\mathcal{D}_1$.
- STEP 2: Construct its formal adjoint $ad(\mathcal{D}_1)$.
- STEP 3: Construct the generating CC of such an operator as a new operator and use the fact that $ad(ad(P)) = P, \forall P \in D$ in order to denote it by $ad(\mathcal{D})$.
- STEP 4: Write down $\mathcal{D} = ad(ad(\mathcal{D}))$.
- STEP 5: As $ad(\mathcal{D}) \circ ad(\mathcal{D}_1) = ad(\mathcal{D}_1 \circ \mathcal{D}) = ad(0) = 0 \Leftrightarrow \mathcal{D}_1 \circ \mathcal{D} = 0$, we just need to check whether $\mathcal{D}_1$ generates the CC of $\mathcal{D}$ or not.

If "*yes*", we shall say that $\mathcal{D}_1$ is parametrized by $\mathcal{D}$.

If "*no*", we shall say that $\mathcal{D}_1$ cannot be parametrized.

The purpose of this paper is to extend such a result to a much more general situation, that is *when $M$ is not torsion-free*, by using unexpected results first found by F.S. Macaulay in 1916 ([21]) through his study of "*inverse systems*" for "*unmixed polynomial ideals*".

Introducing $t_r(M) = \{m \in M \mid cd(Dm) > r\}$ where the codimension of $Dm$ is $n$ minus the dimension of the characteristic variety determined by $m$ in the corresponding system for one unknown, we may define the *purity filtration* as in ([1] [3] [4]):

$$0 = t_n(M) \subseteq t_{n-1}(M) \subseteq \cdots \subseteq t_1(M) \subseteq t_0(M) = t(M) \subseteq M$$

The module $M$ is said to be *r-pure* if $t_r(M) = 0$, $t_{r-1}(M) = M$ or, equivalently, if $cd(M) = cd(N) = r, \forall N \subset M$ and a torsion-free module is a 0-pure module. Moreover, when $K = k = cst(K)$ is a field of constants and $m = 1$, a pure module is *unmixed* in the sense of Macaulay, that is defined by an ideal having an equidimensional primary decomposition.

**Example 2.1**: As an elementary example with $K = k = \mathbb{Q}$, $m = 1$, $n = 2$, $p = 2$, the differential module defined by $d_{22}y = 0$, $d_{12}y = 0$ is not pure because $z' = d_2 y$ satisfies $d_2 z' = 0$, $d_1 z' = 0$ while $z'' = d_1 y$ only satisfies $d_2 z'' = 0$ and $\left((\chi_2)^2, \chi_1\chi_2\right) = (\chi_2) \cap (\chi_1, \chi_2)^2$. We obtain therefore the purity filtration $0 = t_2(M) \subset t_1(M) \subset t_0(M) = t(M) = M$ with strict inclusions as $0 \neq z' \in t_1(M)$ while $z'' \in t_0(M)$ but $z'' \notin t_1(M)$.

From the few (difficult) references ([1] [2] [3] [4] [5] [22]-[31]) dealing with the extension modules $ext^r(M) = ext_D^r(M, D)$ and purity in the framework of differential homological algebra, it is known that $M$ is $r$-pure if and only if there is an embedding $0 \to M \to ext^r\left(ext^r(M)\right)$. Indeed, the case $r = 0$ is exactly the one already considered because $ext^0(M) = ext_D^0(M, D) = hom_D(M, D)$ and the ker/coker exact sequence ([4] [5]):

$$0 \to ext^1(N) \to M \to ext^0\left(ext^0(M)\right) \to ext^2(N) \to 0$$

allows to test the torsion-free property of $M$ in actual practice by using the double-duality formula $t(M) = ext^1(N)$ as in ([3] [5]).

Independently of the previous results, the following procedure, *where one may have to change linearly the independent variables if necessary*, is the heart towards the next effective definition of involution. It is intrinsic even though it must be checked in a particular coordinate system called $\delta$*-regular* ([32] [33] [34]) and is quite simple for first order systems without zero order equations.

- *Equations of class n*: Solve the maximum number $\beta_q^n$ of equations with respect to the jets of order $q$ and class $n$. Then call $(x^1, \cdots, x^n)$ *multiplicative variables*.
- *Equations of class* $i \geq 1$: Solve the maximum number $\beta_q^i$ of *remaining* equations with respect to the jets of order $q$ and class $i$. Then call $(x^1, \cdots, x^i)$ *multiplicative variables* and $(x^{i+1}, \cdots, x^n)$ *non-multiplicative variables*.
- *Remaining equations of order* $\leq q-1$: Call $(x^1, \cdots, x^n)$ *non-multiplicative variables*.

In actual practice, we shall use a *Janet tabular* where the multiplicative "variables" are represented by their index in upper left position while the non-multiplicative variables are represented by dots in lower right position ([3] [32] [35]) (compare to ([36]).

**DEFINITION 2.2**: A system of PD equations is said to be *involutive* if its first prolongation can be achieved by prolonging its equations only with respect to the corresponding multiplicative variables. In that case, we may introduce the *Cartan characters* $\alpha_q^i = m\frac{(q+n-i-1)!}{(q-1)!(n-i)!} - \beta_q^i$ for $i = 1,\cdots,n$ and we have $dim(g_q) = \sum \alpha_q = \alpha_q^1 + \cdots + \alpha_q^n$ and $dim(g_{q+1}) = \sum i\alpha_q^i = 1\alpha_q^1 + \cdots + n\alpha_q^n$. Moreover, one can exhibit the *Hilbert polynomial* $dim(R_{q+r})$ in $r$ with leading term $(\alpha/d!)r^d$ with $d \le n$ when $\alpha$ is the smallest non-zero character in the case of an involutive symbol. Such a prolongation allows to compute *in a unique way* the principal ( $pri$ ) jets from the parametric ( $par$ ) other ones. This definition may also be applied to nonlinear systems as well.

**REMARK 2.3**: For an involutive system with $\beta = \beta_q^n < m$, then $(y^{\beta+1},\cdots,y^m)$ can be given arbitrarily and may constitute the *input* variables in control theory, though it is not necessary to make such a choice. *In this case*, the intrinsic number $\alpha = \alpha_q^n = m - \beta > 0$ is called the *n-character* and is the system counterpart of the so-called "*differential transcendence degree*" in differential algebra and the "*rank*" in module theory. As we shall see in the next Section, *the smallest non-zero character and the number of zero characters are intrinsic numbers that can most easily be known by bringing the system to involution* and we have $\alpha_q^1 \ge \cdots \ge \alpha_q^n \ge 0$.

In the situation of the last remark, the following procedure will generalize for PD control systems the well known first order Kalman form of OD control systems where the derivatives of the input do not appear ([3], VI, Remark 1.14, p 802). For this, *we just need to modify the Spencer form* and we provide the procedure that must be followed in the case of a first order involutive system with no zero order equation, for example an involutive Spencer form.

- Look at the equations of class $n$ solved with respect to $y_n^1,\cdots,y_n^\beta$.
- Use integrations by parts like:

$$y_n^1 - a(x)y_n^{\beta+1} = d_n\left(y^1 - a(x)y^{\beta+1}\right) + \partial_n a(x)y^{\beta+1} = \bar{y}_n^1 + \partial_n a(x)y^{\beta+1}$$

- Modify $y^1,\cdots,y^\beta$ to $\bar{y}^1,\cdots,\bar{y}^\beta$ in order to "*absorb*" the various $y_n^{\beta+1},\cdots,y_n^m$ *only appearing in the equations of class n*.

We have the following unexpected result providing what we shall call a *reduced Spencer form*:

**THEOREM 2.4**: The new equations of class $n$ contain $y^1,\cdots,y^\beta$ and their jets but only contain $y_i^{\beta+1},\cdots,y_i^m$ with $0 \le i \le n-1$ while the equations of class $1,\cdots,n-1$ no longer contain $y^{\beta+1},\cdots,y^m$ and their jets. Accordingly, as we shall see in the next section, any torsion element, if it exists, only depends on $\bar{y}^1,\cdots,\bar{y}^\beta$.

If $\chi_1,\cdots,\chi_n$ are $n$ algebraic indeterminates or, in a more intrinsic way, if $\chi = \chi_i dx^i \in T^*$ is a covector and $\mathcal{D}: E \to F: \xi \to a_k^{\tau\mu}(x)\partial_\mu \xi^k(x)$ is a linear *involutive* operator of order $q$, we may introduce the *characteristic matrix* $a(x,\chi) = \left(a_k^{\tau\mu}(x)\chi_\mu, |\mu| = \mu_1 + \cdots + \mu_n = q\right)$ and the resulting map $\sigma_\chi(\mathcal{D}): E \to F$ is called the *symbol* of $\mathcal{D}$ at $\chi$. Then there are two possibilities:

- If $max_\chi rk(\sigma_\chi(\mathcal{D})) < m \Leftrightarrow \alpha_q^n > 0$: the characteristic matrix fails to be injective for any covector.

- If $max_{\chi} rk\left(\sigma_{\chi}(\mathcal{D})\right) = m \Leftrightarrow \alpha_q^n = 0$: the characteristic matrix fails to be injective if and only if all the determinants of the $m \times m$ submatrices vanish. However, one can prove that this algebraic ideal $\mathfrak{a} \in K[\chi]$ is not intrinsically defined and must be replaced by its radical $rad(\mathfrak{a})$ made by all polynomials having a power in $\mathfrak{a}$. This radical ideal is called the *characteristic ideal* of the operator.

**DEFINITION 2.5**: For each $x \in X$, the algebraic set defined by the characteristic ideal is called the *characteristic set* of $\mathcal{D}$ at $x$ and $V = \bigcup_{x \in X} V_x$ is called the *characteristic set* of $\mathcal{D}$ while we keep the word "*variety*" for an irreducible algebraic set defined by a prime ideal.

One has the following important theorem ([3] [13]) that will play an important part later on:

**THEOREM 2.6**: (Hilbert-Serre) The *dimension* $d(V)$ of the characteristic set, that is the maximum dimension of the irreducible components, is equal to the number of non-zero characters while the *codimension* $cd(V) = n - d(V)$ is equal to the number of zero characters, that is to the number of "*full*" classes in the Janet tabular of an involutive system.

If $P = a^{\mu} d_{\mu} \in D = K[d]$ with implicit summation on the multi-index, the highest value of $|\mu|$ with $a^{\mu} \neq 0$ is called the *order* of the *operator* $P$ and the ring $D$ with multiplication $(P,Q) \to P \circ Q = PQ$ is filtred by the order $q$ of the operators. We have the *filtration* $0 \subset K = D_0 \subset D_1 \subset \cdots \subset D_q \subset \cdots \subset D_{\infty} = D$. Moreover, it is clear that $D$, as an algebra, is generated by $K = D_0$ and $T = D_1/D_0$ with $D_1 = K \oplus T$ if we identify an element $\xi = \xi^i d_i \in T$ with the vector field $\xi = \xi^i(x)\partial_i$ of differential geometry, but with $\xi^i \in K$ now. It follows that $D = {}_D D_D$ is a *bimodule* over itself, being at the same time a left $D$-module by the composition $P \to QP$ and a right $D$-module by the composition $P \to PQ$. We define the *adjoint* map $ad : D \to D^{op} : P = a^{\mu} d_{\mu} \to ad(P) = (-1)^{|\mu|} d_{\mu} a^{\mu}$ and we have $ad(ad(P)) = P$. It is easy to check that $ad(PQ) = ad(Q)ad(P), \forall P,Q \in D$. Such a definition can also be extended to any matrix of operators by using the transposed matrix of adjoint operators (see [3] [5] [8] [17] [20] [37] [38] for more details and applications to control theory and mathematical physics).

Accordingly, if $y = (y^1, \cdots, y^m)$ are differential indeterminates, then $D$ acts on $y^k$ by setting $d_{\mu} y^k = y_{\mu}^k$ with $d_i y_{\mu}^k = y_{\mu+1_i}^k$ and $y_0^k = y^k$. We may therefore use the jet coordinates in a formal way as in the previous section. Therefore, if a system of OD/PD equations is written in the form:

$$\Phi^{\tau} \equiv a_k^{\tau\mu} y_{\mu}^k = 0$$

with coefficients $a_k^{\tau\mu} \in K$, we may introduce the free differential module $Dy = Dy^1 + \cdots + Dy^m \simeq D^m$ and consider the differential submodule $I = D\Phi \subset Dy$ which is usually called the *module of equations*, both with the *differential module* $M = Dy/D\Phi$ or $D$-module and we may set $M = {}_D M$ if we want to specify the ring of differential operators. The work of Macaulay only covers the case $m=1$ with $K$ replaced by $k \subseteq cst(K)$. Again, we may introduce the formal *prolongation* with respect to $d_i$ by setting:

$$d_i\Phi^\tau \equiv a_k^{\tau\mu} y_{\mu+1_i}^k + \left(\partial_i a_k^{\tau\mu}\right) y_\mu^k$$

in order to induce maps $d_i : M \to M : \bar{y}_\mu^k \to \bar{y}_{\mu+1_i}^k$ if we use to denote the residue $Dy \to M : y^k \to \bar{y}^k$ by a bar as in algebraic geometry. However, for simplicity, we shall not write down the bar when the background will indicate clearly if we are in $Dy$ or in $M$.

As a byproduct, the differential modules we shall consider will always be *finitely generated* ($k=1,\cdots,m<\infty$) and *finitely presented* ($\tau=1,\cdots,p<\infty$). Equivalently, introducing the *matrix of operators* $\mathcal{D}=\left(a_k^{\tau\mu} d_\mu\right)$ with $m$ columns and $p$ rows, we may introduce the morphism $D^p \xrightarrow{\mathcal{D}} D^m : (P_\tau) \to \left(P_\tau \Phi^\tau\right) : P \to P\Phi = P\mathcal{D}$ over $D$ by acting with $D$ *on the left of these row vectors* while acting with $\mathcal{D}$ *on the right of these row vectors* and the *presentation* of $M$ is defined by the exact cokernel sequence $D^p \to D^m \to M \to 0$. It is essential to notice that the presentation only depends on $K, D$ and $\Phi$ or $\mathcal{D}$, that is to say never refers to the concept of (explicit or formal) solutions. It is at this moment that we have to take into account the results of the previous section in order to understand that certain presentations will be much better than others, in particular to establish a link with formal integrability and involution.

**DEFINITION 2.7**: It follows from its definition that $M$ can be endowed with a *quotient filtration* obtained from that of $D^m$ which is defined by the order of the jet coordinates $y_q$ in $D_q y$. We have therefore the *inductive limit* $0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_q \subseteq \cdots \subseteq M_\infty = M$ with $d_i M_q \subseteq M_{q+1}$ and $M = DM_q$ for $q \gg 0$ with prolongations $D_r M_q \subseteq M_{q+r}, \forall q, r \geq 0$. We shall set $gr\left(M_q\right) = G_q = M_q / M_{q-1}$ and $gr(M) = G = \oplus_q G_q$.

Having in mind that $K$ is a left $D$-module for the action $(D,K) \to K : (d_i, a) \to \partial_i a$ and that $D$ is a bimodule over itself, *we have only two possible constructions*:

**DEFINITION 2.8**: We define the *system* $R = hom_K(M,K) = M^*$ and set $R_q = hom_K\left(M_q, K\right) = M_q^*$ as the *system of order q*. We have the *projective limit* $R = R_\infty \to \cdots \to R_q \to \cdots \to R_1 \to R_0$. It follows that $f_q \in R_q : y_\mu^k \to f_\mu^k \in K$ with $a_k^{\tau\mu} f_\mu^k = 0$ defines a *section at order q* and we may set $f_\infty = f \in R$ for a *section* of $R$. For a ground field of constants $k$, this definition has of course to do with the concept of a formal power series solution. However, for an arbitrary differential field $K$, *the main novelty of this new approach is that such a definition has nothing to do with the concept of a formal power series solution* (*care*) as illustrated in ([39]).

**DEFINITION 2.9**: We may define the right differential module $ext^0(M) = hom_D(M,D)$.

**PROPOSITION 2.10**: When $M$ is a left $D$-module, then $R$ is also a left $D$-module.

*Proof*: As $D$ is generated by $K$ and $T$ as we already said, let us define:

$$(af)(m) = af(m), \quad \forall a \in K, \forall m \in M$$

$$(\xi f)(m) = \xi f(m) - f(\xi m), \quad \forall \xi = a^i d_i \in T, \forall m \in M$$

In the operator sense, it is easy to check that $d_i a = a d_i + \partial_i a$ and that $\xi\eta - \eta\xi = [\xi, \eta]$ is the standard bracket of vector fields. We finally get $(d_i f)^k_\mu = (d_i f)(y^k_\mu) = \partial_i f^k_\mu - f^k_{\mu+1_i}$ and thus recover *exactly* the Spencer operator though *this is not evident at all*. We also get $(d_i d_j f)^k_\mu = \partial_{ij} f^k_\mu - \partial_i f^k_{\mu+1_j} - \partial_j f^k_{\mu+1_i} + f^k_{\mu+1_i+1_j} \Rightarrow d_i d_j = d_j d_i, \forall i, j = 1, \cdots, n$ and thus $d_i R_{q+1} \subseteq R_q \Rightarrow d_i R \subset R$ induces a well defined operator $R \to T^* \otimes R : f \to dx^i \otimes d_i f$. This result has been discovered (up to sign) by Macaulay in 1916 ([21]). For more details on the Spencer operator and its applications, the reader may look at ([14] [40] [41] [42]).

Q.E.D.

**DEFINITION 2.11**: $t_r(M)$ is the greatest differential submodule of *M* having codimension > *r*.

**PROPOSITION 2.12**: $cd(M) = cd(V) = r \Leftrightarrow \alpha_q^{n-r} \neq 0$, $\alpha_q^{n-r+1} = \cdots = \alpha_q^n = 0 \Leftrightarrow t_r(M) \neq M$, $t_{r-1}(M) = \cdots = t_0(M) = t(M) = M$ and this intrinsic result can be most easily checked by using the standard or reduced Spencer form of the system defining *M*.

We are now in a good position for defining and studying purity for differential modules.

**DEFINITION 2.13**: *M* is *r-pure* $\Leftrightarrow t_r(M) = 0$, $t_{r-1}(M) = M \Leftrightarrow cd(Dm) = r$, $\forall m \in M$. More generally, *M* is *pure* if it is *r*-pure for a certain $0 \leq r \leq n$. In particular, *M* is 0-pure if $t(M) = 0$ and, if $cd(M) = r$ but *M* is not *r*-pure, we may call $M/t_r(M)$ the *pure part* of *M*. It follows that $t_{r-1}(M)/t_r(M)$ is equal to zero or is *r*-pure (see the picture in [3], p. 545). When $M = t_{n-1}(M)$ is *n*-pure, its defining system is a finite dimensional vector space over *K* with a symbol of finite type, that is when $g_q = 0$ is (trivially) involutive. Finally, when $t_{r-1}(M) = t_r(M)$, we shall say that there is a "*gap*" in the purity filtration:

$$0 = t_n(M) \subseteq t_{n-1}(M) \subseteq \cdots \subseteq t_1(M) \subseteq t_0(M) = t(M) \subseteq M$$

**PROPOSITION 2.14**: $t_r(M)$ does not depend on the presentation or on the filtration of *M*.

**EXAMPLE 2.15**: If $K = \mathbb{Q}$ and *M* is defined by the involutive system $y_{33} = 0, y_{23} = 0, y_{13} = 0$, then $z = y_3$ satisfies $d_3 z = 0, d_2 z = 0, d_1 z = 0$ and $cd(Dz) = 3$ while $z' = y_2$ only satisfies $d_3 z' = 0$ and $cd(Dz') = 1$. We have the purity filtration $0 = t_3(M) \subset t_2(M) = t_1(M) \subset t_0(M) = t(M) = M$ with one gap and two strict inclusions.

We now recall the definition of the *extension modules* $ext_D^i(M, D)$ that we shall simply denote by $ext^i(M)$ and the way to use their dimension or codimension. We point out once more that these numbers can be most easily obtained by bringing the underlying systems to involution in order to get informations on *M* from informations on *G*. We divide the procedure into four steps that can be achieved by means of computer algebra ([11] [12]):

- STEP 1: Construct a *free resolution* of $M$, say:
$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to M \to 0$$
- STEP 2: Suppress $M$ in order to obtain the *deleted sequence*:
$$\cdots \to F_i \to \cdots \to F_1 \to F_0 \to 0$$
- STEP 3: Apply $hom_D(\bullet, D)$ in order to obtain the *dual sequence* heading backwards:
$$\cdots \leftarrow hom_D(F_i, D) \leftarrow \cdots \leftarrow hom_D(F_1, D) \leftarrow hom_D(F_0, D) \leftarrow 0$$
- STEP 4: Define $ext^i(M)$ to be the cohomology at $hom_D(F_i, D)$ of the dual sequence in such a way that $ext^0(M) = hom_D(M, D)$.

The following nested chain of difficult propositions and theorems can be obtained, *even in the non-commutative case*, by combining the use of extension modules and *bidualizing complexes* in the framework of algebraic analysis. The main difficulty is to obtain first these results for the *graded module* $G = gr(M)$ by using techniques from commutative algebra before extending them to the *filtered module* $M$ as in ([1] [2] [3] [4] [24] [25] [26] [27] [37] [42]).

**THEOREM 2.16**: The extension modules do not depend on the resolution of $M$ used.

**PROPOSITION 2.17**: Applying $hom_D(\bullet, D)$ provides right $D$-modules that can be transformed to left $D$-modules by means of the *side changing functor* and vice-versa. Namely, if $N_D$ is a right $D$-module, then ${}_D N = \wedge^n T \otimes_K N_D$ is the *converted left* $D$-module while, if $N = {}_D N$ is a left $D$-module, then $N_D = \wedge^n T^* \otimes_K N$ is the *converted right* $D$-module.

**PROPOSITION 2.18**: Instead of using $hom_D(\bullet, D)$ and the side changing functor in the module framework, we may use $ad$ in the operator framework. Namely, to any operator $\mathcal{D}: E \to F$ we may associate the formal adjoint $ad(\mathcal{D}): \wedge^n T^* \otimes F^* \to \wedge^n T^* \otimes E^*$ with the useful though striking relation $rk_D(ad(\mathcal{D})) = rk_D(\mathcal{D})$.

**PROPOSITION 2.19**: $ext^i(M)$ is a torsion module $\forall 1 \le i \le n$ but $ext^0(M) = hom_D(M, D)$ may not be a torsion module.

**EXAMPLE 2.20**: When $M$ is a torsion module, we have $hom_D(M, D) = 0$. Indeed, if $m \in M$, we may find $0 \ne P \in D$ such that $Pm = 0$. Hence, if $f \in hom_D(M, D)$, we have $Pf(m) = f(Pm) = f(0) = 0$ in $D$ and thus $f(m) = 0, \forall m \in M$, that is to say $f = 0$ because $D$ is an integral domain. When $n = 3$ and the torsion-free module $M$ is defined by the formally surjective $div$ operator, the formal adjoint of $div$ is $-grad$ which defines a torsion module. Also, when $n = 1$ as in classical control theory, a controllable system with coefficients in a differential field allows to define a torsion-free module $M$ which is free in that case because a finitely generated module over a principal ideal domain is free if and only if it is torsion-free and $hom_D(M, D)$ is thus also a free module.

**THEOREM 2.21**: $ext^i(M) = 0, \forall i < cd(M)$ and $\forall i \ge n+1$.

**THEOREM 2.22**: $cd\left(ext^i\left(M\right)\right) \geq i$.

**THEOREM 2.23**: $cd\left(M\right) \geq r \Leftrightarrow ext^i\left(M\right) = 0, \forall i < r$.

**PROPOSITION 2.24**: $cd\left(M\right) = r \Rightarrow cd\left(ext^r\left(M\right)\right) = r$ and $ext^r\left(M\right)$ is *r*-pure.

**PROPOSITION 2.25**: $ext^r\left(ext^r\left(M\right)\right)$ is equal to 0 or is *r*-pure, $\forall 0 \leq r \leq n$.

**PROPOSITION 2.26**: If we set $t_{-1}\left(M\right) = M$, there are exact sequences $\forall 0 \leq r \leq n$:

$$0 \to t_r\left(M\right) \to t_{r-1}\left(M\right) \to ext^r\left(ext^r\left(M\right)\right)$$

**THEOREM 2.27**: If $cd\left(M\right) = r$, then *M* is *r*-pure if and only if there is a monomorphism $0 \to M \to ext^r\left(ext^r\left(M\right)\right)$ of left differential modules.

**THEOREM 2.28**: *M* is pure $\Leftrightarrow ext^s\left(ext^s\left(M\right)\right) = 0, \forall s \neq cd\left(M\right)$.

**COROLLARY 2.29**: If *M* is *r*-pure with $r \geq 1$, then it can be embedded into a differential module *L* having a free resolution with only *r* operators.

The previous theorems are known to characterize purity but it is however evident that they are not very useful in actual practice. For more details on these two results which are absolutely out of the scope of this paper, see ([2], pp. 490-491) and ([3], p. 547). Proposition 3.24 and Theorem 3.25 come from the Cohen-Macaulay property of *M*, namely $cd\left(M\right) = g\left(M\right) = inf\left\{i \mid ext^i\left(M\right) \neq 0\right\}$ where $g\left(M\right)$ is called the *grade* of *M* (see [2] and [3] [4] for more details).

**THEOREM 2.30**: When *M* is *r*-pure, the characteristic ideal is thus *unmixed*, that is a finite intersection of prime ideals having the same codimension *r* and the characteristic set is *equidimensional*, that is the union of irreducible algebraic varieties having the same codimension *r*.

In 2012 we have provided a new effective test for checking purity while using the involutivity of the Spencer form with four steps as follows ([4]):

- STEP 1: Compute the involutive Spencer form of the system and the number *r* of full classes.
- STEP 2: Select only the equations of class 1 to $d\left(M\right) = n - r$ of this Spencer form which are making an involutive system over $K\left[d_1, \cdots, d_{(n-r)}\right]$.
- STEP 3: Using differential biduality for such a system, check if it defines a torsion-free module $M_{(n-r)}$ and work out a parametrization.
- STEP 4: Substitute the above parametrization in the remaning equations of class $n - r + 1, \cdots, n$ of the Spencer form in order to get a system of PD equations which provides the *parametrizing module L* in such a way that $M \subseteq L$ and *L* has a resolution with *r* operators.

**THEOREM 2.31**: As purity is an intrinsic property, we may work with an involutive Spencer form and *M* is *r*-pure if the classes $n - r + 1, \cdots, n$ are full and the module $M_{(n-r)}$ defined by the equations of class 1 + $\cdots$ + class $\left(n - r\right)$ is torsion-free. Hence *M* is 0-pure if it is torsion-free.

We shall now illustrate and apply this new procedure in the next two sections.

## 3. Motivating Examples

**EXAMPLE 3.1**: With $n = 3, m = 1$ and $K = \mathbb{Q}$, let us consider the following polynomial ideal:

$$\mathfrak{a} = \left( (\chi_3)^2, \chi_2\chi_3 - \chi_1\chi_3, (\chi_2)^2 - \chi_1\chi_2 \right) \subset K[\chi_1, \chi_2, \chi_3] = K[\chi]$$

We shall discover that it is not evident to prove that it is an unmixed polynomial ideal and that the corresponding differential module is 1-pure.

The first result is provided by the existence of the *primary decomposition* obtained from the two existing factorizations ([23]):

$$\mathfrak{a} = \left( (\chi_3)^2, \chi_2 - \chi_1 \right) \cap (\chi_3, \chi_2) = \mathfrak{q}' \cap \mathfrak{q}''$$

Taking the respective radical ideals, we get the *prime decomposition*:

$$rad(\mathfrak{a}) = (\chi_3, \chi_2 - \chi_1) \cap (\chi_3, \chi_2) = \mathfrak{p}' \cap \mathfrak{p}'' = rad(\mathfrak{q}') \cap rad(\mathfrak{q}'')$$

The corresponding involutive system is:

$$\left\{ \begin{array}{l} y_{33} = 0 \\ y_{23} - y_{13} = 0 \\ y_{22} - y_{12} = 0 \end{array} \right. \quad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{array}}$$

with characters $\alpha_2^3 = 1 - 1 = 0$, $\alpha_2^2 = 2 - 2 = 0$, $\alpha_2^1 = 3 - 0 = 3$ and $dim(g_2) = \sum \alpha = 3$.

Setting ($z^1 = y, z^2 = y_1, z^3 = y_2, z^4 = y_3$), we obtain the involutive first order Spencer form:

$$\left\{ \begin{array}{l} z_3^4 = 0, z_3^3 - z_1^4 = 0, z_3^2 - z_1^4 = 0, z_3^1 - z^4 = 0 \\ z_2^4 - z_1^4 = 0, z_2^3 - z_1^3 = 0, z_2^2 - z_1^3 = 0, z_2^1 - z^3 = 0 \\ z_1^1 - z^2 = 0 \end{array} \right. \quad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \end{array}}$$

with new characters $\alpha_1^3 = 4 - 4 = 0$, $\alpha_1^3 = 4 - 4 = 0$, $\alpha_1^1 = 4 - 1 = 3$ and similarly $dim(g_1) = \sum \alpha = 3$. Both class 3 and class 2 are full while class 1 is defining a torsion-free module $M_{(1)}$ over $K[d_1]$ by means of a trivially involutive system of class 1. Hence the differential module $M$ is such that $cd(M) = 2$ and is 1-pure because it is 1-pure in *this* presentation.

Suppressing the bar for the various residues, we are ready to exhibit the *relative parametrization* defining the *parametrization module* $L$ because we may choose the 3 potentials $(z^1 = y, z^3, z^4)$ while taking into account that $z^2 = y_1 = d_1 y$:

$$\left\{ \begin{array}{l} z_3^4 = 0 \\ z_3^3 - z_1^4 = 0 \\ z_3^1 - z^4 = 0 \\ z_2^4 - z_1^4 = 0 \\ z_2^3 - z_1^3 = 0 \\ z_2^1 - z^3 = 0 \end{array} \right. \quad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{array}}$$

Both $(y, z^3, z^4)$ are torsion elements and we can eliminate $(z^3, z^4)$ in order to find the desired system that must be satisfied by $y$ which is showing the inclusion $M \subset L$ but we have indeed $M = L$ because $z^3 = y_2, z^4 = y_3$. It follows that $M$ admits a free resolution with only 2 operators, a result following at once from the last Janet tabular, *contrary to the previous one*.

The reader may treat similarly the example $\mathfrak{a} = (\chi_1, \chi_2) \cap (\chi_3, \chi_4)$ and look at ([39]) for details. (Hint: use the involutive system $y_{44} + y_{14} = 0$, $y_{34} + y_{13} = 0$, $y_{33} + y_{23} = 0$, $y_{24} - y_{13} = 0$).

**EXAMPLE 3.2**: With $n = 3$, $m = 1$, $q = 2$, $K = \mathbb{Q}$, $D = K[d_1, d_2, d_3]$, let us consider the differential module $M$ defined by the second order system $Py \equiv y_{33} = 0$, $Qy \equiv y_{13} - y_2 = 0$ first considered by Macaulay in 1916 ([19] [21]). We shall prove that $M$ is 2-pure through the inclusion $0 \to M \to ext^2(ext^2(M))$ directly and by finding out a relative parametrization, a result highly not evident at first sight.

First of all, in order to find out the codimension $cd(M) = 2$, we have to consider the equivalent involutive system:

$$\left\{\begin{array}{l} \Phi^4 \equiv y_{33} = u \\ \Phi^3 \equiv y_{23} = d_1 u - d_3 v \\ \Phi^2 \equiv y_{22} = d_{11} u - d_{13} v - d_2 v \\ \Phi^1 \equiv y_{13} - y_2 = v \end{array}\right. \qquad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \end{array}$$

The Janet tabular on the right allows at once to compute the characters $\alpha_2^3 = 0$, $\alpha_2^2 = 0$, $\alpha_2^1 = \alpha = 3 - 1 = 2$ and to construct the following strictly exact sequence of differential modules:

$$0 \to D \to D^4 \to D^4 \to D \xrightarrow{p} M \to 0$$

Also, we have

$$rad(\mathfrak{a}) = rad\left((\chi_3)^2, \chi_2\chi_3, (\chi_2)^2, \chi_1\chi_3\right) = (\chi_3, \chi_2) = \mathfrak{p} \Rightarrow dim(V) = 1 .$$

As the classes 3 and 2 are full, it follows that $d(M) = d(Dy) = 1 \Rightarrow cd(M) = n - 1 = 2$ if we denote simply by $y$ the canonical residue $\bar{y}$ of $y$ after identifying $D$ with $Dy$. We have constructed explicitly in ([29]) a finite length resolution of $N = ext^2(M)$ by pointing out that $N$ does not depend on the resolution of $M$ used and one can refer to the single compatibility condition (CC) $P \circ Qy - Q \circ Py = 0$ for the initial system in the exact sequence made by second order operators:

$$0 \to D \xrightarrow[2]{\mathcal{D}_1} D^2 \xrightarrow{\mathcal{D}} D \xrightarrow{p} M \to 0$$

Indeed, introducing differential duality through the functor $hom_D(\bullet, D)$ and the respective adjoint operators, we may define the torsion left differential module $N$ by the long exact sequence:

$$0 \leftarrow N \xleftarrow{q} D \xleftarrow{ad(\mathcal{D}_1)} D^2 \xleftarrow{ad(\mathcal{D})} D \leftarrow 0$$

showing that $rk_D(M) = 0 \Rightarrow rk_D(N) = 1 - 2 + 1 = 0$ because of the additivity property of the differential rank and the vanishing of the Euler-Poincaré characteristic of the full sequence. It follows that $M = ext^2(N) = ext^2(ext^2(M))$.

Similarly, using certain parametric jet variables as new unknowns, we may set $z^1 = y$, $z^2 = y_1$, $z^3 = y_2$, $z^4 = y_3$ in order to obtain the following involutive first order system with no zero order equation:

$$\begin{cases} \text{class 3} & d_3z^1 - z^4 = 0, d_3z^2 - z^3 = 0, d_3z^3 = 0, d_3z^4 = 0 \\ \text{class 2} & d_2z^1 - z^3 = 0, d_2z^2 - d_1z^3 = 0, d_2z^3 = 0, d_2z^4 = 0 \\ \text{class 1} & d_1z^1 - z^2 = 0, d_1z^4 - z^3 = 0 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \end{matrix}}$$

where we have separated the classes while using standard computer algebra notations this time instead of the jet notations used in the previous example. Contrary to what could be believed, this operator does not describe the Spencer sequence that could be obtained from the previous Janet sequence but we can use it exactly like a Janet sequence or exactly like a Spencer sequence. We obtain therefore a long strictly exact sequence of differential modules *with only first order operators* while replacing $Dy$ by $Dz = Dz^1 + Dz^2 + Dz^3 + Dz^4$ as follows:

$$0 \to D^2 \underset{1}{\longrightarrow} D^8 \underset{1}{\longrightarrow} D^{10} \underset{1}{\longrightarrow} D^4 \to M \to 0$$

and we still have the vanishing Euler-Poincaré characteristic $2 - 8 + 10 - 4 = 0$.

The differential module $M_1$ is defined over $K[d_1]$ by the two PD equations of class 1 and is easily seen to be torsion-free with the two potentials $(z^1 = y, z^4)$. Substituting into the PD equations of class 2 and 3, we obtain the generating differential constraints:

$$\begin{cases} d_3z^1 - z^4 = 0 \\ d_3z^4 = 0 \\ d_2z^1 - d_1z^4 = 0 \\ d_2z^4 = 0 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \end{matrix}}$$

They define the *parametrization module* $L$ and the inclusion $M \subseteq L$ is obtained by eliminating $z^4$ but we have indeed $M = L$ because $z^4 = d_3y$.

**EXAMPLE 3.3**: We have provided in ([4], Example 4.2) a case leading to a strict inclusion $M \subset L$ that we revisit now totally in this new framework. With $K = \mathbb{Q}, m = 1, n = 4, q = 2$, let us study the 2-pure differential module $M$ defined by the involutive second order system:

$$\begin{cases} y_{44} = 0 \\ y_{34} = 0 \\ y_{33} = 0 \\ y_{24} - y_{13} = 0 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & \bullet & \bullet \end{matrix}}$$

From the Janet tabular we may construct at once the Janet sequence:

$$0 \to \Theta \to 1 \xrightarrow{\mathcal{D}} 4 \xrightarrow{\mathcal{D}_1} 4 \xrightarrow{\mathcal{D}_2} 1 \to 0$$

where $\mathcal{D}_1$ is defined by the involutive system:

$$\begin{cases} d_4(y_{34}) - d_3(y_{44}) = 0 \\ d_4(y_{33}) - d_3(y_{34}) = 0 \\ d_4(y_{24} - y_{13}) - d_2(y_{44}) + d_1(y_{34}) = 0 \\ d_3(y_{24} - y_{13}) - d_2(y_{34}) + d_1(y_{33}) = 0 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \end{matrix}}$$

and so on. We have therefore a free resolution of $M$ with 3 operators:

$$0 \to D \to D^4 \to D^4 \to D \to M \to 0$$

and thus discover that $pd(M) \le 3$.

However, we have

$$rad\left((\chi_4)^2, \chi_3\chi_4, (\chi_3)^2, \chi_2\chi_4 - \chi_1\chi_3\right) = (\chi_4, \chi_3) = \mathfrak{p} \Rightarrow dim(V) = 1.$$

Let us transform the initial second order involutive system for $y$ into a first order involutive system for $\left(z^1 = y, z^2 = y_1, z^3 = y_2, z^4 = y_3, z^5 = y_4\right)$ as follows:

$$\begin{cases} d_4 z^1 - z^5 = 0, d_4 z^2 - d_1 z^5 = 0, d_4 z^3 - d_1 z^4 = 0, d_4 z^4 = 0, d_4 z^5 = 0 \\ d_3 z^1 - z^4 = 0, d_3 z^2 - d_1 z^4 = 0, d_3 z^3 - d_2 z^4 = 0, d_3 z^4 = 0, d_3 z^5 = 0 \\ d_2 z^1 - z^3 = 0, d_2 z^2 - d_1 z^3 = 0, d_2 z^5 - d_1 z^4 = 0 \\ d_1 z^1 - z^2 = 0 \end{cases} \quad \begin{array}{|cccc|} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & \bullet & \bullet \\ 1 & \bullet & \bullet & \bullet \\ \hline \end{array}$$

with five equations of full class 4, five equations of full class 3, three equations of class 2 and finally one equation of class 1. The equations of classes 2 and 1 are providing an involutive system over $\mathbb{Q}[d_1, d_2]$ defining a torsion-free module $M_{(2)}$ that can be parametrized by setting $z^1 = y, z^2 = d_1 y, z^3 = d_2 y, z^4 = d_2 z, z^5 = d_1 z$ with only 2 arbitrary potentials $(y, z)$. Substituting in the other equations of classes 3 and 4, we finally discover that $L$ is defined by the involutive system describing the relative parametrization:

$$\begin{cases} d_4 y - d_1 z = 0 \\ d_4 z = 0 \\ d_3 y - d_2 z = 0 \\ d_3 z = 0 \end{cases} \quad \begin{array}{|cccc|} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & 3 & \bullet \\ \hline \end{array}$$

We have the strict inclusion $M \subset L$ obtained by eliminating $z$ because now $z \notin Dy$ if we take the residue or, equivalently, the residue of $z$ does not belong to $M$. The differential module $L$ defined by the above system is therefore 2-pure with a strict inclusion $M \subset L$ and admits a free resolution with only 2 operators according to its Janet tabular.

**EXAMPLE 3.4**: (*Contact structure*) With $n = m = 3$ and $K = \mathbb{Q}\left(x^1, x^2, x^3\right)$ let us introduce the so-called *contact* 1-form $\alpha = dx^1 - x^3 dx^2$ and consider the first order system of infinitesimal Lie equations obtained by eliminating the contact factor $\rho$ from the equations $L(\xi)\alpha = \rho\alpha$. We let the reader check that he will obtain only the two equations $\Phi^1 = 0, \Phi^2 = 0$ which is nevertheless neither formally integrable nor even involutive. Using crossed derivatives one obtains the involutive system:

$$\begin{cases} \Phi^3 \equiv \partial_3\xi^3 + \partial_2\xi^2 + 2x^3\partial_1\xi^2 = 0 \\ \Phi^2 \equiv \partial_3\xi^1 - x^3\partial_3\xi^2 = 0 \\ \Phi^1 \equiv \partial_2\xi^1 - x^3\partial_2\xi^2 + x^3\partial_1\xi^1 - \left(x^3\right)^2\partial_1\xi^2 - \xi^3 = 0 \end{cases} \quad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ \hline \end{array}$$

with the unique CC $\Psi \equiv \partial_3\Phi^1 - \partial_2\Phi^2 - x^3\partial_1\Phi^2 + \Phi^3 = 0$. The following injective absolute parametrization is well known and we let the reader find it by using differential double duality:

$$\phi - x^3\partial_3\phi = \xi^1, -\partial_3\phi = \xi^2, \partial_2\phi + x^3\partial_1\phi = \xi^3 \Rightarrow \xi^1 - x^3\xi^2 = \phi$$

We obtain the Janet sequence

$$\begin{array}{ccccccccc} 0 \rightarrow & 1 & \xrightarrow{\mathcal{D}_{-1}} & 3 & \xrightarrow{\mathcal{D}} & 3 & \xrightarrow{\mathcal{D}_1} & 1 & \rightarrow 0 \\ & \phi & & \xi & & \Phi & & \Psi & \end{array}$$

with formally exact adjoint sequence:

$$\begin{array}{ccccccccc} 0 \leftarrow & 1 & \xleftarrow{ad(\mathcal{D}_{-1})} & 3 & \xleftarrow{ad(\mathcal{D})} & 3 & \xleftarrow{ad(\mathcal{D}_1)} & 1 & \rightarrow 0 \\ & \theta & & \nu & & \mu & & \lambda & \end{array}$$

and the resolution of the trivially torsion-free module $M \simeq D$:

$$0 \rightarrow D \rightarrow D^3 \rightarrow D^3 \rightarrow M \rightarrow 0$$

which splits totally because it is made with free and thus projective modules.

**EXAMPLE 3.5**: (*Unimodular contact structure*) With $n = m = 3$ and $K = \mathbb{Q}(x^1, x^2, x^3)$ let us introduce the 1-form $\omega = dx^1 - x^3dx^2$ used as a geometric object and consider the first order system of infinitesimal Lie equations from the equations $L(\xi)\omega = 0$. One obtains the system using jet notations:

$$\xi^1_3 - x^3\xi^2_3 = 0, \xi^1_2 - x^3\xi^2_2 - \xi^3 = 0, \xi^1_1 - x^3\xi^2_1 = 0$$

We let the reader prove that these three PD equations are differentially independent and we obtain the free resolution of *M*:

$$0 \rightarrow D^3 \xrightarrow{\mathcal{D}} D^3 \rightarrow M \rightarrow 0$$

and its adjoint sequence is:

$$0 \leftarrow N \leftarrow D^3 \xleftarrow{ad(\mathcal{D})} D^3 \leftarrow 0$$

because $rk_D(M) = rk_D(N) = 3 - 3 = 0$, that is both *M* and *N* are torsion modules with $N = ext^1(M) \Rightarrow M = ext^1(N) = ext^1(ext^1(M))$ and $M$ is surely 1-pure. However, this system is not formally integrable, as it can be checked directly through crossed derivatives or by noticing that $\mathcal{L}(\xi)d\omega = 0$ with $d\omega = dx^2 \wedge dx^3$ and $\mathcal{L}(\xi)(\omega \wedge d\omega) = 0$ with $\omega \wedge d\omega = dx^1 \wedge dx^2 \wedge dx^3$. Hence, we have to add the 3 first order equations:

$$\xi^2_2 + \xi^3_3 = 0, \xi^3_1 = 0, \xi^2_1 = 0 \Rightarrow \xi^1_1 = 0$$

Exchanging $x^1$ and $x^3$, we obtain the equivalent involutive system in $\delta$ -regular coordinates:

$$\left\{\begin{array}{l} \xi^3_3 = 0 \\ \xi^2_3 = 0 \\ \xi^1_3 = 0 \\ \xi^2_2 + \xi^3_1 = 0 \\ \xi^1_2 + x^1\xi^3_1 - \xi^3 = 0 \\ \xi^1_1 - x^1\xi^2_1 = 0 \end{array}\right. \quad \begin{array}{|ccc|} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ \hline \end{array}$$

The differential module $M_{(2)}$ over $K[d_1, d_2]$ is defined by the three bottom equations. Setting now $\phi = \xi^1 - x^1\xi^2$, we deduce from the last bottom equation that $\xi^2 = -d_1\phi$ and thus $\xi^1 = \phi - x^1d_1\phi$. Finally, substituting in the equation before the last, we get $\xi^3 = d_2\phi$. We have thus obtained an injective parametrization of $M_{(2)}$ which is therefore torsion-free and $M$ is 2-pure in a coherent way. Substituting into the three upper equations, we obtain the desired relative parametrization by adding the differential constraint $d_3\phi = 0$. Coming back to the original coordinates, we obtain the relative parametrization:

$$\phi - x^3 d_3\phi = \xi^1, -d_3\phi = \xi^2, d_2\phi = \xi^3 \text{ with } d_1\phi = 0 \Rightarrow d_2\phi + x^3 d_1\phi = \xi^3$$

which is thus strikingly obtained from the previous contact parametrization by adding the only differential constraint $d_1\phi = 0$ obtained by substituting it in the new system of Lie equations.

## 4. Applications

Before studying applications to mathematical physics, we shall start with an example describing in an explicit way the Janet and Spencer sequences used thereafter, both with their link, namely the relations existing between the dimensions of the respective Janet and Spencer bundles.

**EXAMPLE 4.1**: When $n = m = 2, q = 2$, $\omega$ is the Euclidean metric of $X = \mathbb{R}^2$ with Christoffel symbols $\gamma$ and metric density $\tilde{\omega} = \omega / \sqrt{det(\omega)}$, we consider the two involutive systems of linear infinitesimal Lie equations $R_2 \subset \tilde{R}_2 \subset J_2(T)$ respectively defined by $\{L(\xi)\omega = 0, L(\xi)\gamma = 0\}$ and $\{L(\xi)\tilde{\omega} = 0, L(\xi)\gamma = 0\}$. We have $g_2 = \tilde{g}_2 = 0$ and construct the following successive commutative and exact diagrams followed by the corresponding dimensional diagrams that are used in order to construct effectively the respective Janet and Spencer differential sequences while comparing them.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & & \searrow & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & S_2T^* \otimes T & \rightarrow & F_0'' & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \rightarrow & R_2 & \rightarrow & J_2(T) & \rightarrow & F_0 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_1 & \rightarrow & J_1(T) & \rightarrow & F_0' & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & & \searrow & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 6 & \rightarrow & 6 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & \searrow & \downarrow & & \\
0 & \rightarrow & 3 & \rightarrow & 12 & \rightarrow & 9 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 3 & \rightarrow & 6 & \rightarrow & 3 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In the present situation we notice that

$R_2=\rho_1(R_1)=J_1(R_1)\cap J_2(T)\subset J_1(J_1(T))$ and thus $F_0\simeq J_1(F_0')$ with $F_0''\simeq T^*\otimes F_0'\simeq S_2T^*\otimes T$ by counting the dimensions because we have surely $F_0\subset J_1(F_0')$ with $g_2=0$.

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & S_3T^*\otimes T & \rightarrow & T^*\otimes F_0 & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & R_3 & \rightarrow & J_3(T) & \rightarrow & J_1(F_0) & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & R_2 & \rightarrow & J_2(T) & \rightarrow & F_0 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & 8 & \rightarrow & 18 & \rightarrow & 10 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & 3 & \rightarrow & 20 & \rightarrow & 27 & \rightarrow & 10 & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 3 & \rightarrow & 12 & \rightarrow & 9 & \rightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}
$$

SPENCER

$$
\begin{array}{cccccccccccc}
 & & & & & & 0 & & 0 & & 0 & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & \rightarrow & \Theta & \xrightarrow{j_q} & C_0 & \xrightarrow{D_1} & C_1 & \xrightarrow{D_2} & C_2 & \rightarrow 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & \rightarrow & T & \xrightarrow{j_q} & C_0(T) & \xrightarrow{D_1} & C_1(T) & \xrightarrow{D_2} & C_2(T) & \rightarrow 0 \\
 & & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & \\
0 & \rightarrow & \Theta & \rightarrow & T & \xrightarrow{\mathcal{D}} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & \xrightarrow{\mathcal{D}_2} & F_2 & \rightarrow 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & & & 0 & & 0 & & 0 &
\end{array}
$$

JANET

$$
\begin{array}{ccccccccccccc}
 & 0 & \rightarrow & \tilde{\Theta} & \xrightarrow{j_2} & 4 & \xrightarrow{D_1} & 8 & \xrightarrow{D_2} & 4 & \rightarrow 0 & & \\
 & 0 & \rightarrow & \Theta & \xrightarrow{j_2} & 3 & \xrightarrow{D_1} & 6 & \xrightarrow{D_2} & 3 & \rightarrow 0 & \text{SPENCER} \\
 & & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 & \rightarrow & 2 & \xrightarrow{j_2} & 12 & \xrightarrow{D_1} & 16 & \xrightarrow{D_2} & 6 & \rightarrow 0 & \text{HYBRID} \\
 & & & \| & & \downarrow \Phi_0 & & \downarrow \Phi_1 & & \downarrow \Phi_2 & & \\
0 \rightarrow & \Theta & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 9 & \xrightarrow{\mathcal{D}_1} & 10 & \xrightarrow{\mathcal{D}_2} & 3 & \rightarrow 0 & \text{JANET} \\
0 \rightarrow & \tilde{\Theta} & \rightarrow & 2 & \xrightarrow{\mathcal{D}} & 8 & \xrightarrow{\mathcal{D}_1} & 8 & \xrightarrow{\mathcal{D}_2} & 2 & \rightarrow 0 &
\end{array}
$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & T^*\otimes(\tilde{C}_0/C_0) & = & \tilde{C}_1/C_1 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_3T^*\otimes T & \to & T^*\otimes F_0 & \to & F_1 & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & S_3T^*\otimes T & \to & T^*\otimes \tilde{F}_0 & \to & \tilde{F}_1 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & 2 & \to & 2 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 8 & \to & 18 & \to & 10 & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & 8 & \to & 16 & \to & 8 & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

$$\tilde{C}_r/C_r \simeq ker\left(F_r \to \tilde{F}_r\right) \simeq \wedge^r T^* \otimes \left(\tilde{C}_0/C_0\right)$$
$$\Rightarrow dim\left(\tilde{C}_r\right) - dim\left(C_r\right) = dim\left(F_r\right) - dim\left(\tilde{F}_r\right)$$
$$\Rightarrow dim\left(C_r\right) + dim\left(F_r\right) = dim\left(\tilde{C}_r\right) + dim\left(\tilde{F}_r\right)$$

In this new situation, we now notice that

$$\tilde{R}_2 \subsetneq \rho_1\left(\tilde{R}_1\right) = J_1\left(\tilde{R}_1\right) \cap J_2\left(T\right) \subset J_1\left(J_1\left(T\right)\right)$$

and the induced morphism $\tilde{F}_0 \to J_1\left(\tilde{F}_0'\right)$ is thus no longer a monomorphism though we still have an isomorphism $\tilde{F}_0'' \simeq S_2T^* \otimes T$ because $\tilde{g}_2 = 0$ again. Finally, we may extend such a procedure to the conformal group of space-time by considering the system of infinitesimal conformal transformations of the Minkowski metric defined by the first order system $\hat{R}_1 \subset J_1\left(T\right)$ in such a way that we have the strict inclusions $R_2 \subset \tilde{R}_2 \subset \hat{R}_2 \subset J_2\left(T\right)$ with $dim\left(\hat{g}_2\right) = n = 4$. For this, we just need to introduce the metric density $\hat{\omega} = \omega\left(\left|det\left(\omega\right)\right|\right)^{-\frac{1}{n}}$ and consider the system $L\left(\xi\right)\hat{\omega} = 0$ ([13]):

$$\hat{\Omega}_{ij} \equiv \hat{\omega}_{rj}\partial_i\xi^r + \hat{\omega}_{ir}\partial_j\xi^r - \frac{2}{n}\hat{\omega}_{ij}\partial_r\xi^r + \xi^r\partial_r\hat{\omega}_{ij} = 0$$

**A) Control Theory**:

**EXAMPLE 4.A.1**: (*OD control theory*) In classical control theory we have $n = 1$ and the only independent variable is the time, simply denoted by $x$ but we may choose any ground differential field like $K = \mathbb{Q}\left(x\right)$. In that case, we shall refer to ([5] or [14]) for the proof of the following technical results that will be used in this case (compare to [43]). Instead of the standard "upper dot" notation for derivative we shall identify the formal and the jet notations, setting thus

$d_x y = dy = y_x$. With $m=2$, let us consider the elementary Single Input/Single Output (SISO) second order system $y^1_{xx} - y^2_x + a(x)y^2 = 0$ with a variable coefficient $a \in K$. The corresponding formally surjective operator is $\partial_{xx}\eta^1 - \partial_x\eta^2 + a(x)\eta^2 = \zeta$. Treating such a system by using classical methods is not so easy when $a$ is not constant as it cannot be possible to transform it to the standard Kalman form. On the contrary, multiplying by a test function (or Lagrange multiplier) $\lambda$ and integrating by parts, we obtain the adjoint system/operator:

$$\begin{cases} y^1 \to \lambda_{xx} = \mu^1 \\ y^2 \to \lambda_x + a\lambda = \mu^2 \end{cases} \quad \boxed{\begin{matrix} 1 \\ \bullet \end{matrix}}$$

This system has a trivially involutive zero symbol but is not even formally integrable and we have to consider:

$$\begin{cases} \lambda_{xx} = \mu^1 \\ \lambda_x + a\lambda = \mu^2 \\ \left(\partial_x a - a^2\right)\lambda = \mu^2_x - \mu^1 - a\mu^2 \end{cases} \quad \boxed{\begin{matrix} 1 \\ \bullet \\ \bullet \end{matrix}}$$

We have thus two possibilities:

- We have $a_x - a^2 \neq 0$ and the adjoint system has the only zero solution, that is the adjoint operator is injective. In this case $N=0$ and thus $t(M) = ext^1(N) = 0$ that is $M$ is torsion-free. However, as $n=1$ it follows that $D = K[d]$ is a principal ideal ring which is therefore free and thus projective ([26] [30]), that is *$M$ is torsion-free if and only if* $N=0$ and the system is controllable.
- The Riccati equation $a_x - a^2 = 0$ is satisfied, for example if $a = -1/x$, and we get the CC $\mu^2_x - \mu^1 - a\mu^2 = 0$. Multiplying by a test function $\xi$ and integrating by parts, we get the adjoint operator:

$$\begin{cases} \mu^1 \to -\xi = \eta^1 \\ \mu^2 \to -\xi_x - a\xi = \eta^2 \end{cases}$$

with only one first order generating CC, namely $\partial_x\eta^1 - \eta^2 + a\eta^1 = 0$. It follows that $N \neq 0 \Rightarrow ext^1(N) \neq 0$ is a torsion module generated by the residue of $z = y^1_x - y^2 + ay^1$, even though $y^1$ and $y^2$ are free *separately* and $M$ is not pure. We obtain indeed a torsion element as we can check at once that $z_x - az = 0$ and wish good luck for control people to recover simply this result even on such an elementary example because the Kalman criterion is only working for systems with constant coefficients (compare [5] and [43]).

**EXAMPLE 4.A.2**: (*PD control theory*) With $n=2$, let us consider the (trivially involutive) inhomogeneous single first order PD equations with two independent variables $(x^1, x^2)$, two unknown functions $(\eta^1, \eta^2)$ and a second member $\zeta$:

$$\partial_2\eta^1 - \partial_1\eta^2 + x^2\eta^2 = \zeta \quad \Leftrightarrow \quad \mathcal{D}_1\eta = \zeta$$

The ring of differential operators is $D = K[d_1, d_2]$ with $K = \mathbb{Q}(x^1, x^2)$. Multiplying on the left by a test function $\lambda$ and integrating by parts, the corresponding adjoint operator is described by:

$$\begin{cases} \eta^1 \to -\partial_2\lambda = \mu^1 \\ \eta^2 \to \quad \partial_1\lambda + x^2\lambda = \mu^2 \end{cases} \Leftrightarrow \quad ad\left(\mathcal{D}_1\right)\lambda = \mu$$

Using crossed derivatives, this operator is injective because $\lambda = \partial_2\mu^2 + \partial_1\mu^1 + x^2\mu^1$ and we even obtain a lift $\lambda \to \mu \to \lambda$. Substituting, we get the two CC:

$$\begin{cases} \partial_{22}\mu^2 + \partial_{12}\mu^1 + x^2\partial_2\mu^1 + 2\mu^1 = \nu^1 \\ \partial_{12}\mu^2 + \partial_{11}\mu^1 + 2x^2\partial_1\mu^1 + x^2\partial_2\mu^2 + \left(x^2\right)^2\mu^1 - \mu^2 = \nu^2 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 \\ 1 & \bullet \end{matrix}}$$

This system is involutive and the corresponding generating CC for the second member $\left(\nu^1,\nu^2\right)$ is:

$$\partial_2\nu^2 - \partial_1\nu^1 - x^2\nu^1 = 0$$

Therefore $\nu^2$ is differentially dependent on $\nu^1$ but $\nu^1$ is also differentially dependent on $\nu^2$. Multiplying on the left by a test function $\theta$ and integrating by parts, the corresponding adjoint system of PD equations is:

$$\begin{cases} \nu^1 \to \partial_1\theta - x^2\theta = \xi^1 \\ \nu^2 \to -\partial_2\theta = \xi^2 \end{cases} \Leftrightarrow \quad ad\left(\mathcal{D}_{-1}\right)\theta = \xi$$

Multiplying now the first equation by the test function $\xi^1$, the second equation by the test function $\xi^2$, adding and integrating by parts, we get the *canonical parametrization* $D\xi = \eta$:

$$\begin{cases} \mu^2 \to \partial_{22}\xi^1 + \partial_{12}\xi^2 - x^2\partial_2\xi^2 - 2\xi^2 = \eta^2 \\ \mu^1 \to \partial_{12}\xi^1 - x^2\partial_2\xi^1 + \xi^1 + \partial_{11}\xi^2 - 2x^2\partial_1\xi^2 + \left(x^2\right)^2\xi^2 = \eta^1 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 \\ 1 & \bullet \end{matrix}}$$

of the initial system with zero second member. This system is involutive and the kernel of this parametrization has differential rank equal to 1 because $\xi^1$ or $\xi^2$ can be given arbitrarily.

Keeping now $\xi^1 = \xi$ while setting $\xi^2 = 0$, we get the *first second order minimal parametrization* $\xi \to \left(\eta^1,\eta^2\right)$:

$$\begin{cases} \partial_{22}\xi = \eta^2 \\ \partial_{12}\xi - x^2\partial_2\xi + \xi = \eta^1 \end{cases} \quad \boxed{\begin{matrix} 1 & 2 \\ 1 & \bullet \end{matrix}}$$

This system is again involutive and the parametrization is minimal because the kernel of this parametrization has differential rank equal to 0. With a similar comment, setting now $\xi^1 = 0$ while keeping $\xi^2 = \xi'$, we get the *second order minimal parametrization* $\xi' \to \left(\eta^1,\eta^2\right)$:

$$\begin{cases} \partial_{11}\xi' - 2x^2\partial_1\xi' + \left(x^2\right)^2\xi' = \eta^1 \\ \partial_{12}\xi' - x^2\partial_2\xi' - 2\xi' = \eta^2 \end{cases}$$

which is again easily seen to be involutive by exchanging $x^1$ with $x^2$.

With again a similar comment, setting now $\xi^1 = \partial_1\phi, \xi_2 = -\partial_2\phi$ in the canonical parametrization, we obtain the *third different second order minimal parametrization*:

$$\left\{\begin{array}{l} x^2\partial_{22}\phi + 2\partial_2\phi = \eta^2 \\ x^2\partial_{12}\phi - (x^2)^2\partial_2\phi + \partial_1\phi = \eta^1 \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & \bullet \end{array}}$$

We are now ready for understanding the meaning and usefulness of what we have called "*relative parametrization*" in ([4]) by imposing the *differential constraint* $\partial_2\xi^1 + \partial_1\xi^2 = 0$ which is compatible as we obtain indeed the new *first order relative parametrization*:

$$\left\{\begin{array}{l} \partial_2\xi^1 + \partial_1\xi^2 = 0 \\ -x^2\partial_2\xi^2 - 2\xi^2 = \eta^2 \\ -x^2\partial_1\xi^2 + \left(x^2\right)^2\xi^2 + \xi^1 = \eta^1 \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & 2 \\ 1 & \bullet \end{array}}$$

with 2 equations of class 2 (thus with class 2 full) and only 1 equation of class 1.

In a different way, we may add the differential constraint $\partial_1\xi^1 + \partial_2\xi^2 = 0$ but we have to check that it is compatible with the previous parametrization. For this, we have to consider the following second order system which is easily seen to be involutive with 2 second order equations of (full) class 2, (only) 2 second order equations of class 1 and 1 equation of order 1:

$$\left\{\begin{array}{l} \partial_{22}\xi^2 + \partial_{12}\xi^1 = 0 \\ \partial_{22}\xi^1 + \partial_{12}\xi^2 - x^2\partial_2\xi^2 - 2\xi^2 = \eta^2 \\ \partial_{12}\xi^2 + \partial_{11}\xi^1 = 0 \\ \partial_{12}\xi^1 - x^2\partial_2\xi^1 + \xi^1 + \partial_{11}\xi^2 - 2x^2\partial_1\xi^2 + \left(x^2\right)^2\xi^2 = \eta^1 \\ \partial_2\xi^2 + \partial_1\xi^1 = 0 \end{array}\right. \quad \boxed{\begin{array}{cc} 1 & 2 \\ 1 & 2 \\ 1 & \bullet \\ 1 & \bullet \\ \bullet & \bullet \end{array}}$$

The 4 generating CC only produce the desired system for $\left(\eta^1,\eta^2\right)$ as we wished.

We cannot impose the condition $\mathcal{D}_{-1}\theta = \xi$ already found as it should give the identity $0 = \eta$.

It is however also important to notice that the strictly exact long exact sequence:

$$0 \to D \xrightarrow{\mathcal{D}_1} D^2 \xrightarrow{\mathcal{D}} D^2 \xrightarrow{\mathcal{D}_{-1}} D \to 0$$

splits because we have a lift $\zeta \to \eta \to \zeta$, namely:

$$\zeta \to \left(-\partial_1\zeta + x^2\zeta = \eta^1, -\partial_2\eta^2 = \eta^2\right) \to \partial_2\eta^1 - \partial_1\eta^2 + x^2\eta^2 = \zeta$$

We have thus an isomorphism $D^2 \simeq D \oplus M$ in the resolution $0 \to D \xrightarrow{\mathcal{D}_1} D^2 \xrightarrow{p} M \to 0$ and all the differential modules defined from the operators involved are projective, thus torsion-free or 0-pure with vanishing *r*-extension modules $ext^r(\ ) = 0, \forall r \geq 1$.

As an exercise, we finally invite the reader to study the situation met with the system $\partial_2\eta^1 - \partial_1\eta^2 + a(x)\eta^2$ whenever $a \in K$ (*Hint*: The controllability condition is now $\partial_1 a \neq 0$). The comparison with the previous OD case needs no comment.

**B) Electromagnetism**:

Most physicists know the Maxwell equations in vacuum, eventually in dielectrics and magnets, but are largely unaware of the more delicate constitutive laws involved in field-matter couplings like piezzoelectricity, photoelasticity or streaming birefringence. In particular they do not know that the phenomenological laws of these phenomena have been given ... by Maxwell ([7]). The situation is even more critical when they deal with invariance properties of Maxwell equations because of the previous comments ([44]). Therefore, we shall first quickly recall what the use of adjoint operators and differential duality can bring when studying Maxwell equations as a first step before providing comments on the so-called gauge condition brought by the Danish physicist Ludwig Lorenz in 1867 and not by Hendrik Lorentz with name associated with the Lorentz transformations.

Though it is quite useful in actual practice, the following approach to Maxwell equations cannot be found in any textbook. Namely, avoiding any variational calculus based on given Minkowski constitutive laws $\mathcal{F} \sim F$ between *field* $F$ and *induction* $\mathcal{F}$ for dielectric or magnets, let us use differential duality and define the first set $M_1$ of Maxwell equations by $d : \wedge^2 T^* \rightarrow \wedge^3 T^*$ while the second set $M_2$ will be defined by $ad(d) : \wedge^4 T^* \otimes \wedge^2 T \rightarrow \wedge^4 T^* \otimes T$ with $d : T^* \rightarrow \wedge^2 T^*$, in a totally independent and intrinsic manner, *using now contravariant tensor densities in place of covariant tensors*. As we have already proved since a long time in ([3] [7] [13] [14] [33] [40]), the key result is that *these two sets of Maxwell equations are invariant by any diffeomorphism*, contrary to what is generally believed ([44]). We recapitulate below this procedure in the form of a (locally exact) differential sequence and its (locally exact) formal adjoint sequences where the dotted down arrow in the left square is the standard composition of operators:

$$\begin{array}{ccccccc} \text{potential} & \xrightarrow{d} & \text{field} & \rightarrow & \text{induction} & \xrightarrow{ad(d)} & \text{current} \\ A & \rightarrow & F & \rightarrow & \mathcal{F} & \rightarrow & \mathcal{J} \end{array}$$

which is responsible for EM waves, though it is equivalent to the composition in the right square:

$$\text{pseudopotential} \xrightarrow{ad(d)} \text{induction} \rightarrow \text{field} \xrightarrow{d} \wedge^3 T^*$$

The main difference is that we need to set $\mathcal{J} = 0$ in the first approach because of $M_2$ while we get *automatically* such a vanishing assumption in the second approach because of $M_1$, avoiding therefore the Lorenz condition as in ([35], Remark 5.5) and below.

$$\begin{array}{ccccc} \text{potential} = (A_i) & \xrightarrow{d} & (\partial_i A_j - \partial_j A_i = F_{ij}) = \text{field} & \xrightarrow{M_1} & (\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0) \\ \text{current} = (\partial_i \mathcal{F}^{ij} = \mathcal{J}^j) & \xleftarrow{M_2} & (\mathcal{F}^{ij}) = \text{induction} & \xleftarrow{ad(d)} & \text{pseudopotential} \end{array}$$

$$\begin{array}{ccccc}
A & & F & & \\
T^* & \xrightarrow{d} & \wedge^2 T^* & \xrightarrow{d=M_1} & \wedge^3 T^* \\
\vdots & & \updownarrow & & \vdots \\
\wedge^4 T^* \otimes T & \xleftarrow{ad(d)=M_2} & \wedge^4 T^* \otimes \wedge^2 T & \xleftarrow{ad(d)} & \wedge^4 T^* \otimes \wedge^3 T \\
\updownarrow & & \updownarrow & & \updownarrow \\
\wedge^3 T^* & \xleftarrow{d} & \wedge^2 T^* & \xleftarrow{d} & T^* \\
\mathcal{J} & & \mathcal{F} & &
\end{array}$$

Using symbolic notations with an Euclidean metric instead of the Minkowski one because they are both locally constant while using the constitutive law $\mathcal{F} = F$ for simplicity in vacuum while raising or lowering the indices by means of the metric, we have the parametrization $d_i A_j - d_j A_i = F_{ij}$ and obtain by composition in the left upper square:

$$d_i\left(d^i A^j - d^j A^i\right) = d_i d^i A^j - d^j\left(d_i A^i\right) = \mathcal{J}^j$$
$$\Rightarrow d_j\left(d_i d^i A^j - d^j d_i A^i\right) = d_j \mathcal{J}^j = 0$$

with implicit summations on *i* and *j*. However, such a result does not prove at all that the density of current does not satisfy other CC. Nevertheless, we have:

**LEMMA 4.B.1:** The system $d_i d^i A^j - d^j\left(d_i A^i\right) = \mathcal{J}^j$ is involutive whenever $d_j \mathcal{J}^j = 0$ but the differential module defined by the corresponding homogeneous system is *not* torsion-free.

*Proof*: When $\mathcal{J} = 0$, this second order system is formally integrable because it is homogeneous. However, even if we know *a priori* that *necessarily* $d_j \mathcal{J}^j = 0$, it is not evident that such a condition is also *sufficient*, contrary to what is claimed in the literature. When $n = 4$, using the Euclidean metric for simplicity, one can rewrite the system in the form:

$$\left\{\begin{array}{l}
d_{44}A^3 + d_{33}A^3 + d_{22}A^3 + d_{11}A^3 - d_3\left(d_4A^4 + d_3A^3 + d_2A^2 + d_1A^1\right) = \mathcal{J}^3 \\
d_{44}A^2 + d_{33}A^2 + d_{22}A^2 + d_{11}A^3 - d_2\left(d_4A^4 + d_3A^3 + d_2A^2 + d_1A^1\right) = \mathcal{J}^2 \\
d_{44}A^1 + d_{33}A^1 + d_{22}A^1 + d_{11}A^3 - d_1\left(d_4A^4 + d_3A^3 + d_2A^2 + d_1A^1\right) = \mathcal{J}^1 \\
d_{34}A^3 + d_{24}A^2 + d_{14}A^1 - d_{33}A^4 - d_{22}A^4 - d_{11}A^4 = -\mathcal{J}^4
\end{array}\right. \qquad \boxed{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \end{array}}$$

Let us check that the second order symbol is involutive with three equations of class 4 and only one equation of class 3. Indeed, we have successively:

$$d_{344}A^3 + d_{244}A^2 + d_{144}A^1 - d_{334}A^4 - d_{224}A^4 - d_{114}A^4 = -d_4\mathcal{J}^4$$
$$-d_{344}A^3 - d_{333}A^3 - d_{223}A^3 - d_{113}A^3 + d_{334}A^4 + d_{333}A^3 + d_{233}A^2 + d_{133}A^1 = -d_3\mathcal{J}^3$$
$$-d_{244}A^2 - d_{233}A^2 - d_{222}A^2 - d_{112}A^2 + d_{224}A^4 + d_{223}A^3 + d_{222}A^2 + d_{122}A^1 = -d_2\mathcal{J}^2$$
$$-d_{144}A^1 - d_{133}A^1 - d_{122}A^1 - d_{111}A^1 + d_{114}A^4 + d_{113}A^3 + d_{112}A^2 + d_{111}A^1 = -d_1\mathcal{J}^1$$

Summing the four equations, the left member is vanishing and we get the desired only CC. The four characters are $(1 < 7 < 12 < 16)$ and we check that $dim\left(g_2\right) = 1 + 7 + 12 + 16 = 36 = (10 \times 4) - 4$. *This procedure cannot be avoided though it cannot be found in the literature*.

In addition, rewriting the homogeneous system in the form $d_i d_i A_j - d_i d_j A_i = 0$

with implicit summation on $i$, we obtain successively ([10]):

$$\begin{aligned} d_i d_i F_{jk} &= d_i d_i \left(d_j A_k - d_k A_j\right) \\ &= d_j \left(d_i d_i A_k\right) - d_k \left(d_i d_i A_j\right) \\ &= d_j \left(d_i d_k A_i\right) - d_k \left(d_i d_j A_i\right) \\ &= d_j d_k \left(d_i A_i\right) - d_k d_j \left(d_i A_i\right) \\ &= 0 \end{aligned}$$

It follows that *each component of the field is a torsion element* of the corresponding differential module $M$, which is killed by the Dalembertian. Accordingly, *$M$ is not torsion-free and thus not pure because we have just proved that* $rk_D(M)=1$. In particular, *each component of the potential is free by itself.* Such a situation is similar to the one met in the study of Single Input/Single Output (SISO) ordinary differential control systems. For example, if we have $y_{xx}-u_x=0$, then $z=y_x-u$ is a torsion element with $z_x=0$ even though both the input $u$ and the output $y$ are free *separately.* Bringing the system to first order by choosing the new unknowns $\left(z^1=y, z^2=y_x, z^3=u\right)$, we obtain the Spencer form over $K=\mathbb{Q}$:

$$z_x^1 - z^2 = 0,\ z_x^2 - z_x^3 = 0$$

Setting a new $\bar z^1 = z^1, \bar z^2 = z^2 - z^3, \bar z^3 = z^3$, we obtain the reduced Spencer form:

$$\bar z_x^1 - \bar z^2 - \bar z^3 = 0,\ \bar z_x^2 = 0 \Leftrightarrow \bar z = Az + Bu \Rightarrow rk_K(B, AB) = 1 < 2$$

which is a Kalman form because no jet of the input $\bar z^3 = z^3 = u$ is indeed appearing.

Finally, the character $\alpha_2^n$ is obtained by considering $d_{nn}A_j - d_{jn}A_n$ for the equation giving $\mathcal{J}_j$. For $\mathcal{J}_n$ we get $d_{nn}A_n - d_{nn}A_n = 0$ and thus $\alpha_2^n = n-(n-1) = 1$ a result showing that the corresponding differential module has rank 1 and there is thus only one CC, namely $d_j\mathcal{J}^j = 0$ with implicit summation on $j$.

Q.E.D.

We now prove that we may add the Lorenz condition $d_iA^i=0$ to bring the rank to zero. Indeed, we have now the inhomogeneous system $d_id^iA^j=\mathcal{J}^j$ and the differential constraint thus brought is compatible with the conservation of current. The corresponding homogeneous system obtained by adding the Lorenz constraint has second order symbol obtained by considering *both* $d_id^iA^j=0$ *and* $d_{ij}A^i=0$ *or* $d_{ij}A^j=0$. We obtain therefore $d_{nn}A_j=0, d_{nn}A_n=0$ showing that we have now $\alpha_2^n=0$ and a torsion differential module. As a more important and effective result that does not seem to be known, we have:

**PROPOSITION 4.B.2**: When $n=4$, the system:

$$\left\{\begin{array}{l} \Psi^j \equiv d_{44}A^j + \cdots + d_{11}A^j = \mathcal{J}^j \\ d_4\Phi - \Psi^4 \equiv d_{34}A^3 + d_{24}A^2 + d_{14}A^1 - d_{33}A^4 - d_{22}A^2 - d_{11}A^4 = -\mathcal{J}^4 \\ d_3\Phi \equiv d_{34}A^4 + \cdots + d_{13}A^1 = 0 \\ d_2\Phi \equiv d_{24}A^4 + \cdots + d_{12}A^1 = 0 \\ d_1\Phi \equiv d_{14}A^4 + \cdots + d_{11}A^1 = 0 \\ \Phi \equiv d_4A^4 + \cdots + d_1A^1 = 0 \end{array}\right. \qquad \begin{array}{|cccc|} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & 3 & \bullet \\ 1 & 2 & \bullet & \bullet \\ 1 & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

is involutive with four equations of class 4, two equations of class 3, one equation of class 2 and one equation of class 1. The 11 resulting CC only provide the conservation of current.

*Proof*: Using the corresponding *Janet tabular* on the right, one can check at once that the 4 CC brought by the only first order equation $\Phi = 0$ do not bring anything new, as they amount to crossed derivatives, and that *we are only left with the 4 upper dots on the right side*. However, for $i = 1,2,3$, we have $d_4(d_i\Phi) = d_i(d_4\Phi - \Psi^4) + d_i\Psi^4$ and we are thus only left with a single CC, getting successively:

$$d_4(d_4\Phi - \Psi^4) \equiv d_{344}A^3 + d_{244}A^2 + d_{144}A^1 - d_{334}A^4 - d_{224}A^2 - d_{114}A^1$$
$$-d_3\Psi^3 \equiv -d_{344}A^3 - d_{333}A^3 - d_{223}A^3 - d_{113}A^3$$
$$-d_2\Psi^2 \equiv -d_{244}A^2 - d_{233}A^2 - d_{222}A^2 - d_{112}A^2$$
$$-d_1\Psi^1 \equiv -d_{144}A^1 - d_{133}A^1 - d_{122}A^1 - d_{111}A^1$$
$$d_3(d_3\Phi) \equiv d_{334}A^4 + d_{333}A^3 + d_{233}A^2 + d_{133}A^1$$
$$d_2(d_2\Phi) \equiv d_{224}A^4 + d_{223}A^3 + d_{222}A^2 + d_{122}A^1$$
$$d_1(d_1\Phi) \equiv d_{114}A^4 + d_{113}A^3 + d_{112}A^2 + d_{111}A^1$$

Summing these 7 equations, we are left with the identity $-(d_{44}\Phi + \cdots + d_{11}\Phi) + d_j\Psi^j = d_j\mathcal{J}^j = 0$. It is important to notice that *no other procedure* can prove that we have an involutive symbol in $\delta$-regular coordinates and this is the only way to compute *effectively* all the four characters ($0 < 6 < 11 < 15$) with $6 + 11 + 15 = 32 = (4 \times 10) - (4 + 4)$ for the dimension of the symbol of order 2, a result not evident at first sight. Accordingly, the so-called Lorenz gauge condition is only a pure "*artifact*" amounting to a relative minimum parametrization with no important physical meaning as it can be avoided by using only the EM field $F$ ([6] [10] [45]).

Q.E.D.

**REMARK 4.B.3**: Using the exactness of the Euler-Poincaré characteristic for the Poincaré sequence and its adjoint sequence, we have:

$$0 = 1 - n + n(n-1)/2 - rk_D(d = M_1) \Rightarrow rk_D(d = M_1) = (n-1)(n-2)/2$$
$$0 = 1 - n + n(n-1)/2 - rk_D(d = M_2) \Rightarrow rk_D(ad(d) = M_2) = (n-1)(n-2)/2$$

by counting the ranks from left to right in the upper row and then from right to left in lower row of the previous diagram. We obtain therefore:

$$\begin{aligned} & rk_D(d) + rk_D(d = M_1) \\ & = rk_D(ad(d) = M_2) + rk_D(ad(d)) = 3 + 3 = dim(\wedge^2 T^*) = 6 \end{aligned}$$

when $n = 4$ by using successively the exactness of the upper and lower rows of the diagram. Hence, the image of the dotted arrow on the left is equal to the image of $ad(d) = M_2$, a result explaining why both operators have the same CC, namely the conservation of current obtained by extending the previous diagram one step more to the left. As the linear Spencer sequence for a Lie group action (isometries, isometries plus dilatation, conformal isometries in Example 4.1) is (locally) isomorphic to the tensor product of the Poincaré sequence by a finite dimensional Lie algebra ($10 < 11 < 15$ when $n = 4$ in Example 4.1), the same comment remains valid. This result justifies by itself the specific importance of the linear Spencer sequence for infinitesimal Lie equations.

Such a new approach to a classical result is nevertheless bringing a totally unsatisfactory consequence. Using the well known correspondence between electromagnetism (EM) and elasticity (EL) used for all engineering computations with finite elements:

$$\text{EM potential} \leftrightarrow \text{EL displacement}, \quad \text{EM field} \leftrightarrow \text{EL strain},$$
$$\text{EM induction} \leftrightarrow \text{EL stress}$$

where EL means elasticity, and instead of the left upper square in the diagram, ... *we have to consider the right upper square*.

We finally prove that the use of the linear *and* nonlinear Spencer operators drastically changes the previous standard procedure in a way that could not even be imagined with classical methods. For such a purpose, we make a few comments on the implicit summation appearing in differential duality. For example, we have, up to a divergence:

$$\mathcal{X}_k^{,r} X_{,r}^k = \mathcal{X}_k^{,r}\left(\partial_r \xi^k - \xi_r^k\right) = -\partial_r\left(\mathcal{X}_k^{,r}\right)\xi^k - \mathcal{X}_k^{,r}\xi_r^k + \cdots$$

In the conformal situation, we have $\xi_1^1 = \xi_2^2 = \cdots = \xi_n^n = \frac{1}{n}\xi_r^r$ and obtain therefore, as factor of the firs jets:

$$\mathcal{X}_1^{,1}\xi_1^1 + \mathcal{X}_2^{,2}\xi_2^2 + \cdots + \mathcal{X}_n^{,n}\xi_n^n = \left(\mathcal{X}_r^{,r}\right)\frac{1}{n}\xi_r^r = \left(\mathcal{X}_r^{,r}\right)\xi_1^1$$

Going to the next order, we get as in ([26]), up to a divergence:

$$\mathcal{X}_1^{1,r}\partial_r\xi_1^1 = -\left(\partial_r\mathcal{X}_1^{1,r}\right)\xi_1^1 + \cdots$$

Collecting the results and changing the sign, we obtain for the first time the *Cosserat equation for the dilatation*, namely the so-called *virial equation* that we provided in 2016 ([34], p. 35):

$$\partial_r\mathcal{X}_1^{1,r} + \mathcal{X}_r^{,r} = 0$$

generalizing the well known *Cosserat equations for the rotations* provided in 1909 ([46], p. 137):

$$\partial_r\mathcal{X}^{ij,r} + \mathcal{X}^{i,j} - \mathcal{X}^{j,i} = 0$$

As for EM, substituting $\partial_i\xi_{rj}^r - \partial_j\xi_{ri}^r$ in the dual sum $\mathcal{F}^{i<j}F_{i<j} = \frac{1}{2}\mathcal{F}^{ij}F_{ij}$ and integrating by parts, we get a part of the *Cosserat equations for the elations*, namely:

$$\partial_r\mathcal{F}^{ir} - \mathcal{J}^i = 0 \Rightarrow \partial_i\mathcal{J}^i = 0 \Rightarrow \mathcal{X}_r^{,r} = 0$$

saying that *the trace of the EM impulsion-energy tensor must vanish* ([45], p. 37).

**REMARK 4.B.4**: When $g_{q+1} = 0$, introducing the linear Spencer sequence with Spencer bundles $C_r = \wedge^r T^* \otimes \hat{R}_q$, we obtain the following diagram describing the Cosserat procedure ([4] [40] [46]):

$$
\begin{array}{ccccc}
\text{potential} & & \text{field} & & \\
C_0 & \overset{D_1}{\rightarrow} & C_1 & \overset{D_2}{\rightarrow} & C_2 \\
\vdots & & \updownarrow & & \vdots \\
\wedge^n T^* \otimes C_0^* & \overset{ad(D_1)}{\leftarrow} & \wedge^n T^* \otimes C_1^* & \overset{ad(D_2)}{\leftarrow} & \wedge^n T^* \otimes C_2^* \\
\updownarrow & & \updownarrow & & \updownarrow \\
\wedge^n T^* \otimes M_q & \overset{ad(D_1)}{\leftarrow} & \wedge^{n-1} T^* \otimes M_q & \overset{ad(D_2)}{\leftarrow} & \wedge^{n-2} T^* \otimes M_q \\
 & & \text{induction} & & \text{pseudopotential}
\end{array}
$$

where we have used the isomorphism $R_q^* \simeq M_q$. It just remains to consider the various Spencer bundles $C_r \subset \tilde{C}_r \subset \hat{C}_r$ that we have already considered with $q = 2$ (see [17] for an explicit example).

We sum up all these results in the following tabular only depending on the Spencer operator:

| FIELD | | INDUCTION |
|---|---|---|
| NONLINEAR | LINEAR | DUAL |
| $\bar{D} f_{q+1} = \chi_q \in T^* \otimes R_q$ | $D\xi_{q+1} = X_q \in T^* \otimes R_q$ | $\mathcal{X}_q \in \wedge^n T^* \otimes T \otimes R_q^*$ |
| $\chi^k_{,r}$ | $\partial_r \xi^k - \xi^k_r = X^k_{,r}$ | $\mathcal{X}_k^{,r}$ |
| $\chi^k_{i,r}$ | $\partial_r \xi^k_i - \xi^k_{i,r} = X^k_{i,r}$ | $\mathcal{X}_k^{i,r}$ |
| $\chi^r_{r,i}$ | $\partial_i \xi^r_r - \xi^r_{ri} = X^r_{r,i} = X_i$ | $\mathcal{J}^i$ |
| $\chi^k_{ij,r}$ | $\partial_r \xi^k_{ij} - \xi^k_{ijr} = \partial_r \xi^k_{ij} = X^k_{ij,r}$ | $\mathcal{X}_k^{ij,r}$ |
| $\partial_i \chi^r_{r,j} - \partial_j \chi^r_{r,i} = \varphi_{ij}$ | $\partial_i \xi^r_{r,j} - \partial_j \xi^r_{r,i} = F_{ij}$ | $\mathcal{F}^{ij}$ |
| | $\frac{1}{2}\left(\partial_i \xi^r_{r,j} + \partial_j \xi^r_{r,i}\right) = R_{ij}$ | ? |

We notice that $F = (F_{ij}) \in \wedge^2 T^*, R = (R_{ij}) \in S_2 T^*$ and the part of the first Spencer bundle made by the 1-forms with value in the $n = 4$ elations provides the splitting:

$$(F, R) \in \wedge^2 T^* \oplus S_2 T^* \simeq T^* \otimes T^*$$

because of the well known exactness of the Spencer $\delta$-sequence:

$$0 \rightarrow S_2 T^* \xrightarrow{\delta} T^* \otimes T^* \xrightarrow{\delta} \wedge^2 T^* \rightarrow 0$$

As a byproduct, the surprising shift of exterior powers is simply obtained by applying the Spencer $\delta$-map to $\hat{C}_r / \tilde{C}_r = \wedge^r T^* \otimes \hat{g}_2 \simeq \wedge^r T^* \otimes T^*$ because $\delta\left(\wedge^r T^* \otimes T^*\right) \subset \wedge^{(r+1)} T^*$.

**C) General Relativity**:

Roughly speaking, we shall say that a parametrization of an operator is *minimal* if its corresponding operator defines a torsion module or, equivalently, if the kernel of the parametrizing operator has differential rank equal to 0. It is not so well known even today that, up to an isomorphism, the *Cauchy* stress operator essentially admits only one parametrization in dimension $n=2$ which is minimum but the situation is quite different in dimension $n=3$. Indeed, the parametrization found by E. Beltrami in 1892 with 6 potentials ([9]) is not minimal as the kernel of the *Beltrami* operator has differential rank 3 while the *two other* parametrizations respectively found by J.C. Maxwell in 1870 and by G. Morera in 1892 are both minimal with only 3 potentials even though they are quite different because the first is cancelling 3 among the 6 potentials while the other is cancelling the 3 others. In particular, we point out the technical fact that it is quite difficult to prove that the Morera parametrization is providing an involutive system. These three tricky examples are proving that the possibility to exhibit different parametrizations of the stress equations that we have presented has surely nothing to do with the proper mathematical background of elasticity theory as it provides an explicit application of double differential duality in differential homological algebra. Also, the example presented in Section 3. A is proving that the existence of many different minimal parametrizations has surely nothing to do with the mathematical foundations of control theory. Similarly, we have just seen in the previous section that the so-called Lorenz condition has surely nothing to do with the mathematical foundations of EM. Such a comment will be now extended in a natural manner to GR.

With standard notations, denoting by $\Omega \in S_2T^*$ a perturbation of the non-degenerate metric $\omega$, it is well known (see [8] [10] and [47] for more details) that the linearization of the *Ricci* tensor $R=(R_{ij}) \in S_2T^*$ over the Minkowski metric, considered as a second order operator $\Omega \to R$, may be written *with four terms* as:

$$2R_{ij} = \omega^{rs}\left(d_{ij}\Omega_{rs} + d_{rs}\Omega_{ij} - d_{ri}\Omega_{sj} - d_{sj}\Omega_{ri}\right) = 2R_{ji}$$

Multiplying by test functions $(\lambda^{ij}) \in \wedge^4 T^* \otimes S_2T$ and integrating by parts on space-time, we obtain the following *four terms* describing the so-called *gravitational waves equations*:

$$\left(\Box\lambda^{rs} + \omega^{rs}d_{ij}\lambda^{ij} - \omega^{sj}d_{ij}\lambda^{ri} - \omega^{ri}d_{ij}\lambda^{sj}\right)\Omega_{rs} = \sigma^{rs}\Omega_{rs}$$

where $\Box$ is the standard Dalembertian. Accordingly, we have:

$$d_r\sigma^{rs} = \omega^{ij}d_{rij}\lambda^{rs} + \omega^{rs}d_{rij}\lambda^{ij} - \omega^{sj}d_{rij}\lambda^{ri} - \omega^{ri}d_{rij}\lambda^{sj} = 0$$

The basic idea used in GR has been to simplify these equations by adding the *differential constraints* $d_r\lambda^{rs}=0$ in order to find only $\Box\lambda^{rs} = \sigma^{rs}$, exactly like in the Lorenz condition for EM. Before going ahead, it is important to notice that when $n=2$, the only Lagrange multiplier $\lambda$ is just the Airy function $\phi$ and, using an integration by parts, we have the identity:

$$\phi\left(d_{11}\Omega_{22} + d_{22}\Omega_{11} - 2d_{12}\Omega_{12}\right) = d_{22}\phi\Omega_{11} - 2d_{12}\phi\Omega_{12} + d_{11}\phi\Omega_{22} + div(\ )$$

providing the *Airy parametrization* of the Cauchy stress equations:

$$\sigma^{11} = d_{22}\phi,\ \sigma^{12} = \sigma^{21} = -d_{12}\phi,\ \sigma^{22} = d_{11}\phi$$

where *the Airy function has, of course, nothing to do with any perturbation of* *the metric*.

However, even if it is clear that the constraints are compatible with the Cauchy equations, we do believe that the following result is not known as it does not contain any reference to the usual *Einstein* tensor $E_{ij} = R_{ij} - \frac{1}{2}\omega_{ij} tr(R)$ where $tr(R) = \omega^{rs} R_{rs}$, which is therefore *useless* because it contains 6 terms instead of 4 terms only, *even though the corresponding operator is self-adjoint*.

**PROPOSITION 4.C.1**: The system made by $\Box\lambda^{rs} = \sigma^{rs}$ and $d_r\lambda^{rs} = 0$ is a relative minimum involutive parametrization of the Cauchy equations describing the formal adjoint of the Killing operator, that is $\text{Cauchy} = ad(\text{Killing})$ as operators.

*Proof*: For each *given* $s = 1,2,3,4$ the system under study is *exactly* the system used for studying the Lorenz condition in Proposition 4.B.1. Accordingly, nothing has to be changed in the proof of this proposition and we get an involutive second order system with $d_r\sigma^{rs} = 0$ as only CC in place of the conservation of current. Needless to say that this result has nothing to do with any concept of gauge theory as it is sometimes claimed ([8] [47]).

Q.E.D.

## 5. Conclusion

In 1916, F.S. Macaulay used a new localization technique for studying unmixed polynomial ideals. In 2012, we have generalized this procedure in order to study pure differential modules, obtaining therefore a relative parametrization in place of the absolute parametrization already known for torsion-free modules and equivalent to controllability in the study of OD or PD control systems. Such a result is showing that *controllability does not depend on the choice of the control variables*, despite what engineers still believe. Meanwhile, we have pointed out the existence of minimum parametrizations obtained by adding, in a convenient but generally not intrinsic way, certain compatible differential constraints on the potentials. We have proved that this is *exactly* the kind of situation met in control theory, in EM with the Lorenz condition and in GR with gravitational waves. However, the systematic use of *adjoint operators* and *differential duality* is proving that the physical meaning of the potentials involved has *absolutely nothing to do* with the one usually adopted in these domains. Therefore, these results bring the need to revisit the mathematical foundations of Electromagnetism and Gravitation, thus of Gauge Theory and General Relativity, in particular Maxwell and Einstein equations, even if they seem apparently well established.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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## Main Notations

$K$ differential field containing $\mathbb{Q}$ with $n$ commuting derivations $\partial_1, \cdots, \partial_n$.
$d_1, \cdots, d_n$ formal derivatives acting on the $m$ differential indeterminates $y^1, \cdots, y^m$.
$D = K[d_1, \cdots, d_n] = K[d]$ ring of differential operators $P, Q$ with coefficients in $K$.
$L, M, N$ filtered differential modules over $D$.
$\mu = (\mu_1, \cdots, \mu_n)$ multi-index with length $|\mu| = \mu_1 + \cdots + \mu_n$ and
$\mu + 1_i = (\mu_1, \cdots, \mu_i + 1, \cdots, \mu_n)$.
$X$ manifold with tangent, cotangent, symmetric and exterior bundles
$T, T^*, S_q T^*, \wedge^r T^*$.
$y_q = (y^k, y_i^k, y_{ij}^k, \cdots) = (y_\mu^k), 0 \le |\mu| \le q$ jet coordinates up to order $q$.
$ad(\mathcal{D})$ formal adjoint defined by $\langle \lambda, \mathcal{D}\xi \rangle = \langle ad(\mathcal{D})\lambda, \xi \rangle + div(\ )$ for any test row vector $\lambda$.
$E, F$ vector bundles over $X$ or free modules over $D$.
$V$ characteristic algebraic variety with dimension $d$ and codimension $cd = n - d$ over $K$.
$0 = t_n(M) \subseteq \cdots \subseteq t_0(M) \subseteq M$ purity filtration with
$t_r(M) = \{m \in M \mid cd(Dm) > r\}$.
$t_0(M) = t(M) = \{m \in M \mid \exists 0 \ne P \in D, Pm = 0\}$ the torsion submodule of $M$.
$R = hom_K(M, K)$ differential module over $D$ associated with $M$, also called inverse system.
$d : R \to T^* \otimes_K R = f \to dx^i \otimes d_i f$ Spencer operator with $(d_i f)_\mu^k = \partial_i f_\mu^k - f_{\mu+1_i}^k$.
$T^* \otimes_X T^* \simeq S_2 T^* \oplus \wedge^2 T^*$ with $R = (R_{ij}) \in S_2 T^*$ Ricci tensor and
$F = (F_{ij}) \in \wedge^2 T^*$ EM field.

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