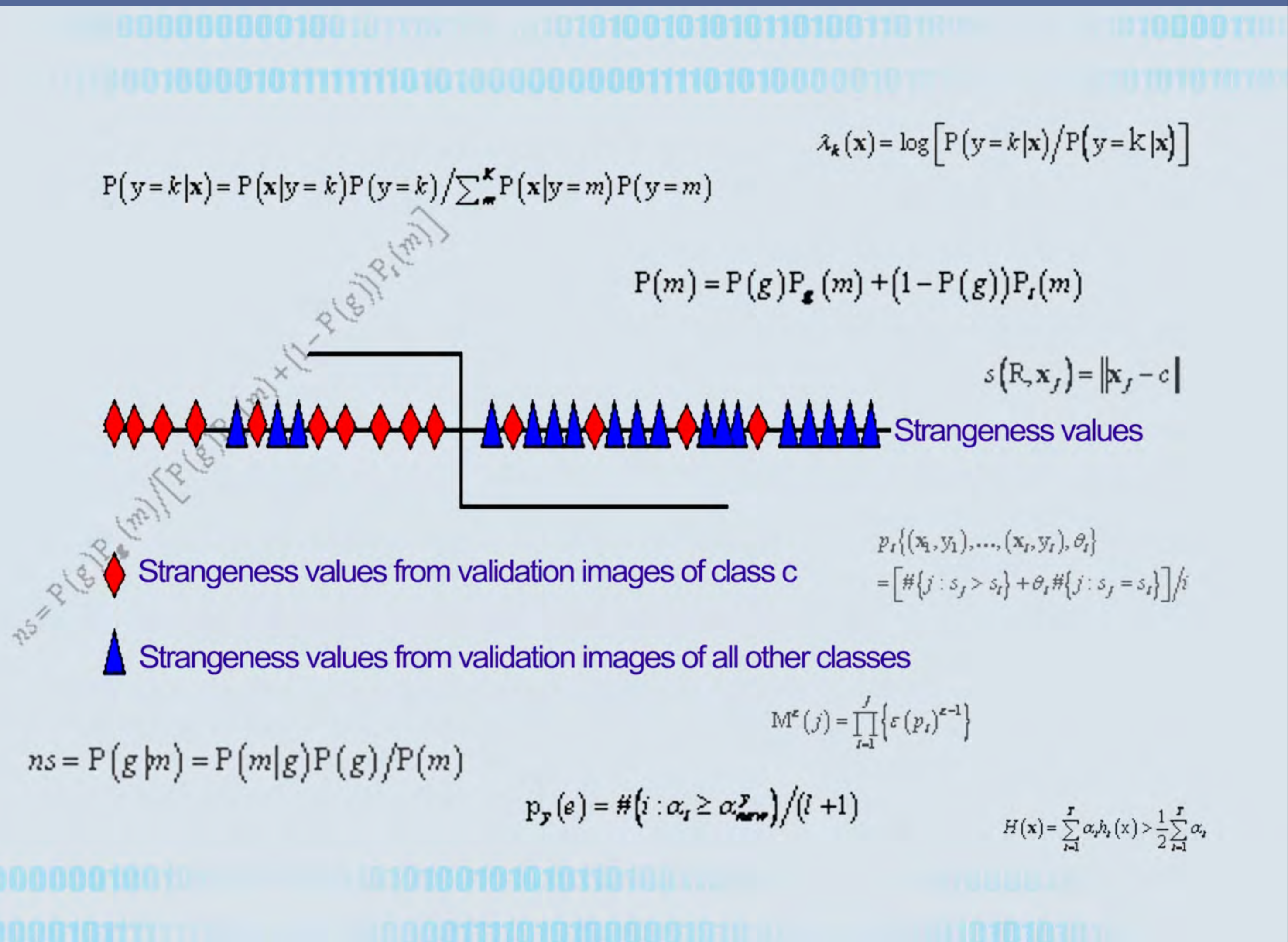
ISSN: 2150-8194 Vol. 2, No. 9, September 2010

Scientific Research

Intelligent Information Management

Editor-in-Chief: Dr.Bin Wu



ISSN: 2150-8194

www.scirp.org/journal/iim

Journal Editorial Board

ISSN: 2150-8194 (Print), 2150-8208 (Online)

http://www.scirp.org/journal/iim

Intelligent Information Management, **2010, 2, 499-568**
Published Online September 2010 in SciRes (http://www.SciRP.org/journal/iim/)

TABLE OF CONTENTS

Volume 2 Number 9 September 2010

The figure on the front cover is from the article published in *Intelligent Information Management*, 2010, Vol. 2, No. 9, pp. 499-511 by Harry Wechsler.

Intelligent Information Management (IIM)

Journal Information

SUBSCRIPTIONS

The *Intelligent Information Management* (Online at Scientific Research Publishing, www.SciRP.org) is published monthly by Scientific Research Publishing, Inc., USA.

Subscription rates:

Print: $50 per issue.

To subscribe, please contact Journals Subscriptions Department, E-mail: sub@scirp.org

SERVICES

Advertisements

Advertisement Sales Department, E-mail: service@scirp.org

Reprints (minimum quantity 100 copies)

Reprints Co-ordinator, Scientific Research Publishing, Inc., USA.

E-mail: sub@scirp.org

COPYRIGHT

PRODUCTION INFORMATION

For manuscripts that have been accepted for publication, please contact:

E-mail: iim@scirp.org

Intelligent Information Management, **2010, 2, 499-511**
doi:10.4236/iim.2010.29060 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Intelligent Biometric Information Management

Harry Wechsler
Department of Computer Science, George Mason University, Fairfax, USA
E-mail: *wechsler@gmu.edu*
Received June 10, 2010; *revised July* 29, 2010; *accepted August* 30, 2010.

Abstract

We advance here a novel methodology for robust intelligent biometric information management with inferences and predictions made using randomness and complexity concepts. Intelligence refers to learning, adaptation, and functionality, and robustness refers to the ability to handle incomplete and/or corrupt adversarial information, on one side, and image and or device variability, on the other side. The proposed methodology is model-free and non-parametric. It draws support from discriminative methods using likelihood ratios to link at the conceptual level biometrics and forensics. It further links, at the modeling and implementation level, the Bayesian framework, statistical learning theory (SLT) using transduction and semi-supervised learning, and Information Theory (IY) using mutual information. The key concepts supporting the proposed methodology are a) local estimation to facilitate learning and prediction using both labeled and unlabeled data; b) similarity metrics using regularity of patterns, randomness deficiency, and Kolmogorov complexity (similar to MDL) using strangeness/typicality and ranking p-values; and c) the Cover – Hart theorem on the asymptotical performance of k-nearest neighbors approaching the optimal Bayes error. Several topics on biometric inference and prediction related to 1) multi-level and multi-layer data fusion including quality and multi-modal biometrics; 2) score normalization and revision theory; 3) face selection and tracking; and 4) identity management, are described here using an integrated approach that includes transduction and boosting for ranking and sequential fusion/aggregation, respectively, on one side, and active learning and change/outlier/intrusion detection realized using information gain and martingale, respectively, on the other side. The methodology proposed can be mapped to additional types of information beyond biometrics.

Keywords: Authentication, Biometrics, Boosting, Change Detection, Complexity, Cross-Matching, Data Fusion, Ensemble Methods, Forensics, Identity Management, Imposters, Inference, Intelligent Information Management, Margin gain, MDL, Multi-Sensory Integration, Outlier Detection, P-Values, Quality, Randomness, Ranking, Score Normalization, Semi-Supervised Learning, Spectral Clustering, Strangeness, Surveillance, Tracking, Typicality, Transduction

1. Introduction

Information can be viewed as an asset, in general, and resource or commodity, in particular. Information management [using information technology] stands for the "application of management principles to the acquisition, organization, control, alert and dissemination and strategic use of information for the effective operation of organizations [including information architectures and management information systems] of all kinds. 'Information' here refers to all types of information of value, whether having their origin inside or outside the organization, including data resources, such as production data; records and files related, for example, to the personnel [subject] biometric function; market research data; and competitive intelligence from a wide range of sources. In formation management deals with the value, quality, ownership, use and security of information in the context of organizational performance" [1]. The life-cycle of information includes 1) *creation and acquisition*; 2) *management of information*, e.g., creation of (biometric and forensic) databases, storage, retrieval, sharing and dissemination, leading to and including full-fledged information systems; and purposeful *use of information*. Intelligent information management, according to HP, enables near real-time business intelligence with robust, scalable data management, data-intensive analytics and fusion of structured and unstructured information.

We start with parsing and understanding the meaning of intelligent information management when information refers to biometrics where physical characteristics, e.g., appearance, are used to verify and/or authenticate individuals. Physical appearance and characteristics can include both internal ones, e.g., DNA and iris, and external ones, e.g., face and fingerprints. Behavior, e.g., face expression and gait, and cognitive state, e.g., intent, can further expand the scope of what biometrics stand for and are expected to render. Note that some biometrics, e.g., face expression (see smile) can stand for both appearance, inner cognitive state, and/or medical condition. With information referring here to biometrics one should consider identity management as a particular instantiation of information management. Identity management (IM) is then responsible, among others, with authentication, e.g., (ATM) verification, identification, and large scale screening and surveillance. IM is also involved with change detection, destruction, retention, and/or revision of biometric information, e.g., as people age and/or experience illness. Identity management is most import- ant among others to (homeland) security, commerce, fi nance, mobile networks, and education. A central infrastructure needs to be designed and implemented to enforce and guarantee robust and efficient enterprise-wide policies and audits. Biometric information need to be safeguarded to ensure regulatory compliance with privacy and anonymity best (lawful) practices.

The intelligent aspect is directly related to what biometrics provides us with and the means and ways used to accomplish it. It is mostly about management principles related to robust inference and prediction, e.g., authentication via classification and discrimination, using incremental and progressive evidence ("information") accumulation and disambiguation, learning, adaptation, and closed-loop control. Towards that end, the specific means advocated here include discriminative methods (for practical intelligence) linking the Bayesian framework, forensics, statistical learning, and information theory, on one side, and likelihoods (and odds), randomness, and complexity, on the other side. The challenges that have to be met include coping with incomplete ("occlusion") and corrupt ("disguise") information, image variability, e.g., pose, illumination, and expression (PIE) and temporal change. The "human subject" stands at the center of any IM system. The *subject* interfaces and mediates between *biometric tasks*, e.g., filtering and indexing data, searching for identity, categorization for taxonomy purposes or alternatively classification and discrimination using information retrieval and search engines crawling the web, data mining and business intelligence (for abstraction, aggregation, and analysis purposes) and knowledge discovery, and multi-sensory integration and data fusion; *biometric contents*, e.g., data, information, knowledge, and intelligence / wisdom / meta-knowledge; *biometric organization*, e.g., features, models, and ontologies and semantics; and last but not least, *biometric applications*, e.g., face selection and CCTV surveillance, and mass screening for security purposes. We note here that data streams (subject to exponential growth) and their metamorphosis are important assets and processes, respectively, for each business enterprise. Data and processes using proper context and web services facilitate decision-making and provide value-added to the users. Intelligent information management is further related to autonomic computing and self-managing operations. Intelligent biometric information management revolves mostly around robust predictions, despite interferences in data capture, using adaptive inference. For the remainder of this paper the biometric of interest is the human face with predictions on face identities, and reasoning and inference as the aggregate means to make predictions.

There has been much realization that face recognition is still lacking. The recent call for papers (CFP) for a Special Issue on *Real-World Face Recognition* issued in March 2010 by *IEEE Transactions on Pattern Analysis and Machine Intelligence*, includes as a matter-of-fact the statement "Face recognition in well-controlled environments is relatively mature and has been heavily studied, but face recognition in uncontrolled or moderately controlled environments is still in its early stages." Two significant efforts have been undertaken over the last several years to alleviate the concerns expressed above and to advance the state-of-the art of biometric authentication. One effort is the use of multi-modal but hopefully complementary (relative to authentication) biometrics, while the other effort is geared toward finding better ways and means to fuse the data that such suites of multimodal biometrics acquire and/or derive. The latter effort of fusing data is performed at different levels of functionality and granularity. Data fusion (including multisensory integration), however, is just a euphemism for reasoning and sound inferences. As it is practiced today it involves ad-hoc combination rules. The goal for this paper is to advance an integrated, principled and unified methodology for biometric inference, whose realization interfaces between the Bayesian framework (including forensics), Statistical Learning Theory (SLT), and Information Theory (IT) using randomness and complexity concepts. The interface and its implementation are built around discriminative methods whose realization takes place using likelihood ratios (LR) using model-free and non-parametric concepts borrowed from transduction and semi-supervised learning (SSL), e.g., strangeness ("typicality") and p-values (for relative ranking but different from distribution tails). Reasoning and inference are known as iterative processes that accrue evidence to lessen ambiguity and to eventually reach the stage where decisions can be made. Evidence accumulation involves many channels of information and takes place over time and varying contexts. It further requires the means for information aggregation, which are provided in our methodology by boosting methods. Additional expansions to our methodology are expected. In particular, it is apparent that the methodology should involve stage-wise the mutual information, between the input signals and the eventual output labels, and links channel capacity with data compression and expected performance. Compression is after all comprehension (practical inference/intelligence for decision-making) according to Leibniz. The potential connections between the Bayesian framework and SLT, on one side, and Information Theory (IT), on the other side, were recently explored [2].

Complementary to biometric face information is biometric process, in general, and data fusion, in particular. Recent work related to data fusion [3,4] is concerned among others with cross-device matching and device interoperability, and quality dependent and cost-sensitive score level multi-modal fusion. The solutions offered are quality dependent cluster indexing and sequential fusion, respectively. The tasks considered and the solutions proffered can be subsumed by our proposed SLT & IT methodology. In terms of functionality and granularity biometric inference can address multi-level fusion: feature / parts, score ("match"), and detection ("decision"); multi-layer fusion: modality, quality, and method (algorithm); and multi-system fusion using boosting for aggregation and transduction for local estimation and score normalization. This is explained, motivated, and detailed throughout the remaining of the paper as outlined next. Section 2 is a brief on discriminative methods and forensics. Background on randomness and complexity comes in Section 3. Discussion continues in Section 4 on strangeness and p-values, in Section 5 on transduction and open set inference, and in Section 6 on aggregation using boosting. Biometric inference to address specific data fusion tasks is presented in Section 7 on generic multi-level and multi-layer fusion, in Section 8 on score normalization and revision theory, in Section 9 on face selection (tracking mode 1), and in Sect. 10 on identity management (tracking mode 2) using martingale for change detection and active learning. The paper concludes in Section 11 with suggestions for promising venues for future research at the intersection between the Bayesian framework, Information Theory (IT), and Statistical Learning Theory (SLT) using the temporal dimension as the medium of choice.

2. Discriminative Methods and Forensics

Discriminative methods support practical intelligence, in general, and biometric inference and prediction, in particular. Progressive processing, evidence accumulation, and fast decisions are their hallmarks. There is no time for expensive density estimation and marginalization characteristic of generative methods. There are additional philosophical and linguistic arguments that support the discriminative approach. It has to do with practical reasoning and epistemology, when recalling from Hume, that "all kinds of reasoning consist in nothing but a comparison and a discovery of those relations, either constant or inconstant, which two or more objects bear to each other," similar to non-accidental coincidences and sparse but discriminative codes for association [5]. Formally, "the goal of pattern classification can be approached from two points of view: informative [generative] - where the classifier learns the class densities, [e.g., HMM] or discriminative – where the focus is on learning the class boundaries without regard to the underlying class densities, [e.g., logistic regression and neural networks]" [6]. Discriminative methods avoid estimating how the data has been generated and instead focus on estimating the posteriors similar to the use of likelihood ratios (LR) and odds. The informative approach for 0/1 loss assigns some input $\boldsymbol{x}$ to the class k ε K for whom the class posterior probability P(y = k | **x**)

$$\mathrm{P}\left(\mathrm{y}=k \middle| \mathbf{x}\right)=\mathrm{P}\left(\mathbf{x} \middle| \mathrm{y}=k\right)\mathrm{P}\left(\mathrm{y}=k\right)\Big/\sum_{m}^{K}\mathrm{P}\left(\mathbf{x} \middle| \mathrm{y}=m\right)\mathrm{P}\left(\mathrm{y}=m\right)$$

yields the maximum. The MAP decision requires access to the log-likelihood $\mathrm{P}_\theta(\mathbf{x}, \mathrm{y})$. The optimal (hyper) parameters $\boldsymbol{\theta}$ are learned using maximum likelihood (ML) and a decision boundary is then induced, which corresponds to a minimum distance classifier. The discriminative approach models directly the conditional log-likelihood or posteriors $\mathrm{P}_\theta(\mathrm{y} \mid \mathbf{x})$. The optimal parameters are estimated using ML leading to the discriminative function

$$\lambda_k(\mathbf{x})=\log\left[\mathrm{P}\left(\mathrm{y}=k \middle| \mathbf{x}\right)\Big/\mathrm{P}\left(\mathrm{y}=\mathrm{k} \middle| \mathbf{x}\right)\right]$$

that is similar to the use of the Universal Background Model (UBM) for score normalization and LR definition. The comparison takes place between some specific class membership k and a generic distribution (over K) that describes everything known about the population at large. The discriminative approach was found [6] to be more flexible and robust compared to informative/generative methods because fewer assumptions are made. One possible drawback for discriminative methods comes from ignoring the marginal distribution P(**x**), which is difficult to estimate anyway. Note that the informative approach is biased when the distribution chosen is incorrect.

The likelihood ratio LR provides straightforward means for discriminative methods using optimal hypothesis testing. Assume that the null "H0" and alternative "H1" hypotheses correspond to impostor ***i*** and genuine ***g*** subjects, respectively. The probability to reject the null hypothesis, known as the false accept rate (FAR) or type I error, describes the situation when impostors are authenticated as genuine subjects by mistake. The probability for correctly rejecting the null hypothesis (in favor of the alternative hypothesis) is known as the hit or genuine acceptance ("hit") rate (GAR). It defines the power of the test $1 - \beta$ with β the type II error when the test fails to reject the null hypothesis when it is false. The Neyman-Pearson (NP) statistical test $\psi(\mathbf{x})$ compares in an optimal fashion the null hypothesis against the alternative hypothesis, e.g., $P(\psi(\mathbf{x}) = H1 | H0\} = \alpha$, $\psi(\mathbf{x}) = 1$ when $f_{\mathbf{g}}(\mathbf{x}) / f_{\mathbf{i}}(\mathbf{x}) > \tau$, and $\psi(\mathbf{x}) = H0$ when $f_{\mathbf{g}}(\mathbf{x}) / f_{\mathbf{i}}(\mathbf{x}) < \tau$ with τ some constant. The Neyman-Pearson lemma states that for some fixed FAR = α one can select the threshold τ such that the $\psi(\mathbf{x})$ test maximizes GAR and it is the most powerful test for testing the null hypothesis against the alternative hypothesis at significance level α. Specific implementations for $\psi(\mathbf{x})$ during cascade classification are possible and they are driven by boosting and strangeness (transduction) (see Section 4).

Gonzales-Rodriguez *et al.* [7] provide strong motivation from forensic sciences for the evidential and discriminative use of the likelihood ratio (LR). They make the case for rigorous quantification of the process leading from evidence (and expert testimony) to decisions. Classical forensic reporting provides only "identification" or "exclusion/elimination" decisions and it requires the use of subjective thresholds. If the forensic scientist is the one choosing the thresholds, he will be ignoring the prior probabilities related to the case, disregarding the evidence under analysis and usurping the role of the Court in taking the decision, "… *the use of thresholds is in essence a qualification of the acceptable level of reasonable doubt adopted by the expert*" [8].

The Bayesian approach's use of the likelihood ratio avoids the above drawbacks. The roles of the forensic scientist and the judge/jury are now clearly separated. What the Court wants to know are the posterior odds in favor of the prosecution proposition (*P*) against the defense (*D*) [posterior odds = LR × prior odds]. The prior odds concern the Court (background information relative to the case), while the likelihood ratio, which indicates the strength of support from the evidence, is provided by the forensic scientist. The forensic scientist cannot infer the identity of the probe from the analysis of the scientific evidence, but gives the Court the likelihood ratio for the two competing hypothesis (*P* and *D*). The likelihood ratio serves as an indicator of the discriminating power (similar to Tippett plots) for the forensic system, e.g., the face recognition engine, and it can be used to comparatively assess authentication performance.

The use of the likelihood ratio has been recently motivated by similar inferences holding between biometrics and forensics [9] with evidence evaluated using a probabilistic framework. Forensic inferences correspond to authentication, exclusion, or inconclusive outcomes, and are based on the strength of biometric (filtering) evidence accrued by prosecution and defense competing against each other. The use of the LR draws further support from the US Supreme Court Daubert ruling on the admissibility of scientific evidence [10]. The Daubert ruling called for a common framework that is both transparent and testable and can be the subject of further calibration ("normalization"). Transparency comes from the Bayesian approach, which includes likelihood ratios as mechanisms for evidence assessment ("weighting") and aggregation ("interpretation"). The likelihood ratio LR is the quotient of a similarity factor, which supports the evidence that the query sample belongs to a given suspect (assuming that the null hypothesis is made by the prosecution *P*), and a typicality factor, e.g., UBM (Universal Background Model) which quantifies support for the alternative hypothesis made by the defense *D* that the query sample belongs to someone else (see Sect. 4 for the similarity between LR and the strangeness measure provided by transduction).

3. Randomness and Complexity

Let **#(**z**)** be the length of the binary string *z* and ***K(z)*** be its Kolmogorov complexity, which is the length of the smallest program (up to an additive constant) that a Universal Turing Machine needs as input in order to output *z*. The randomness deficiency ***D(z)*** for string *z* [11] is $D(z) = \#(z) - K(z)$ with $D(z)$ a measure of how random the binary string *z* is. The larger the randomness deficiency is the more regular and more probable the string *z* is. Kolmogorov complexity and randomness using MDL (minimum description length) are closely related. Transduction (see Section 4) chooses from all the possible labeling ("identities") for test data the one that yields the largest randomness deficiency, *i.e.*, the most probable labeling. The biometric inference engine is built around randomness and complexity with similarity metrics and corresponding rankings driven by strangeness and p-values throughout the remaining of the paper.

4. Strangeness and p-Values

The strangeness measures the lack of typicality (for a face or face component) with respect to its true or putative (assumed) identity label and the labels for all the other faces or parts thereof. Formally, the strangeness measure α_i is the (likelihood) ratio of the sum of the *k* nearest neighbor (k-nn) distances *d* from the same class *y* divided by the sum of the *k* nearest neighbor (k-nn) distances from all the other classes ($\bar{y}$). The smaller the strangeness, the larger its typicality and the more probable its (putative) label *y* is. The strangeness facilitates both feature selection (similar to Markov blankets) and variable selection (dimensionality reduction). One finds empirically that the strangeness, classification margin, sample and hypothesis margin, posteriors, and odds are all related via a monotonically non-decreasing function with a small strangeness amounting to a large margin.

Additional relations that link the strangeness and the Bayesian approach using the likelihood ratio can be observed, e.g., the logit of the probability is the logarithm of the odds, logit (p) = log (p/(1-p)), the difference between the logits of two probabilities is the logarithm of the odds ratio, *i.e.*, log (p/(1-p)/ q/(1-q)) = logit (p) – logit (q) (see also logistic regression and the Kullback-Leibler (KL) divergence). The logit function is the inverse of the "sigmoid" or "logistic" function. Another relevant observation that buttresses the use of the strangeness comes from the fact that unbiased learning of Bayes classifiers is impractical due to the large number of parameters that have to be estimated. The alternative to the unbiased Bayes classifier is logistic regression, which implements the equivalent of a discriminative classifier.

The likelihood-like definitions for strangeness are intimately related to discriminative methods. The p-values suggested next compare ("rank") the strangeness values to determine the credibility and confidence in the putative classifications ("labeling") made. The p-values bear resemblance to their counterparts from statistics but are not the same [12]. p-values are determined according to the relative rankings of putative authentications against each one of the identity classes known to the enrolled gallery using the strangeness. The standard p-value construction shown below, where *l* is the cardinality of the training set *T*, constitutes a valid randomness (deficiency) test approximation [13] for some putative label *y* hypothesis assigned to a *new* sample

$$\mathrm{p}_y(e) = \#\left(i : \alpha_i \geq \alpha_{new}^{y}\right) \big/ (l+1)$$

P-values are used to assess the extent to which the biometric data supports or discredits the null hypothesis H0 (for some specific authentication). When the null hypothesis is rejected for each identity class known, one declares that the test image lacks mates in the gallery and therefore the identity query is answered with "none of the above." This corresponds to forensic exclusion with rejection. It is characteristic of open set recognition with authentication implemented using *Open Set Transduction Confidence Machine (TCM) – k-nearest neighbor (k-nn)* [14]. TCM facilitates outlier detection, in general, and imposters detection, in particular.

5. Transduction

Transduction is different from inductive inference. It is local inference ("estimation") that moves from particular(s) to particular(s). In contrast to inductive inference, where one uses empirical data to approximate a functionnal dependency (the inductive step [that moves from particular to general] and then uses the dependency learned to evaluate the values of the function at points of interest (the deductive step [that moves from general to particular]), one now directly infers (using transduction) the values of the function only at the points of interest from the training data [15]. Inference now takes place using both labeled and unlabeled data, which are complementary to each other. Transduction incorporates unlabeled data, characteristic of test samples, in the decision-making process responsible for their labeling for prediction, and seeks for a consistent and stable labeling across both (near-by) training ("labeled data") and test data. Transduction seeks to authenticate unknown faces in a fashion that is most consistent with the given identities of known but similar faces (from an enrolled gallery/data base of raw images and/or face templates). The search for putative labels (for unlabeled samples) seeks to make the labels for both training and test data compatible or equivalently to make the training and test error consistent.

Transduction "works because the test set provides a nontrivial factorization of the [discrimination] function class" [16]. One key concept behind transduction (and consistency) is the symmetrization lemma [15], which replaces the true (inference) risk by an estimate computed on an independent set of data, e.g., unlabeled or test data, referred to as 'virtual' or 'ghost samples'. The simplest realization for transductive inference is the method of *k* – nearest neighbors. The Cover – Hart theorem [17] proves that asymptotically, the one nearest neighbor classification algorithm is bounded above by twice the Bayes' minimum probability of error. This mediates between the Bayesian approach and likelihood ratios, on one side, and strangeness / p- values and transduction, on the other side (see below). Similar and complementary to transduction is semi-supervised learning (SSL) [16]. Face recognition requires (for discrimination purposes) to compare face images according to the way they are different from each other and to rank them accordingly. Scoring, ranking and inference are done using the *strangeness* and *p – values*, respectively, as explained below.

Similar to semi-supervised learning, changing the class assignments (characteristic of impostor behavior) provides the bias needed to determine ("infer") the rejection threshold required to make an authentication or to decline making one. Towards that end one re-labels the training exemplars, one at a time, with all the ("impostor") putative labels except the one originally assigned to it. The PSR (peak-to-side) ratio, PSR = $(p_{max} - p_{min}) / p_{stdev}$, traces the characteristics of the resulting p-value distribution and determines, using cross validation, the [a priori] threshold used to identify ("infer") impostors. The PSR values found for impostors are low because impostors do not mate and their relative strangeness is high (and p-value low). Impostors are deemed as outliers and are thus rejected [14]. The same cross-validation is used for similar purposes during boosting.

6. Boosting

The motivation for boosting goes back to Marvin Minsky and Levin Kanal who have claimed at an earlier time that "It is time to stop arguing over what is best [for decision-making] because that depends on context and goal. Instead we should work at a higher level of [information] organization and discover how to build [decision-level] managerial [fusion] systems to exploit the different virtues and evade the different limitations of each of these ways of comparing things" and "No single model exists for all pattern recognition problems and no single technique is applicable for all problems. Rather what we have in pattern recognition is a bag of tools and a bag of problems", respectively. This is exactly what data fusion is expected to do with biometric samples that need to be authenticated. The combination rule for data fusion is now principled. It makes inferences using sequential aggregation (similar to [4]) of different components, which are referred to in the boosting framework as weak learners (see below). Inference takes now advantage of both localization and specialization to combine expertise. This corresponds to an ensemble of method and mixtures of experts.

Logistic regression is a sigmoid function that directly estimates the parameters of P $(y|\mathbf{x})$ to learn mappings $f : \mathbf{x} \rightarrow y$ or $P(y|\mathbf{x})$, e.g., $P\{y = 1 \mid \mathbf{x}\}$ for the case when y is Boolean. Logistic regression is behind discriminative methods and likelihood ratios, e.g., label y = 1 if $P\{y = 1 \mid \mathbf{x}\} / P\{y = 0 \mid \mathbf{x}\} > 1$ (see Section 2). Finally, logistic regression can be approximated by Support Vector Machines (SVM). AdaBoost [18] (see below) minimizes (using greedy optimization) some functional whose minimum defines logistic regression [19], while an ensemble of SVM is functionally similar to AdaBoost [20]. The strangeness is thus quite powerful as it provides alternative but simpler realizations for a wide range of well known discriminative methods for inference, in general, and classification, in particular.

The basic assumption behind boosting is that "weak" learners can be combined to learn any target concept with probability 1 – η. Weak learners, usually built around simple features but here built using the full range of components available for data fusion, learn to classify at better than chance (with probability 1/2 + η for η > 0). AdaBoost [18] works by adaptively and iteratively re-sampling the data to focus learning on samples that the previous weak (learner) classifier could not master, with the relative weights of misclassified samples increased ("refocused") after each iteration. AdaBoost involves choosing T effective components h_t to serve as weak (learners) classifiers and using them to construct the separating hyper-planes. The mixture of experts or final boosted (stump) strong classifier H is

$$H(\mathbf{x}) = \sum_{t=1}^{T} \alpha_t h_t(\mathrm{x}) > \frac{1}{2} \sum_{t=1}^{T} \alpha_t$$

with α the reliability or strength of the weak learner. The constant 1/2 comes in because the boundary is located mid – point between 0 and 1. If the negative and positive examples are labeled as – 1 and +1 the constant used is 0 rather than 1/2. The goal for AdaBoost is margin optimization with the margin viewed as a measure of confidence or predictive ability. The weights taken by the data samples are related to their margin and explain the AdaBoost's generalization ability. AdaBoost minimizes (using greedy optimization) some risk functional whose minimum defines logistic regression. AdaBoost converges to the posterior distribution of y conditioned on $\boldsymbol{x}$, and the strong but greedy classifier H in the limit becomes the log-likelihood ratio test.

The multi-class extensions for AdaBoost are AdaBoost.M1 and .M2, the latter one used to learn strong classifiers with the focus now on both difficult samples to recognize and labels hard to discriminate. The use of features or in our case (fusion) components as weak learners is justified by their apparent simplicity. The drawback for AdaBoost.M1 comes from its expectation that the performance for the weak learners selected is better than chance. When the number of classes is k > 2, the condition on error is, however, hard to be met in practice. The expected error for random guessing is 1 – 1/k; for k = 2 the weak learners need to be just slightly better than chance. AdaBoost.M2 addresses this problem by allowing the weak learner to generate instead a set of plausible labels together with their plausibility (not probability), *i.e.*, $[0, 1]^k$. The AdaBoost.M2 version focuses on the incorrect labels that are hard to discriminate. Towards that end, AdaBoost.M2 introduces a pseudo-loss e_t for hypotheses h_t such that for a given distribution D_t one seeks $h_t : \mathbf{x} \times Y \rightarrow [0,1]$ that is better than chance. "The pseudo-loss is computed with respect to a distribution over the set of all pairs of examples and incorrect labels. By manipulating this distribution, the boosting algorithm can focus the weak learner not only on hard-to-classify examples, but more specifically, on the incorrect labels y that are hardest to discriminate" [18]. The use of Neyman-Pearson is complementary to AdaBoost.M2 training (see Section 2) and can meet pre-specified hit and false alarm rates during weak learner selection.

7. Multi-Level and Multi-Layer Fusion

We discuss here biometric inference and address specific data fusion tasks. The discussion is relevant to both generic multi-level and multi-layer fusion in terms of functionality and granularity. Multi-level fusion involves feature/parts, score ("match"), and detection ("decision"), while multi-layer fusion involves modality, quality, and method (algorithm). The components are realized as weak learners whose relative performance is driven by transduction using strangeness and p-value (see Section 5), while their aggregation is achieved using boosting (see Section 6). Additional data fusion-like tasks are discussed in subsequent sections.

The strangeness is the thread to implement both representation and boosting (learning, inference, and prediction regarding classification). The strangeness, which implements the interface between the biometric representation (including its attributes and/or parts) and boosting, combines the merits of filter and wrapper classification methods. The coefficients and thresholds for the weak learners, including the thresholds needed for open set recognition and rejection are learned using validation images, which are described in terms of components similar to those found during enrollment [21]. The best feature correspondence for each component is sought between a validation and a training biometric image over the component ("parts" or "attributes") defining that component. The strangeness of the best component found during training is computed for each validation biometric image under all its putative class labels c (c = 1,…,C). Assuming M validation biometric images from each class, one derives M positive strangeness values for each class c, and M(C – 1) negative strangeness values. The positive and negative strangeness values correspond to the case when the putative label of the validation and training image are the same or not, respectively. The strangeness values are ranked for all the components available, and the best weak learner h_i is the one that maximizes the recognition rate over the whole set of validation biometric images V for some component i and threshold θ_i. Boosting execution is equivalent to cascade classification [22]. A component is chosen as a weak learner on each iteration (see **Figure 1**).

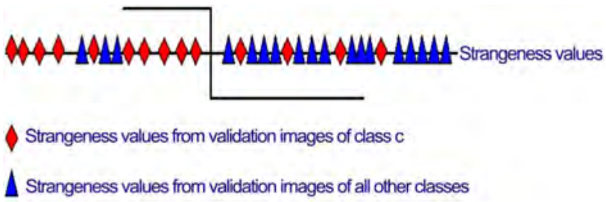


Figure 1. Learning weak learners ("Biometric Components") as stump functions.

The level of significance α determines the scope for the null hypothesis H0. Different but specific alternatives can be used to minimize Type II error or equivalently to maximize the power $(1 - \beta)$ of the weak learner [23]. During cascade learning each weak learner ("classifier") is trained to achieve (minimum acceptable) hit rate h = $(1 - \beta)$ and (maximum acceptable) false alarm rate α (see Sect. 2) Upon completion, boosting yields the strong classifier H(**x**), which is a collection of discriminative biometric components playing the role of weak learners. The hit rate after T iterations is h^T and the false alarm α^T.

8. Score Normalization, Revision Theory, and CMC Estimation

The practice of score normalization in biometrics aims at countering subject/client variability during verification. It is used (a) to draw sharper class or client boundaries for better authentication and (b) to make the similarity scores compatible and suitable for integration. The emphasis in this section is the former rather than the latter, which was already discussed in Section 7. Score normalization is concerned with adjusting both the client dependent scores and the thresholds needed for decision – making during post-processing. The context for score normalization includes clients S and impostors $\neg S$. One should not confuse post processing score normalization with normalization implemented during preprocessing, which is used to overcome the inherent variability in the image acquisition process. Such preprocessing type of normalization usually takes place by subtracting the mean (image) and dividing the result by the standard deviation. This leads to biometric data within the normalized range of [0, 1]. Score normalization during post-processing can be adaptive or empirical, and it requires access to additional biometric data prior and/or during the decision-making process.

The details for empirical score normalization and its effects are as follows [24]. Assume that the PDF of match ("similarity") scores is available for both genuine transactions (for the same client), *i.e.*, P_g, and impostor transactions (between different clients), *i.e.*, P_i. Such information can be gleaned from sets maintained during enrollment or gained during the evaluation itself. One way to calculate the normalized similarity score *ns* for a match score *m* is to use Bayes' rule

$$ns = \mathrm{P}(g|m) = \mathrm{P}(m|g)\mathrm{P}(g)/\mathrm{P}(m)$$

where P(g) is the a priori probability of a genuine event and P($m \mid g$) is the conditional probability of match score m for some genuine event g. The probability of m for all events, both genuine and impostor transactions, is

$$\mathrm{P}(m) = \mathrm{P}(g)\mathrm{P}_g(m) + (1 - \mathrm{P}(g))\mathrm{P}_i(m)$$

The normalized score *ns* is then

$$ns = \mathrm{P}(g)\mathrm{P}_g(m) / \left[\mathrm{P}(g)\mathrm{P}_g(m) + (1 - \mathrm{P}(g))\mathrm{P}_i(m)\right]$$

The accuracy for the match similarity scores depends on the degree to which the genuine and impostor PDF approximate ground truth. Bayesian theory can determine optimal decision thresholds for verification only when the two (genuine and impostor) PDF are known. To compensate for such PDF estimation errors one should fit for the "overall shape" of the normalized score distribution, while at the same time seek to discount for "discrepancies at low match scores due to outliers" [25]. The normalized score serves to convert the match score into a more reliable value.

The motivation behind empirical score normalization using evaluation data can be explained as follows. The evaluation data available during testing attempts to overcome the mismatch between the estimated and the real conditional probabilities referred to above. New (on – line) estimates are obtained for both $\mathrm{P}_g(m)$ and $\mathrm{P}_i(m)$, and the similarity scores are changed accordingly. As a result, the similarity score between a probe and its gallery counterpart varies. Estimates for the genuine and impostor PDF, however, should still be obtained at enrollment time and/or during training rather than during testing. One of the innovations advanced by FRVT 2002 was the concept of virtual image sets. The availability of the similarity (between queries Q and targets T) matrix enables one to conduct different "virtual" experiments by choosing specific query P and gallery G sets as subsets of Q and T. Examples of virtual experiments include assessing the influence of demographics and/or elapsed time on face recognition performance. Performance scores relevant to a virtual experiment correspond to the P × G similarity scores. Empirical score normalization compromises, however, the very concept of virtual experiments. The explanation is quite simple. Empirical score normalization has no access to the information needed to define the virtual experiment. As a result, the updated or normalized similarity scores depend now on additional information whose origin is outside the specific gallery and probe subsets.

Revision theory expands on score normalization in a principled way and furthers the scope and quality of biometric inference. Semi-supervised learning operates under the smoothness assumption (of supervised learning) that similar patterns should yield similar matching scores; and under the low-density separation assumption for both labeled and unlabeled examples. Training and testing are complementary to each other and one can revise both the labels and matching scores to better accommodate the smoothness assumption. Genuine and imposter individual contributions, ranked using strangeness and p-values, are updated, if there is need to do so, in a fashion similar to that used during open set recognition. This contrasts with the holistic approach where matching scores are re-estimated using the Bayes rule and generative models as described earlier.

The basic tools for revision theory are those of perturbation, relearning, and stability to achieve better learning and predictions and therefore to make better inferences [26]. Perturbations to change labels and/or matching scores in regions of relatively high-density and then re-estimate the margin together with quality measures for the putative assignments made, e.g., credibility and confidence (see Section 10). Gradient-descent and stochastic optimization are the methods of choice for choosing and implementing among perturbations using the regularization framework. Re-learning and stability are relevant as explained next. Transduction seeks consistent labels and matching scores for both training (labeled) and test (unlabeled) data. Poggio *et al.* [26] suggest it is the stability of the learning process that leads to good predictions. In particular, the stability property says that "when the training set is perturbed by deleting one example, the learned hypothesis does not change much. This stability property stipulates conditions on the learning map rather than on the hypothesis space." Perturbations ("what if") should therefore include relabeling, exemplar deletion (s), and updating matching scores. As a result of guided perturbations more reliable and robust biometric inference and predictions become possible.

Identification is different from verification in both functionality and implementation. The closed set recognition case is 1 – MANY rather than 1 – 1 and it retrieves, using repeated 1 – 1 verifications a rank – based list of candidates ordered according to their similarity to the unknown test probe. Rank one corresponds to the gallery image found most similar to the test probe. The percentage of probes for which the top ranked candidate is the correct one defines the probability for rank one. The probabilities for rank $\boldsymbol{r}$ record the likelihood that the correct gallery image shows at rank $\boldsymbol{r}$ or lower. The probability points trace the Cumulative Match Curve (CMC). CMC are useful for (ranked) identification only when there is exactly one mate for each probe query, *i.e.*, the gallery sets are singletons. Assume *A* classes enrolled in the gallery, *N* query example, the strangeness ("odds") defined for $k = 1$ to yield (similar to cross TCM validation) *NA* values, *A* "valid", and *N*(*A-1*) invalid (kind of imposters). Determine for each query and for their correct putative class assignment ***a***, the corresponding p-value rank $\boldsymbol{r}$ ε (1, ..., A). The lower the rank ***r***, the more typical the biometric sample is to its true class ***a***. Tabulate the number of queries for each class *A* and normalize by the total number of queries *N*. This yields an estimate for CMC. The presence of more than one mate for some or all of the probes in the gallery, e.g., Smart Gate and FRGC, which employ $k > 1$ and are thus more in tune with the strangeness and p-values definitions, can be handled in several ways. One way is to declare a probe matched at rank ***r*** if at least one of its mated gallery images shows up at that rank. Similar to the singleton case, tabulate the minimum among the p-values for all samples across their correct mates. Other possibilities would include retrieving all the mates or a fraction thereof at rank ***r*** or better and/or using a weighted metric that combines the similarity between the probe and its mates. There is also the possibility that the test probe set itself consists of several images and/or that both the gallery and the probe sets include several images for each subject. This can be dealt using the equivalent of the Hausdorff distance with the minimum over gallery sets performed in an iterative fashion for query sets or using the minimum over both the gallery and query sets pair wise distances. Last but not least, recall and precision (sensitivity and specificity) and F1 are additional (information retrieval) indexes that can be estimated using the strangeness and p-values in a fashion similar to CMC estimation.

9. Face Selection and Tracking

Face selection expands on the traditional role of face authentication. It assumes that multiple still image sets and/or video sequences for each enrollee are available during training, and that a data streaming video sequence of face images, usually acquired from CCTV, becomes available during surveillance. The goal is to identify the subset of (CCTV) frames, if any, where each enrolled subject, if any, shows up. Subjects can appear and disappear as time progresses and the presence of any face is not necessarily continuous across (video) frames. Faces belonging to different subjects thus appear in a sporadic fashion across the video sequence. Some of the CCTV frames could actually be void of any face, while other frames could include occluded or disguised faces from different subjects. Kernel k-means and/or spectral clustering [27] using biometric patches, parts, and strangeness and p-values for typicality and ranking are proposed for face selection and tracking. This corresponds to the usual use of tracking during surveillance, while another use of tracking for identity management is deferred to the next section. Face selection counts as biometric inference. Biometric evidence accumulates and inferences on authentication can be made for familiar ("enrolled") faces.

Spectral clustering is a recent methodology for segmentation and clustering. The inspiration for spectral clustering comes from graph theory (minimum spanning trees (MST) and normalized cuts) and the spectral (eigen decomposition) of the adjacency/proximity ("similarity") matrix and its subsequent projection to a lower dimensional space. This describes in a succinct fashion the graph induced by the set of biometric data samples ("patterns"). Minimizing the "cut" (over the set of edges connecting *k* clusters) yields "pure" (homogeneous) clusters.

Similar to decision trees, where information gain is replaced by gain ratio to prevent spurious fragmentation, one substitutes the "normalized cut" (that minimizes the cut while keeping the size of the clusters large) for "cut." To minimize the normal cut (for $k = 2$) is equivalent to minimize the Raleigh quotient of the normalized graph Laplace matrix L* where $L^* = D^{-1/2} L D^{-1/2}$ with $L = D - W$; W is the proximity ("similarity") matrix and the (diagonal) degree matrix D is the "index" matrix that measures the "significance" for each node. The Raleigh quotient (for $k = 2$) is minimized for the eigenvector $\boldsymbol{z}$ corresponding to the second smallest eigenvalue of L*. Given *n* data samples and the number of clusters expected *k*, spectral clustering (for $k > 2$) employs the Raleigh – Ritz theorem and leads among others to algorithms such as Ng, Jordan, and Weiss [28] where one (1) computes W, D, L, and L*; (2) derives the largest *k* eigenvectors $\boldsymbol{z_i}$ of L*; (3) forms the matrix $\mathbf{U} \in \mathbf{R}^{n \times k}$ by normalizing the row sums of $\boldsymbol{z_i}$ to have norm 1; (4) cluster the samples $\boldsymbol{x_i}$ corresponding to $\boldsymbol{z_i}$ using K-means.

An expanded framework that integrates graph-based semi-supervised learning and spectral clustering for the purpose of grouping and classification, *i.e.*, label propagation, can be developed. One takes now advantage of both labeled and mostly unlabeled biometric patterns. The graphs reflect domain knowledge characteristics over nodes (and sets of nodes) to define their proximity ("similarity") across links ("edges"). The solution proposed is built around label propagation and relaxation. The graph and the corresponding Laplacian, weight, and diagonal matrices L, W, and D are defined over both labeled and unlabeled biometric patterns. The harmonic function solution [29] finds (and iterates) on the (cluster) assignment for the unlabeled biometric patterns $\boldsymbol{Y_u}$ as $Y = -(L_{uu})^{-1} L_{ul} Y_l$ with $\boldsymbol{L_{uu}}$ the submatrix of $\boldsymbol{L}$ on unlabeled nodes and $\boldsymbol{Y_l}$ the group indicator over the labeled nodes. Each row of $\boldsymbol{Y_u}$ reports on the posteriors for the Cartesian product between *k* clusters and *n* biometric samples. Class proportions for the labeled patterns can be estimated and used to scale the posteriors for the unlabeled biometric patterns. The harmonic solution is in sync with a random (gradient) walk on the graph that makes predictions on the unlabeled biometric patterns according to the weighted average of their labeled neighbors.

An application of semi-supervised learning to person identification in low-quality webcam images is described by Balcan *et al.* [30]. Learning takes place over both (few) labeled and (mostly) unlabeled face images using spectral clustering and class mass normalization. The functionality involved is similar to face selection from CCTV. The graphs involved consists of i) time edges for adjacent frames likely to contain the same person (if moving at moderate speed); ii) color edges (over short time interval) assume person's apparel the same; and iii) face edges for similarity over longer time spans. Label propagation and scalability become feasible using ranking [31] and semi-supervised learning and parallel Map-Reduce [32] (in a fashion similar to Page Rank).

10. Identity Management

Identity management stands for another form of tracking. It involves monitoring the gallery of biometric templates in order to maintain an accurate and faithful rendition of enrolled subjects as times moves on. This facilitates reliable and robust temporal mass screening and it represents yet another for of biometric inference. The surveillance aspect is complementary and its role is to prevent imposters from subverting the security arrangements in place. Open set recognition [14], discussed earlier, is integral to surveillance but it does not provide the whole answer. The effective and efficient proper management of the gallery is the main challenge here and it is discussed next.

There are two main problems with identity management. One is to actively monitor the rendering of biometrics signatures and/or templates and the other is to update them if and when significant changes take place. The two problems correspond to active learning [12] and change detection [25], respectively. Active learning is concerned with choosing the most relevant examples needed to improve on classification both in terms of effectiveness ("accuracy") and efficiency ("number of signatures needed"). The scope for active learning can be expanded to include additional aspects including but not limited to choosing the ways and means to accomplish effectiveness and efficiency, on one side, and adversarial learning, characteristic to imposters, on the other side. Our active learning solution [12] is driven by transduction and it is built using strangeness and p-values. The p-values provide a measure of diversity and disagreement in opinion regarding the true label of an unlabeled example when it is assigned all the possible labels. Let p_i be the p-values obtained for a particular example x_{n+1} using all possible labels $i = 1, \ldots, M$. Sort the sequence of p-values in descending order so that the first two p-values, say, p_j and p_k are the two highest p-values with labels *j* and *k*, respectively. The label assigned to the unknown example is *j* with a p-value of p_j. This value defines the credibility of the classification. If p_j (credibility) is not high enough, the prediction is rejected. The difference between the two p-values can be used as a confidence value on the prediction. Note that, the smaller the confidence, the larger the ambiguity regarding the proposed label. We consider three possible cases of p-values, p_j and p_k, assuming $p_j > p_k$: Case 1. p_j high and p_k low. Prediction "*j*" has high credibility and high-confidence value; Case 2. p_j high and p_k high. Prediction "*j*" has high credibility but low-confidence value; Case 3. p_j low and p_k low. Prediction "*j*" has low credibility and low-confidence value. High uncertainty in prediction occurs for both Case 2 and Case 3. Note that uncertainty of prediction occurs when $p_j \approx p_k$. Define "closeness" as $I(x_{n+1}) = p_j - p_k$ to indicate the quality of information possessed by the example. As $I(x_{n+1})$ approaches 0, the more uncertain we are about classifying the example, and the larger the margin information gain from "advise" is. Active learning will add this example, with its true label, to the training set because it provides new information about the structure of the biometric data model. Extensions to the active learning inference strategy describe above will incorporate error analysis and population diversity characteristic of pattern specific error inhomogeneities (PSEI) [14].

The basics for change detection using martingale are as follows [33]. Assume time-varying multi-dimensional data (stream) matrix $\mathbf{R} = \{R(j) = \mathbf{x}_j\}$ where $R(j)$ are "columns" and stand for time-varying (data stream) biometric vectors. Assume that seeding provides some initial $R(j)$ with $j = 1, \ldots, 10$. K-means clustering will find (in an iterative fashion) center "prototypes" $Q(k)$ for the data stream (seen so far). Define the strangeness corresponding to $R(j)$ using the cluster model (with $R = \{\mathbf{x}_j\}$ standing for data stream and *c* standing for cluster center) and the Euclidean distance $d(j)$ between $R(j)$ and $Q(k)$ for $j >= 10$ and $k = j - 9$ as

$$s\left(\mathrm{R},\mathbf{x}_j\right) = \left\|\mathbf{x}_j - c\right\|$$

Define p-values as

$$p_i\left\{\left(\mathbf{x}_1,\mathrm{y}_1\right),\ldots,\left(\mathbf{x}_i,\mathrm{y}_i\right),\theta_i\right\} = \left[\#\left\{j : s_j > s_i\right\} + \theta_i \#\left\{j : s_j = s_i\right\}\right] \Big/ i$$

where s_j is the strangeness measure for $(\mathbf{x}_j, \mathrm{y}_j)$, $j = 1$, 2, …, I and θ_i is randomly chosen from [0, 1] at instance i. Define a family of martingale starting with $M^{\varepsilon}(0) = 1$ and continuing with $M^{\varepsilon}(j)$ indexed by ε in [0, 1]

$$M^{\varepsilon}(j) = \prod_{i=1}^{j} \left\{ \varepsilon (p_i)^{\varepsilon - 1} \right\}$$

The martingale test

$$0 < M^{\varepsilon}(j) < \lambda$$

rejects the null hypothesis H0 "no change in the data stream" for H1 ("change detected in data stream") when M (j) > = λ with the value for λ (empirically chosen to be greater than 2) determined by the FAR one is willing to accept, *i.e.*, $1/\lambda$ = FAR. An alternative (parametric) test, e.g., SPRT, will employ the likelihood ratio (LR) with B < LR < A and decide for H0 as soon as LR(j) < B, decide for the alternative H1 ("change") when LR(j) > A, with B ≈ β (1 – α) and A ≈ (1 – β)/α using α for test significance ("size") and (1 – β) for test power. The changes ("spikes") found can identify critical (transition) states, e.g., ageing, and model an appropriate Hidden Markov Model (HMM) for personal authentication and ID management.

11. Conclusions

This paper proposes a novel all encompassing methodology for robust biometric inference and prediction built around randomness and complexity concepts. The methodology proposed can be mapped to other types of information beyond the biometric modality discussed here. The theoretical framework advanced here for biometric information management is model-free and non-parametric. It draws support from discriminative methods using likelihood ratios to link the Bayesian framework, statistical learning theory (SLT), and Information Theory (IT) using transduction, semi-supervised learning, and mutual information between biometric signatures and/or templates and their labels. Several topics on biometric inference related to i) multi-level and multi-layer data fusion including quality and multi-modal biometrics; ii) cross-matching of capture devices, revision theory, and score normalization; iii) face selection and tracking; and iv) identity management and surveillance were addressed using an integrated approach that includes transduction and boosting for ranking and sequential fusion/aggregation, respectively, on one side, and active learning and change /outlier/intrusion detection using information gain and martingale, respectively, on the other side.

One venue for future research would expand the scope of biometric space regarding information contents and processes. Regarding the biometric space, Balas and Sinha [34] have argued that "it may be useful to also employ region-based strategies that can compare noncontiguous image regions." They further show that "under certain circumstances, comparisons [using dissociated dipole operators] between spatially disjoint image regions are, on average, more valuable for recognition than features that measure local contrast." This is consistent with the expectation that recognition-by-parts architectures [21] should learn [using boosting and transduction] "optimal" sets of regions' comparisons for biometric authentication across varying data capture conditions and contexts. The choices made on such combinations for both multi-level and multi-layer fusion amount to "rewiring" operators and processes. Rewiring corresponds to an additional processing and competitive biometric stage. As a result, the repertoire of information available to biometric inference will now range over local, global, and nonlocal (disjoint) data characteristics with an added temporal dimension. Ordinal rather than absolute codes are feasible in order to gain invariance to small changes in inter-region and temporal contrast. Disjoint and "rewired" patches of information contain more diagnostic information and are expected to perform best for "expression", self-occlusion, and varying image capture conditions. The multi-feature and rewired based biometric image representations and processes together with exemplar-based biometric representations enable flexible matching. The added temporal dimension is characteristic of video sequences and it should lead to enhanced biometric authentication and inference performance using set similarity. Cross-matching biometric devices is yet another endeavor that could be approached using score normalization, non-linear mappings, and revision theory (see Section 8).

Another long-term and needed research venue should consider useful linkages between information theory, the Bayesian framework, and statistical learning theory to advance modes of reliable and robust reasoning and inference with directed application to biometric inference. Such an endeavor will be built around the regularization framework using fidelity of representation, compressive sensing, constraints satisfaction and optimization subject to penalties, and margin for prediction. Biometric dictionaries are also needed for biometric processes to choose from for flexible exemplar-based representation, reasoning, and inference, and to synthesize large-scale databases for biometric evaluations. The ultimate goal is to develop powerful and wide scope biometric language(s) and the corresponding biometric reasoning ("inference") apparatus in a fashion similar to the way language and thought are available for human ("practical") intelligence and inference [35].

12. References

[1] T. D. Wilson, "Information Management," in: J. Feather

and P. Sturges Eds., *International Encyclopedia of Information and Library Science*, Routledge, London, 2003, pp. 263-278.

[2] N. Schmid and H. Wechsler, "Information Theoretical (IT) and Statistical Learning Theory (SLT) Characterizations of Biometric Recognition Systems," *SPIE Electronic Imaging: Media Forensics and Security*, San Jose, CA, Vol. 7541, 2010, pp. 75410M-75410M-13.

[3] N. Poh, T. Bourlai, J. Kittler *et al*., "Benchmark Quality-Dependent and Cost-Sensitive Score-Level Multimodal Biometric Fusion Algorithms," *The IEEE Transaction on Information Forensics and Security*, Vol. 4, No. 4, 2009, pp. 849-866.

[4] N. Poh, T. Bourlai and J. Kittler, "A Multimodal Biometric Test Bed for Quality-dependent, Cost-Sensitive and Client-Specific Score-Level Fusion Algorithms," *Pattern Recognition*, Vol. 43, No. 3, 2010, pp. 1094-1105.

[5] H. B. Barlow, "Unsupervised Learning," *Neural Computation*, Vol. 1, 1989, pp. 295-311.

[6] Y. D. Rubinstein and T. Hastie, "Discriminative Versus Informative Learning," *Knowledge and Data Discovery* (*KDD*), 1997, pp. 49-53.

[7] J. Gonzalez-Rodriguez, P. Rose, D. Ramos, D. T. Toledano and J. Ortega-Garcia, "Emulating DNA: Rigorous Quantification of Evidential Weight in Transparent and Testable Forensic Speaker Recognition," *IEEE Transaction on Audio, Speech and Language Processing*, Vol. 15, No. 7, 2007, pp. 2104-2115.

[8] C. Champed and D. Meuwly, "The Inference of Identity in Forensic Speaker Recognition," *Speech Communication*, Vol. 31, No. 2-3, 2000, pp. 193-203.

[9] D. Dessimoz and C. Champod, "Linkages between Biometrics and Forensic Science," in A. K. Jain, Ed., *Handbook of Biometrics*, Springer, New York, 2008.

[10] B. Black, F. J. Ayala and C. Saffran-Brinks, "Science and the Law in the Wake of Daubert: A New Search for Scientific Knowledge," *Texas Law Review*, Vol. 72, No. 4, 1994, pp. 715-761.

[11] M. Li and P. Vitanyi, "An Introduction to Kolmogorov Complexity and Its Applications," 2nd Edition, Springer-Verlag, Germany, 1997.

[12] S. S. Ho and H. Wechsler, "Query by Transduction," *IEEE Transaction on Pattern Analysis and Machine Intelligence*, Vol. 30, No. 9, 2008, pp. 1557-1571.

[13] T. Melluish, C. Saunders, A. Gammerman, and V. Vovk, "The Typicalness Framework: A Comparison with the Bayesian Approach," TR-CS, Royal Holloway College, University of London, 2001.

[14] F. Li and H. Wechsler, "Open Set Face Recognition Using Transduction," *IEEE Transaction on Pattern Analysis and Machine Intelligence*, Vol. 27, No. 11, 2005, pp. 1686-1698.

[15] V. Vapnik, "Statistical Learning Theory," Springer, New York, 1998.

[16] O. Chapelle, B. Scholkopf and A. Zien (Eds.), "Semi-Supervised Learning," MIT Press, USA, 2006.

[17] T. M. Cover and P. Hart, "Nearest Neighbor Pattern Classification," *IEEE Transaction on Information Theory*, Vol. IT-13, 1967, pp. 21-27.

[18] Y. Freund and R. E. Shapire, "Experiments with a New Boosting Algorithm," *Proceedings of 13th International Conference on Machine Learning* (*ICML*), Bari, Italy, 1996, pp. 148-156.

[19] F. H. Friedman, T. Hastie and R. Tibshirani, "Additive Logistic Regression: A Statistical View of Boosting," *Annals of Statistics*, Vol. 28, 2000, pp. 337-407.

[20] V. Vapnik, "The Nature of Statistical Learning Theory" 2nd Edition, Springer, New York, 2000.

[21] F. Li and H. Wechsler, "Face Authentication Using Recognition-by-Parts, Boosting and Transduction," *International Journal of Artificial Intelligence and Pattern Recognition* (*IJPRAI*), Vol. 23, No. 3, 2009, pp. 545-573.

[22] P. Viola and M. Jones, "Rapid Object Detection Using a Boosted Cascade of Simple Features," *Proceedings of the Computer Vision and Pattern Recognition Conference* (*CVPR*), Kauai, Hawaii, 2001, pp. I-511-I-518.

[23] R. O. Duda, P. E. Hart and D. G. Sork, "Pattern Classification," 2nd Edition, Wiley, New York, 2000.

[24] A. Adler, "Sample Images Can be Independently Regenerated from Face Recognition Templates," 2003. http://www.site.uotawa.ca/~adler/publications/2003/adler-2003-fr-templates.pdf.

[25] T. Poggio and S. Smale, "The Mathematics of Learning: Dealing with Data" *Notices of American Mathematical Socity*, Vol. 50, No. 5, 2003, pp. 537-544.

[26] T. Poggio, R. Rifkin, S. Mukherjee and P. Niyogi, "General Conditions for Predictivity of Learning Theory," *Nature*, Vol. 428, No. 6981, 2004, pp. 419-422.

[27] I. S. Dhillon, Y. Guan and B. Kulis, "Kernel k-means, Spectral Clustering and Normalized Cuts," *Proceedings of the Conference on Knowledge and Data Discovery* (*KDD*), Seattle, WA, 2004.

[28] A. Y. Ng, M. I. Jordan and Y. Weiss, "On Spectral Clustering: Analysis and an Algorithm," *Neural Information Processing Systems* (*NIPS*) 14, MIT Press, Boston, MA, 2002, pp. 849-856.

[29] X. Zhu, Z. Ghahramani and L. Lafferty, "Semi-Supervised Learning Using Gaussian Fields and Harmonic Functions," *Proceedings of the 20th International Conference on Machine Learning* (*ICML*), Washington, DC, 2003, pp. 912-919.

[30] M. F. Balcan, A. Blum, P. P. Choi, J. Lafferty, B. Pantano, M. R. Rwebangira and X. Zhu, "Person Identification in Webcam Images: An Application of Semi-Supervised Learning," *Proceedings of 22nd ICML Workshop on Learning with Partially Classified Training Data*, Bonn, Germany, 2005, pp. 1-9.

[31] K. Duh and K. Kirchhoff, "Learning to Rank with Partially Labeled-Data," SIGIR, Singapore, 2008, pp. 20-27.

[32] D. Rao and D. Yarowsky, "Ranking and Semi-Supervised Classification on Large-Scale Graphs Using Map Reduce," *Proceedings of the Workshop on Graph-based*

Methods for Natural Language Processing (*ACL-IJCNLP*), Singapore, 2009, pp. 58-65.

[33] S. S. Ho and H. Wechsler, “A Martingale Framework for Detecting Changes in the Data Generating Model in Data Streams,” *IEEE Transaction on Pattern Analysis and Machine Intelligence*, No. 99, 2010 (to appear).

[34] B. J. Balas and P. Sinha, “Region-Based Representations for Face Recognition,” *ACM Transactions on Applied Perception*, Vol. 3, No. 4, 2006, pp. 354-375.

[35] H. Wechsler, “Linguistics and Face Recognition,” *Journal of Visual Languages and Computation*, Vol. 20, No. 3, 2009, pp. 145-155.

Intelligent Information Management, **2010, 2, 512-525**
doi:10.4236/iim.2010.29061 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Optimal Task Placement of a Serial Robot Manipulator for Manipulability and Mechanical Power Optimization

Rogério Rodrigues dos Santos[1], Valder Steffen, Jr.[1], Sezimária de Fátima Pereira Saramago[2]
[1]*School of Mechanical Engineering, Federal University of Uberlândia, Uberlândia, Brazil*
[2]*Faculty of Mathematics, Federal University of Uberlândia, Uberlândia, Brazil*
E-mail: *rsantos9@gmail.com, vsteffen@mecanica.ufu.br, saramago@ufu.br*
Received January 1, 2010; *revised May* 19, 2010; *accepted July* 20, 2010

Abstract

Power consumption and accuracy are main aspects to be taken into account in the movement executed by high performance robots. The first aspect is important from the economical point of view, while the second is requested to satisfy technical specifications. Aiming at increasing the robot performance, a strategy that maximizes the manipulator accuracy and minimizes the mechanical power consumption is considered in this work. The end-effector is constrained to follow a predefined path during the optimal task positioning. The proposed strategy defines a relation between mechanical power and manipulability as a key element of the manipulator analysis, establishing a performance index for a rigid body transformation. This transformation is used to compute the optimal task positioning through the optimization of a multicriteria objective function. Numerical simulations regarding a serial robot manipulator demonstrate the viability of the proposed methodology.

Keywords: Optimal Task Placement, Optimal Robot Path Planning, Multicriteria Optimization

1. Introduction

Minimization of production costs and maximization of productivity are some of the major objectives of industrial automation. In this scenario, serial robot manipulators have been proved to be very useful tools.

Due to the augmenting use of serial robot manipulators to perform a number of tasks in industry, requirements concerning higher precision, improved productivity, reduced costs, and better manufacturing quality become very important. To effectively explore all the potential of the robot, its path planning is a subject of major concern.

There are several works in the literature dedicated to different approaches concerning this subject. Regarding the off-line path planning approach, in Saramago [1] a solution of moving a robot manipulator with minimum cost along a specified geometric path in the presence of moving obstacles is presented. The optimal traveling time and the minimum mechanical energy of the actuators are considered together to form a multiobjective function to be minimized along the process.

In Santos [2] a strategy to determine the trajectory for a defined movement taking into account the requirements of torque, velocity, operation time and robot positioning along the movement is proposed. The analysis is performed as an off-line path planning through spline interpolation techniques in the manipulator joint space. In the paper of Chiddarwar [3] an off-line path planning for coordinated manipulation is proposed. The swept sphere volume technique is used to model multiple robots and static obstacles. In Santos [4] the trajectory of two manipulators while manipulating a single object in a collaborative task and the object placement are written as an optimization problem. End-effector positioning and torque requirements are considered together in an optimal control formulation. Accuracy and energy consumption are improved during the path planning. Also, the flexibility effects of manipulators working in a vertical workspace are taken into account, and joint limits are considered in a box-constrained objective function to ensure the movement feasibility at the optimal configuration.

In von Stryk [5] an optimal control path-planning strategy for which the dynamics of the robot and the total traveling time are considered as objectives to be optimized is proposed. In Santos [6] an optimal control strategy that considers the presence of moving obstacles in the workspace is presented.

The problem of time-optimal control along a specified path has been investigated for several authors (Bobrow [7], Slotine [8]). In Constantinescu [9] a smooth timeoptimal strategy for serial robot manipulators is presented.

The paper of Seraji [10] addressed a geometric approach to determine the appropriate base location from which a robot can reach a target point. The work of Zeghloul [11] presented some kinematic performance criteria for the optimal placement of robots, and proposed a general optimization method to determine the placement of manipulators automatically.

In Feddema [12], an algorithm for determining the optimal placement of a robotic manipulator for minimum time coordinated motion is proposed. In Park [13] a study that characterizes a set of desired goal positions and a left-invariant distance metric parametrized by length scale is studied.

The paper of Chirikjian [14] proposed several metrics to be used to generate interpolated sequences of motions of a solid body. In Martinez [15] an analysis regarding the metric of a rigid displacement obtained from the kinematic mapping is presented and other metrics for the set of spatial and planar displacements are proposed. The work of Tabarah [16] proposed an optimization strategy for determining the optimal base for a given manipulator, and two cooperating manipulators, following a continuous path with constant velocity. They considered the constant velocity as a constraint. A strategy for velocity minimization is proposed by Zhang [17], based on a neural network solver. A kinematic planning scheme is reformulated as a quadratic problem that is solved by using a real-time algorithm, and applied to a Puma 560 robotic arm.

In Abdel-Malek [18] a strategy regarding the placement of a robot manipulator aiming at reaching a specified set of target points is described. The placement of a serial manipulator in a given workspace is achieved by defining the position and orientation of the manipulator's base with respect to a fixed reference frame. The strategy is based on characterizing the robot placement by adjusting a constrained cost function that represents the workspace to a set of target points.

With the aim of increasing productivity in the path following, industrial robotic applications have been addressed in the literature by determining path-constrained time-optimal motions, by taking into account the torque limits of the actuators. In these formulations, the joint actuator torques are considered as controlled inputs and the open loop control schemes result in bang-bang or bang-singular-bang controls (Bobrow [7], Chen [19], Shiller [20]). The paper of Xidias [21] proposed the generalization of the task scheduling problem for articulated robots in a constrained 2D environment. An algorithm for the optimum collision-free movement is proposed.

However, the increase of the speed may result (in some cases) in increasing the robot end-effector vibration or decreasing its accuracy. This is observed due to several factors, such as the bang-bang nature of the control, joint flexibility, joint friction, gear arrangement, or even a combination of them.

Vibration is not desired because it can both degrade the system performance and decrease the actuator lifetime. A related and negative effect of vibration is the poor accuracy of the system.

The present contribution discusses an alternative way to improve the overall performance of serial robot manipulators. The use of manipulability and mechanical power related indexes to achieve the best task placement is proposed.

As the basic idea of manipulability consists in describing directions in the task or joint space that maximizes the ratio between some measure of effort in joint space and a measure of performance in task space, the present contribution consists in the optimal positioning of the task to be performed, which results in the maximization of the robot manipulability and the minimization of the required mechanical power. This will lead to accuracy maximization and power consumption minimization, since manipulability can be understood as an ability measurement (Fu [22]) and mechanical power is related to power consumption (Saramago [1]).

In the present paper both kinematics and dynamics features are considered in the proposed task placement strategy. From the best of the authors' knowledge, this problem was not previously addressed in the literature, despite its importance and applicability.

The position of the end-effector is fixed according to a Cartesian reference system, and described as a sequence of sucessive positions. The optimization process applies a translational and rotational transformation matrix with respect to a given reference frame. As a result, an optimal sequence of kinematical robot positions is determined.

The outline of the paper is the following. In Section 2, a review about manipulability is presented. In Section 3, the mechanical power concept is discussed. A strategy to describe the task and its positioning in the design space is proposed in Section 4. Additionally in this section the design variables are defined. In Section 5 the multicriteria programming formulation is presented. In Section 6, the optimization strategy is established and the numerical experiments are performed in Section 7. Finally, the conclusions are drawn in Section 8.

2. Manipulability

Some factors should be taken into account when the posture of the manipulator in the workspace is determined for performing a given task during operation. An important factor is the ease of arbitrarily changing the position and orientation of the end-effector at the tip of the manipulator.

As an approach for evaluating quantitatively the ability of manipulators from the viewpoint of kinematics, the concepts of manipulability ellipsoid and manipulability measure (Yoshikawa [23]) are presented.

Consider a manipulator with n degrees of freedom. The joint variables are denoted by an n-dimensional vector, $\mathbf{q} = [q_1, q_2, ..., q_n]^T$. An m-dimensional vector $\mathbf{r} = [r_1, r_2, ..., r_m]^T$, ($m \leq n$) describes the position and orientation of the end-effector.

The relation between the velocity vector $\mathbf{v}$ corresponding to $\mathbf{r}$ and the joint velocity $\dot{\mathbf{q}}$ is given by:

$$\mathbf{v} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}} \tag{1}$$

where $\mathbf{J}(\mathbf{q})$ is the Jacobian matrix, computed from the kinematics description of the robot.

The set of all end-effector velocities $\mathbf{v}$ that are realizable by joint velocities such that the Euclidean norm of $\dot{\mathbf{q}}$ satisfies $\|\dot{\mathbf{q}}\| \leq 1$ defines an ellipsoid in the m-dimensional Euclidean space. In the direction of the major axis of the ellipsoid, the end-effector can move at high speed, and in the direction of the minor axis, the end-effector can move only at low speed. Also, the larger the ellipsoid is, the faster the end-effector can move.

Since this ellipsoid represents an ability of manipulation, it is called the manipulability ellipsoid.

The principal axes of the ellipsoid can be found by making use of the singular-value decomposition of $\mathbf{J}(\mathbf{q})$.

One of the representative measures for the ability of manipulation derived from the manipulability ellipsoid is the volume of the ellipsoid. Since this volume is proportional to the eigenvalues of the Jacobian, it can be seen as a representative measure. Therefore, the manipulability measure, f_1, for the manipulator configuration, $\mathbf{q}$, is defined as

$$f_1 = \sqrt{\det(\mathbf{J}(\mathbf{q})\mathbf{J}(\mathbf{q})^T)} \tag{2}$$

Generally $f_1 \geq 0$ holds, and $f_1 = 0$ if and only if rank $\mathbf{J}(\mathbf{q}) < m$.

There is a direct relation between singular configuration and manipulability (through the Jacobian concept). According to this relationship, it can be shown that the larger the manipulability measure, the larger the ability of avoiding singular configurations.

3. Mechanical Power

The consideration of the dynamics behavior of a serial robot manipulator is of great importance for its path planning. This information allows a detailed analysis and consequently the development of a precise control specification.

Many efficient schemes have been proposed to model the dynamics of rigid multibody mechanical systems (Richard [24], Mata [25], Mata [26]). The dynamics model can be obtained explicitly through algebraically computation, or numerically through iterative computation.

The techniques based on the Newton-Euler method starts from the dynamics of all individual parts of the system. They look at the instantaneous or infinitesimal aspects of the motion, using vector quantities such as Cartesian velocities and forces.

Alternatively, the Euler-Lagrange based methods starts from the kinetic and potential energy of the entire system, by considering the states of the system during a finite time interval. This approach works with scalar quantities, namely the energies.

Independently of the approach used, at the end of the process the generalized forces are determined:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{c}(\dot{\mathbf{q}}, \mathbf{q}) + \mathbf{g}(\mathbf{q}) = \tau \tag{3}$$

where $\mathbf{q}, \dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are the joint position, velocity and acceleration, respectively. $\mathbf{M}(\mathbf{q})$ is the joint space mass matrix and $\mathbf{c}(\dot{\mathbf{q}}, \mathbf{q})$ is the vector of Coriolis and centrifugal forces. Vector $\mathbf{g}(\mathbf{q})$ is the vector of gravitational forces and τ is the generalized force vector.

The energy that is necessary to move the robot is an important design issue, because in real applications energy supply is limited and any energy reduction leads to smaller operational costs. Due to the relationship that exists between energy and force, the minimal energy can be estimated from the generalized force, $\tau_i(t)$, that is associated to each joint i at the time instant $t_0 \geq t \geq t_f$.

In the present contribution, for a manipulator with n degrees of freedom, the interval between the initial time t_0 and the final time t_f is represented by a set of N points and the computation is numerically carried out through a recursive Newton-Euler formulation (Craig [27]). Then, the mechanical power is used for design purposes as defined by:

$$f_2 = \sum_{j=1}^{n} \int_{t_0}^{t_f} \left[\tau_j(t) \;\; \dot{q}_j(t)\right]^2 dt \tag{4}$$

This expression is representative of the phenomenon under study because it considers both the kinematics and the dynamic aspects of the trajectory, simultaneously (Saramago [1]).

4. Task Specification

The task is specified as a set S of N Cartesian points (x, y, z) and the respective time step. Hence,

$$S = \left\{ (t, x, y, z) \middle| t_0 \leq t \leq t_f, (x, y, z) \in \Re^3 \right\}. \tag{5}$$

In the case of non-redundant manipulators there exist a finite number of configurations that satisfies the end-effector positioning specification. This path-following problem is a highly constrained task, and the inverse kinematics problem will have four solutions or less.

Considering the need of movement continuity together with a time-fixed specification at each step and a relative small step size, the point-to-point path planning strategy may fail due to the complexity of the constraints.

Therefore, a task positioning optimization is proposed as an alternative to increase the robot performance without changing the position of the task reference points with respect to each other. In other words, all the points will be moved simultaneously through a rigid body transformation to find the best position from the point of view of the ability of the manipulator.

To achieve this goal, the use of a homogeneous transformation **A** is proposed, which in this case is defined as

$$\mathbf{A} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 & \delta_x \\ \sin\beta & \cos\beta & 0 & \delta_y \\ 0 & 0 & 1 & \delta_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

where β is a rotation angle in the z axis (given in *rad*) and δ_x, δ_y and δ_z are translations along the reference axes (given in meters), respectively. If desired, it is possible to extend this formulation to consider the rotation with respect to the reference axes.

The physical meaning of considering a rotation angle with respect to x or y is that the robot's base plane will not be parallel to the ground's reference plane. In other words, the robot's z axis (its height) is not normal to the ground's reference plane (xy plane). As positioning two planes according to a precise angle may not be possible in a number of practical situations (even when it is possible, it is not an easy procedure to be done, in general terms), the rotation along the z axis only is considered, maintaining the robot's base plane parallel to the task's base plane.

After defining the set S that describes the task, the inverse kinematics with respect to each Cartesian point is computed.

The inverse kinematics computation can be achieved algebraically, by considering the analytical model of the manipulator and its geometry, or numerically. With the aim of presenting a general procedure, in the present contribution the inverse kinematics problem is solved numerically through an inverse problem formulation. This approach results an efficient and general procedure that works satisfactorily for manipulators exhibiting arbitrary complexity (Santos [28]).

The obtained path planning presents an end-effector positioning error, while the robot executes the path. The interest is focused on errors due to task positioning, as the inverse kinematics can generally be solved with a sufficient precision.

Once the objective is specified, it is necessary to define the optimization domain, *i.e.*, the design space. As the design variables are defined in a theoretical space (defining a rigid body transformation) and the positioning constraints are defined in the Cartesian space (the workspace), a constraint parameter is included in the formulation.

Defining the theoretical positioning error at each point as

$$error_i = \begin{cases} 0 \text{ if } \mid P_{end} - T_0^n P_{base} \mid < \varepsilon \\ \mid P_{end} - T_0^n P_{base} \mid \text{ otherwise} \end{cases} \tag{7}$$

where P_{base}, P_{end} and T_0^n are the robot base point, the end-effector point, and the kinematics representation of the manipulator, respectively. The penalty parameter used as a constraint in the following optimization formulation is given by

$$s_E = \sum_{i=1}^{N} error_i \tag{8}$$

and represents the sum of all end-effector positioning errors during the movement.

Equation (7) means that the end-effector error is considered null when the accuracy is better than the one specified by ε (given in meters). Otherwise, this error corresponds to the distance from the end-effector specified position to the position obtained from the direct kinematics calculation.

In some robotic implementations it is necessary to plan the layout of the workspace, *i.e.*, it is required to locate the robot base in such a way that dexterity is maximized at (or around) given targets. It is worth mentioning that such cases are also covered by the present formulation. Since the optimal design ((β, δ_x, δ_y, δ_z) in Equation (6)) means a coordinate transformation of the end-effector path regarding the fixed robot base, it is also possible to adjust the location of the robot base regarding the fixed end-effector reference through the relation ($-\beta$, $-\delta_x$, $-\delta_y$, $-\delta_z$).

5. Multicriteria Programming

In problems with multiple criteria, one deals with a design variable vector **x**, which satisfies all constraints and makes the scalar performance index that is calculated from the m components of an objective function vector **f**(**x**) as small as possible. This goal can be achieved by the vector optimization problem:

$$\min_{\mathbf{x}\in\Omega}\{\mathbf{f}(\mathbf{x}) \mid \mathbf{h}(\mathbf{x}) = \mathbf{0}, \mathbf{g}(\mathbf{x}) \le \mathbf{0}\} \tag{9}$$

where $\Omega \subset \Re^n$ is the function domain (the design space).

An important feature of such multiple criteria optimization problem is that the optimizer has to deal with objective conflicts (Eschenauer [29]). Other authors discuss the same topic by defining compromise programming, because there is no unique solution to the problem (Vanderplaats [30]).

5.1. The Proposed Formulation

To consider together the manipulability and the total mechanical power, the following scalar objective function is proposed

$$f_3 = k\,\frac{f_1^0}{f_1} + (1-k)\,\frac{f_2}{f_2^0}, \quad 0 \le k \le 1 \tag{10}$$

where f_1 and f_2 are given by Equations (2) and (4), respectively. The reference values f_1^0 and f_2^0 are obtained from the computation of the manipulability and mechanical power sum at the initial task positioning. Therefore, in the ideal case the objective function value is lower than 2, where $f_1^0/f_1 \le 1$ (the manipulability was increased) and $f_2/f_2^0 \le 1$ (the mechanical power was decreased).

The specification of a reference value to both the manipulability measure and the mechanical power are of paramount importance in the construction of the multicriteria objective function.

The inverse kinematics error is included in the problem as an equality constraint in the optimization formulation, through the expression $s_E = 0$ (Equation (8)).

It should be pointed out that the proposed formulation is able to handle the case in which the task is outside the robot workspace. This configuration reflects in the inverse kinematics error, which is taken into account as a constraint (Equation (8)) of the optimization formulation.

Additionally, initial and final velocity specifications are considered as

$$\dot{q}_j(t_0) = 0, \quad j = 1,\dots n. \tag{11}$$

$$\dot{q}_j(t_f) = 0, \quad j = 1,\dots n. \tag{12}$$

for each joint j in the initial and final times t_0 and t_f, respectively. To represent a rest-to-rest motion, zero initial and final velocities are considered above.

6. Optimization Strategy

An important feature of the proposed analysis is the computation of highly nonlinear equations. Despite the good performance of classical nonlinear programming techniques, the global optimal design is hardly reached due to the existence of local minima in the design space.

With the aim of improving the robustness of the optimization process, the use of a global methodology is considered. The tunneling algorithm (Levy [31,32]) is a heuristic methodology designed to find the global minimum of a function. It is composed of a sequence of cycles, each cycle consists of two phases: a minimization phase having the purpose of lowering the current function value, and a tunneling phase that is devoted to finding a new initial point (other than the last minimum found) for the next minimization phase. This algorithm was first introduced by Levy [31], and the name derives from its graphic interpretation.

In summary, the computation evolves according to the following phases:

a) Minimization phase: Given an initial point $\mathbf{q}^0$, the optimization procedure computes a local minimum $\mathbf{q}^*$ of $f(\mathbf{q})$. At the end, it is considered that a local minimum is found.

b) Tunneling phase: Given $\mathbf{q}^*$ found above, since it is a local minimum, there exists at least one $\mathbf{q}^0 \in \Omega$, such that

$$f\left(\mathbf{q}^0\right) \le f\left(\mathbf{q}^*\right), \mathbf{q}^0 \ne \mathbf{q}^* \tag{13}$$

In other words, there exists $\mathbf{q}^0 \in Z = \{\mathbf{q} \in \Omega - \{\mathbf{q}^*\} \mid f(\mathbf{q}) \le f(\mathbf{q}^*)\ \}$. To move from $\mathbf{q}^*$ to $\mathbf{q}^0$ along the tunneling phase, it is defined a new initial point $\mathbf{q} = \mathbf{q}^* + \delta$, $\mathbf{q} \in \Omega$ that is used in the auxiliary function

$$F(\mathbf{q}) = \frac{f(\mathbf{q}) - f(\mathbf{q}^*)}{[(\mathbf{q}-\mathbf{q}^*)^T(\mathbf{q}-\mathbf{q}^*)]^\eta} \tag{14}$$

which has a pole in $\mathbf{q}^*$, for a sufficient large value of η By computing both phases iteratively, the sequence of local minima may lead to the desired global minimum..

Different values for η are suggested for being used in Equation (14) to avoid undesirable points and prevent the search algorithm to fail (Levy [31,32]). In the present case the stop criterion is five consecutive iterations without further improvement in the minimal objective value.

The minimization phase is performed by using a constrained formulation, where the objective function is given by Equation (10) and the constraint function is defined by Equation (8). The velocity specification (Equations (11) and (12)) is achieved through the determination of a suitable interpolation polynomial function using the joint position coordinates.

In Equation (14), during the tunneling phase, only the unconstrained formulation given by Equation (10) is considered. This unconstrained nonlinear programming problem is solved by using the BFGS variable metric method (Luenberger [33], Vanderplaats [30]). The constrained formulation of the minimization phase (Equations (10), (11) and (12)) is solved by using the modified method of the feasible directions (Vanderplaats [34]).

The algorithms are implemented in FORTRAN by using the optimization library DOT (Vanderplaats [35]).

7. Numerical Simulations

The numerical simulations performed to illustrate the presented methodology use two robot models, namely the Scara and the Puma 560 robot manipulators. These two types of serial robot manipulators are classical configurations in the field. The case studies have been proposed due to their complexity as compared to those normally found in the literature. The following results considers only optimal profiles whose sum of the end-effector error is zero (Equation (8)), given ε = 0.001 m (Equation (7)). This means that the inverse kinematics solution is found for all positions with the required precision. Different tolerance values may result from different optimal profiles.

As different values for the weighting factor k lead to different optimization problems (Equation (10)), the value of the global minimum is normalized according to the following equation

$$f_4 = \frac{f_1^0}{f_1} + \frac{f_2}{f_2^0} \tag{15}$$

This performance index enables comparing the improvement of each objective without the weighting effect, and is used in the subsequent analysis.

In summary, the global optimization for each weighting factor (k = 0, 0.1, 0.2, 0.3, ..., 1) is performed according to Equation (10) and the comparison among the optimal designs obtained is performed by using Equation (15). Thus, eleven global optimizations for each task have been carried out. The maximum inverse kinematics error was defined as being lower than ε = 0.001 m in Equation (7).

A Scara manipulator was used in the first and second numerical experiments. The third numerical experiment refers to a Puma 560 manipulator. By considering the ability of describing a complex path, the data for a set of Cartesian specification points (Equation (5)) is determined through a cubic spline interpolation of the parametric coordinates of the desired path. Therefore, all the values are obtained from a suited set of Cartesian points.

The path for each experiment is arbitrarily defined with the aim of exploring the efficiency of the methodology in performing a complex task. All the obtained results consider only configurations for which inverse kinematics errors are lower than ε, *i.e.*, $s_E = 0$ (Equation (8)).

Each value k in Equation (10) defines a new optimization problem. To compare the values given by different optimization formulations, the normalization of the results, despite the weighting factor, is taken into account as established by f_4 (Equation (15)).

7.1. First Experiment

In this experiment, a subset of 7 reference points is defined: (0, 0.4, 0, –0.3), (1, 0.4, –0.1, –0.3), (2, 0.2, –0.1, –0.3), (3, 0.2, –0.2, –0.3), (4, 0.4, –0.25, –0.3), (5, 0.4, –0.2, –0.3) and (6, 0.2, –0.25, –0.3). Comparing these values with Equation (5), the initial task is inside a rectangle defined by $0.2 \le x \le 0.4$, $-0.3 \le y \le -0.1$ and $z = -0.3$. The corresponding traveling time is defined by $0 \le t \le 6$. These data are computed by interpolating cubic splines for t, x, y and z, respectively. The interpolated data encompass a set of N = 27 points (Equation (5)) that describes the path-following problem, as presented in **Figure 1**.

At the original position, the values of the design variables are: β= 0 rad, δ_x = 0 m, δ_y = 0 m, and δ_z = 0 m. These values are used as an initial guess for the optimization process (Equation (10)).

The global minimum values obtained are presented in **Figure 2**.

The optimal values of manipulability and mechanical power indexes demonstrate that there is no clear dependence between the variables. Good results are provided for k = 0.2, 0.5, 0.6, and 0.9.

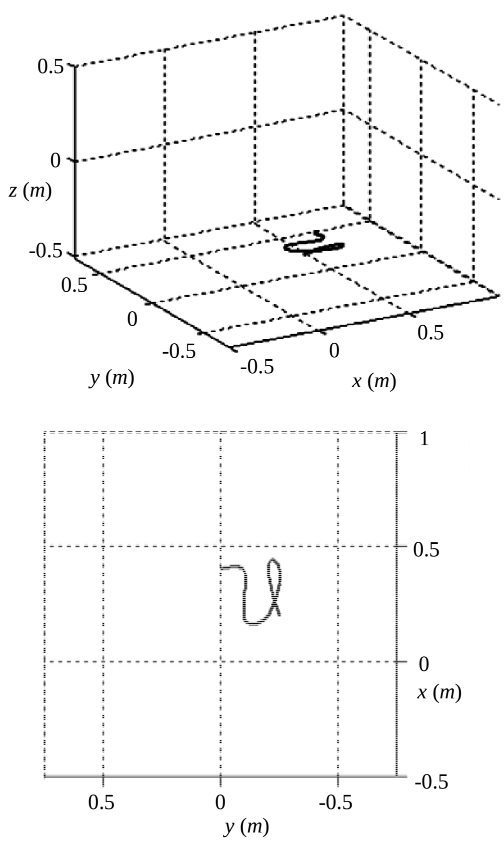


Figure 1. The proposed path-following problem from two different perspectives.

Details about the minimal performance index, $k = 0.9$, are presented in **Table 1**. The overall performance was improved 28%, while the minimal manipulability and the mechanical power were improved 62% and 18%, respectively.

Some of the values provided by other weighting factors may result in further improvement of the manipulability or mechanical power, separately. However, the present configuration is a global minimum of the performance index f_4.

The improvement obtained during the tunneling process is shown in **Figure 3**.

After the 6th iteration of the tunneling process, no further improvement was achieved during the next five consecutive iterations. This behavior defines the stop criteria for the global optimization search. The above result highlights the good performance of the method, as satisfactory results are obtained after a small number of iterations.

Figure 4 shows the robot performing the path following task and **Figure 5** presents the robot optimal positioning, according to the optimal design variable values

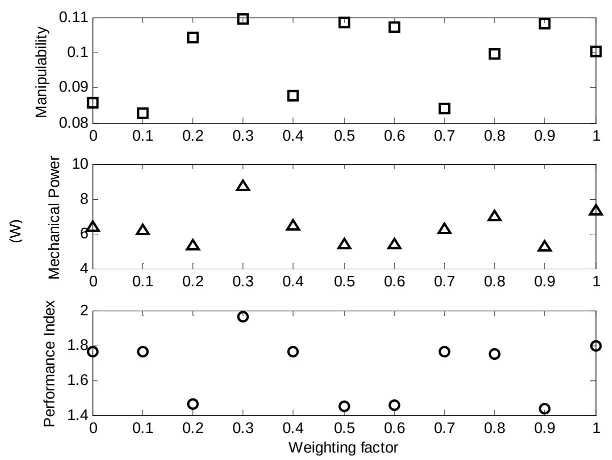


Figure 2. Optimal values obtained by using different weighting factors.

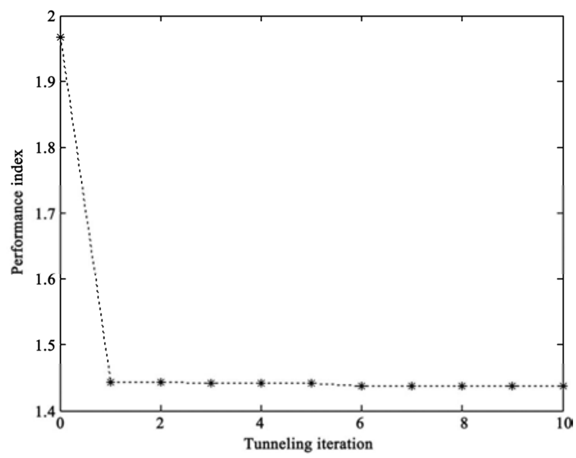


Figure 3. Performance index along the tunneling iterations (for $k = 0.9$).

Table 1. Initial and optimal parameter values (for $k = 0.9$).

Parameter	*Initial value*	*Optimal value*
Minimal manipulability f_1	0.067	0.108
Mechanical power f_2 (*Watts*)	6.451	5.261
Performance index f_4	2.000	1.436
Design variables (β, δ_x, δ_y, δ_z)	(0, 0, 0, 0)	(0.107, -0.013, -0.227, 0.055)

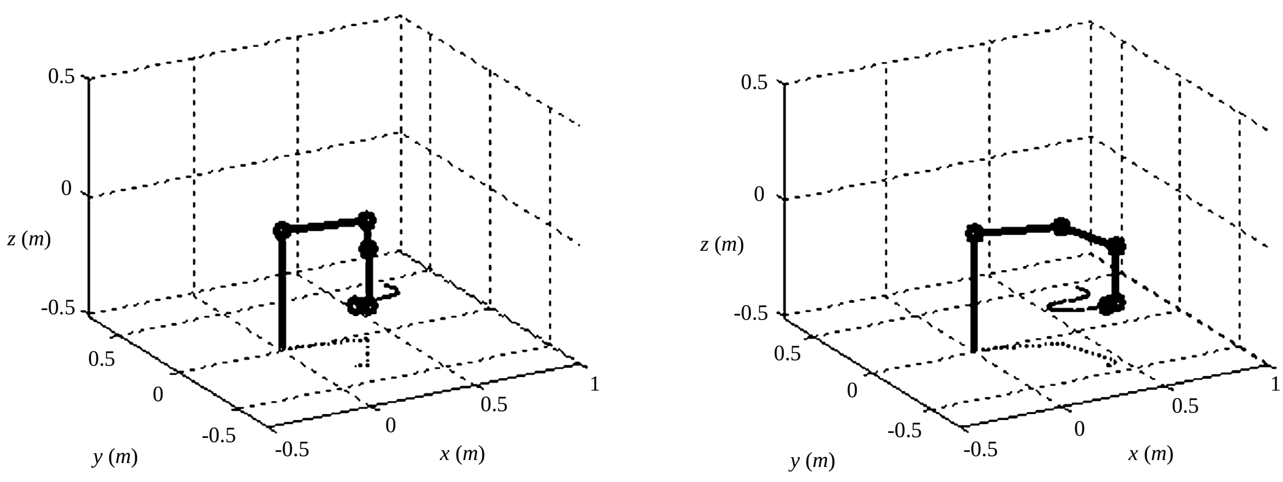


Figure 4. Robot performing the path following task.

given in **Table 1**.

The joint coordinates at each time step are obtained from inverse kinematics computation. Once the task is considered as a set of consecutive Cartesian points with continuous transition between them, the corresponding joint coordinates presents a smooth behavior. **Figure 6** shows the initial and optimal joint coordinates of the first three links and the corresponding manipulability index.

Once the inverse kinematics is iteratively computed for each end-effector positioning, it is possible to include the obstacle avoidance feature in the above strategy. The corresponding procedure for the inverse kinematics computation is addressed in Santos [28], for example.

7.2. Second Experiment

In this experiment a subset of 8 reference points is defined: (0, 0.2, 0, –0.3), (1, 0.4, –0.1, –0.3), (2, 0.4, –0.05, –0.3), (3, 0.2, –0.1, –0.3), (4, 0.25, –0.15, –0.3), (5, 0.3, –0.17, –0.3), (6, 0.27, –0.13, –0.3) and (7, 0.26, –0.2, –0.3). By comparing these values with Equation (10), the initial task is inside a rectangle defined by $0.2 \le x \le 0.4$, $-0.2 \le y \le 0$ and $z = -0.3$. The corresponding traveling time is defined by $0 \le t \le 7$. These data are computed by interpolating cubic splines for t, x, y and z, respectively. The interpolated data encompass a set of $N = 54$ points

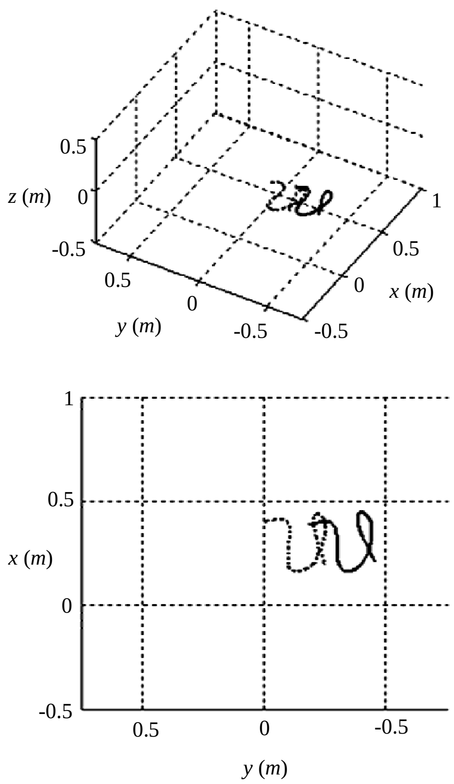


Figure 5. Proposed (---) and optimal (—) paths from two different perspectives.

(Equation (5)) that describes the path-following problem, as presented in **Figure 7**.

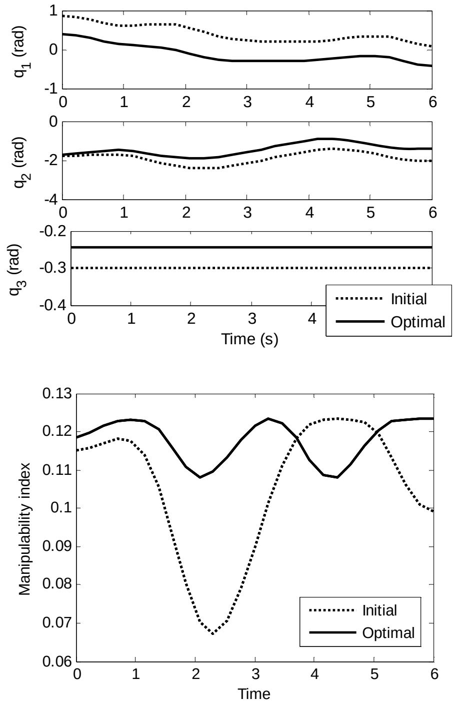


Figure 6. Joint coordinates and the corresponding manipulability index (Scara manipulator).

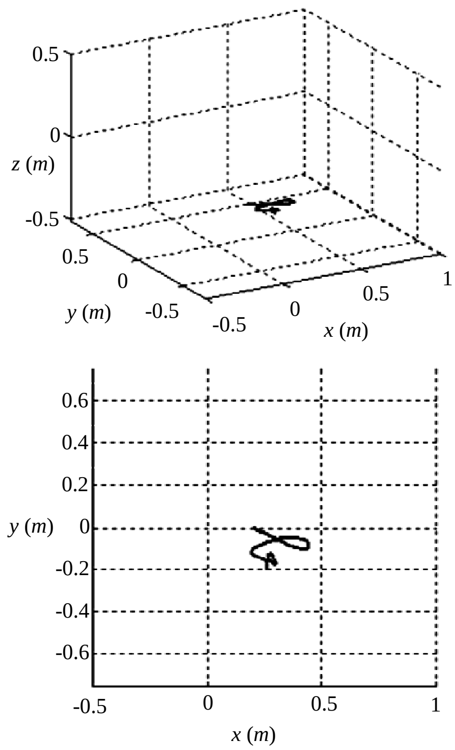


Figure 7. The proposed path-following problem from two different perspectives.

At the original position, the values of the design variables are: $\beta = 0$ *rad*, $\delta_x = 0$ *m*, $\delta_y = 0$ *m*, and $\delta_z = 0$ *m*. These values are used as an initial guess for the optimization process (Equation (10)). Global optimizations were performed for eleven different values of the weighting factor *k*, ($k = 0, 0.1, 0.2, ..., 1$) (see Equation (10)).

The minimum values obtained by the optimization process are presented in **Figure 8**.

In this experiment good results are provided by $k = 0.3$, 0.4, and 0.7. Details about the minimal performance index, $k = 0.7$, are presented in **Table 2**. The overall performance was improved 17%, while the minimal manipulability and the mechanical power were improved 46% and 3%, respectively.

Eventually, the values provided by other weighting factors may result in further improvement either of the manipulability or the mechanical power, separately. However, the above configuration is a global minimum of the performance index f_4.

The design improvement achieved by the tunneling process is shown in **Figure 9**.

After iteration 18 of the tunneling process, no further improvement was achieved for five consecutive iterations, defining the stop criterion of the global optimization search.

Figure 10 shows the robot performing the path-following task and **Figure 11** presents the optimal positioning, according to the optimal design values presented in

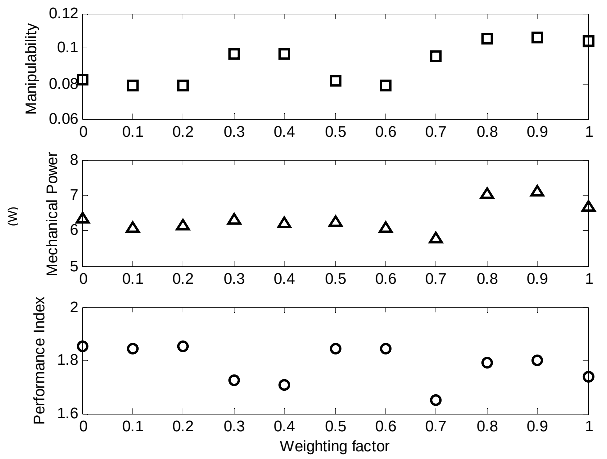


Figure 8. Optimal values obtained by using different weighting factors.

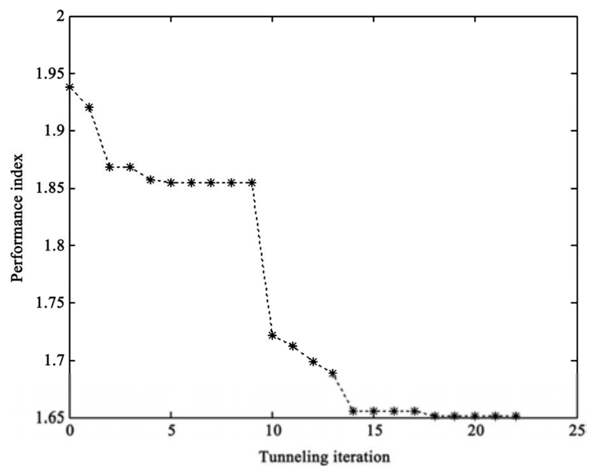


Figure 9. Performance index along the tunneling iterations (for k = 0.7).

Table 2. Initial and optimal parameter values (for k = 0.7).

Parameter	*Initial value*	*Optimal value*
Minimal manipulability f_1	0.065	0.095
Mechanical power f_2 (*Watts*)	5.989	5.773
Performance index f_4	2.000	1.650
Design variables (β, δ_x, δ_y, δ_z)	(0, 0, 0, 0)	(0.044, 0.084, -0.116, 0.066)

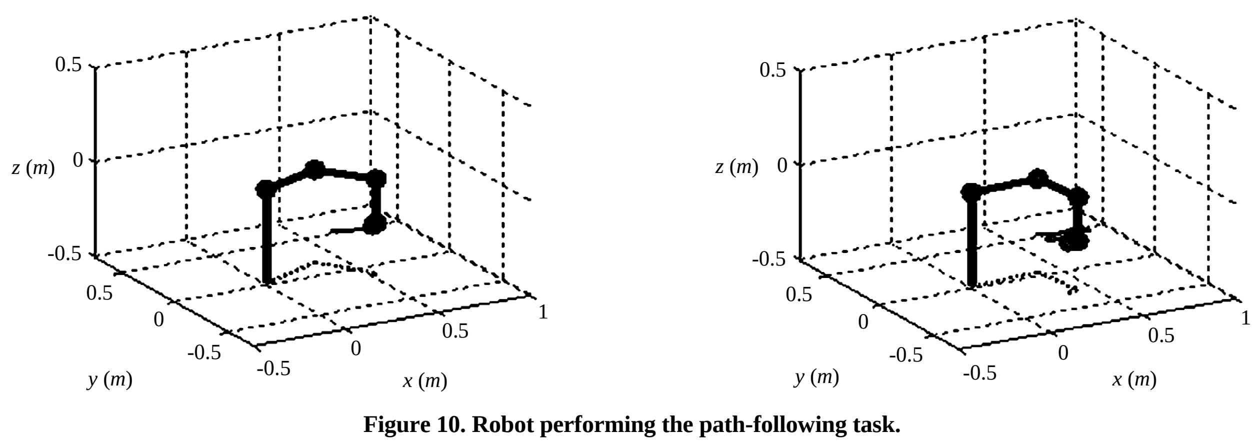


Figure 10. Robot performing the path-following task.

Table 2.

The joint coordinates at each time step are obtained by inverse kinematics computation. As the discrete set of points defining the task (Equation (5)) is supposed to describe a continuous path, the transition between joint coordinates is computed through cubic spline interpolation. **Figure 12** shows the initial and optimal joint coordinates for the first three links and the corresponding manipulability index.

For the examples presented above, the task was defined as being parallel to the *xy* plane, i.e., the path is parallel to the ground.

7.3. Third Experiment

In this experiment, a subset of 7 reference points is defined: (0, 0.4, 0.1, 0), (1, 0.4, 0, 0), (2, 0.4, 0, –0.4), (3, 0.4, –0.3, –0.4), (4, 0.4, –0.4, 0), (5, 0.4, –0.3, 0), and (6, 0.4, –0.4, –0.4). By comparing these values with Equation (5), the initial task is inside a rectangle defined by $x = 0.4$, $-0.4 \le y \le 0.1$, and $-0.4 \le z \le 0$. The corresponding traveling time is defined by $0 \le t \le 6$. These data are computed by interpolating cubic splines for *t*, *x*, *y* and *z*, respectively. The interpolated data encompass a set of $N = 54$ points (Equation (5)) that describe the path-following problem, as presented in **Figure 13**.

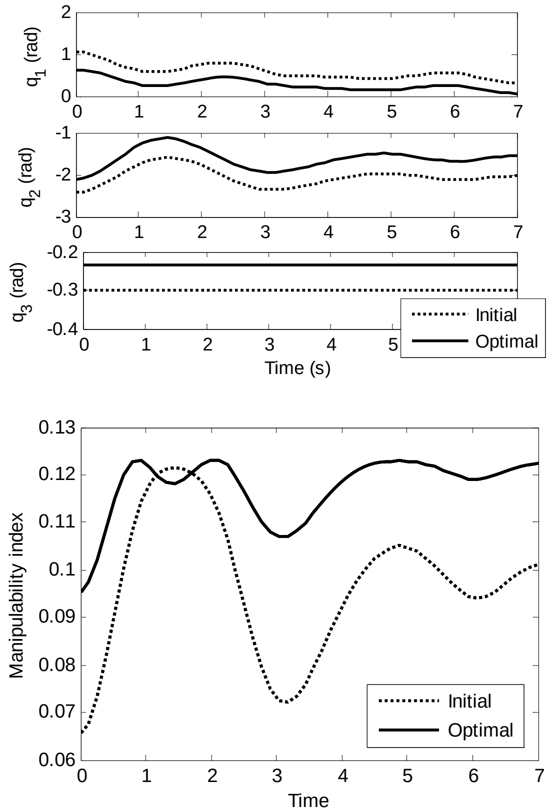


Figure 12. Joint coordinates and the corresponding manipulability index (Scara manipulator).

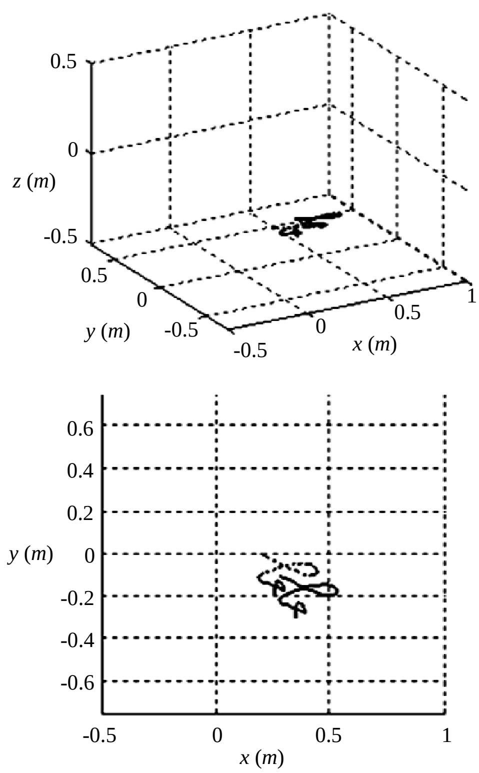


Figure 11. Proposed (---) and optimal (—) paths from two different perspectives.

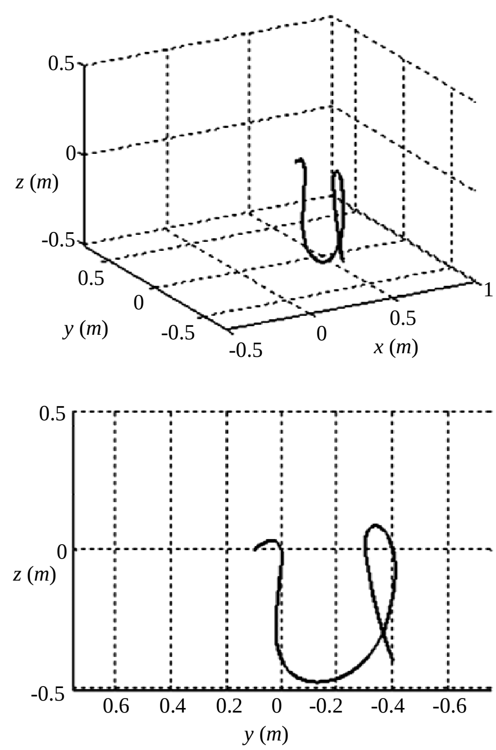


Figure 13. The proposed path-following problem from two different perspectives.

At the original position, the values of the design variables are: $\beta = 0$ *rad*, $\delta_x = 0$ *m*, $\delta_y = 0$ *m*, and $\delta_z = 0$ *m*. These values are used as an initial guess for the optimization process (Equation (10)).

The optimal values of both the manipulability and the mechanical power indexes demonstrate that there is no significant difference among most of the results presented. The best values are provided by $k = 0.4$ and 0.5.

Details about the minimal performance index, $k = 0.5$, are presented in **Table 3**. The overall performance was improved 40%, while the minimal manipulability and the mechanical power were improved 64% and 41%, respectively.

Some of the values obtained by using other weighting factors may result in further improvement of either the manipulability or the mechanical power, separately. However, the above configuration is a global minimum of the performance index f_4.

The performance provided by the tunneling process is shown in **Figure 14**.

After the 5th iteration of the Tunneling process, no further improvement was achieved for the next five consecutive iterations. This behavior defines the stop criteria for the global optimization search. In this case, it is possible to see that good results are obtained in a few iterations.

Figure 15 shows the robot performing the path following task and **Figure 16** presents the optimal positioning, according to the optimal design variable values given in **Table 3**.

Table 3. Initial and optimal parameter values (for *k* = 0.5).

Parameter	*Initial value*	*Optimal value*
Minimal manipulability f_1	0.053	0.087
Mechanical power f_2 (*Watts*)	1461.114	862.289
Performance index f_4	2.000	1.202
Design variables (β, δ_x, δ_y, δ_z)	(0, 0, 0, 0)	(0.391, 0.101, 0.169, 0.418)

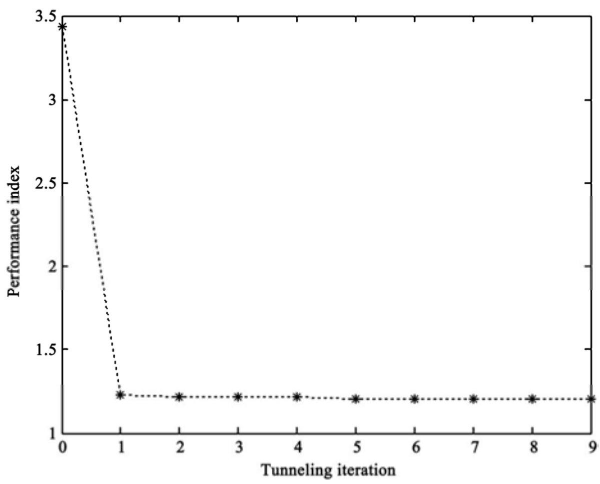


Figure 14. Performance index along the tunneling iterations (for *k* = 0.5).

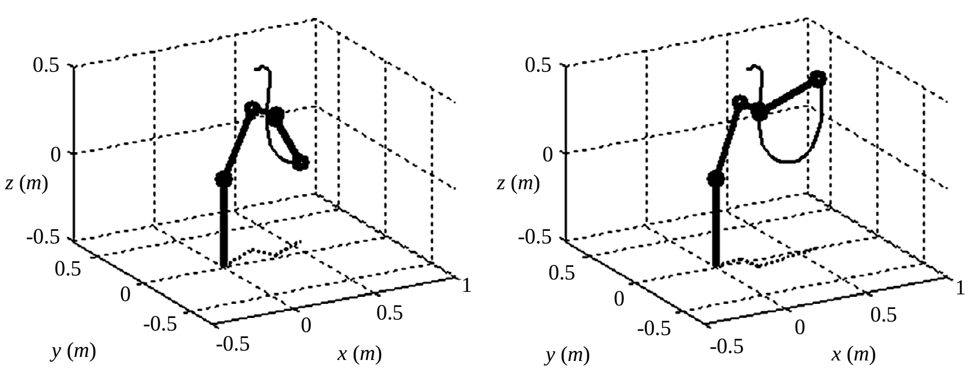


Figure 15. Robot performing the path-following task.

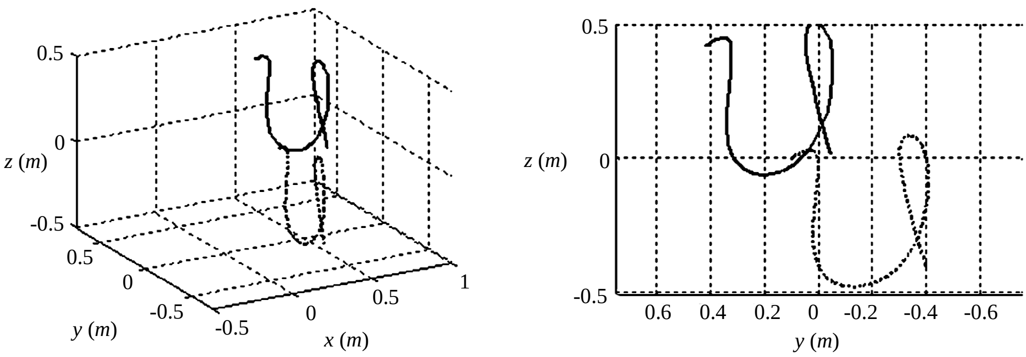


Figure 16. Proposed (---) and optimal (—) paths from two different perspectives.

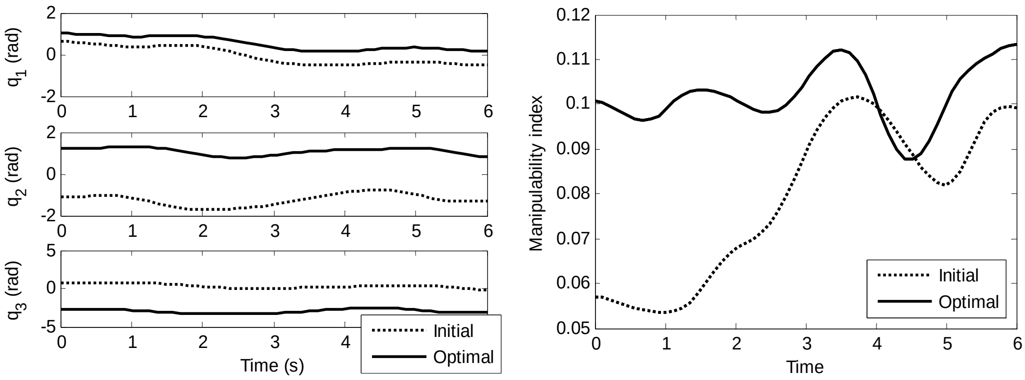


Figure 17. Joint coordinates and the corresponding manipulability index (Puma 560 manipulator).

The joint coordinates at each time step are obtained by inverse kinematics computation. As the task is considered as a set of consecutive Cartesian points with continuous transition, the corresponding joint coordinates presents a smooth behavior. **Figure 17** shows the initial and optimal joint coordinates for the first three links and the corresponding manipulability index.

The previous example considered a task initially defined as being parallel to the *xz* plane, *i.e.*, the path is parallel to a vertical plane. It can be noticed that the optimal design corresponds to rotating the task about 0.391 *rad* anticlockwise with respect to the reference frame.

8. Conclusions

In this work the path-following problem of robot manipulators was addressed. An approach to increase the manipulability while decreasing dynamics requirements was proposed by using optimization techniques.

The concept of manipulability was revisited. The basic idea behind manipulability analysis consists in describing directions either in the task space or the joint space that optimize the ratio between a measure of effort in the joint space and a measure of performance in the task space. In this work a classical formulation to describe the relation between the two spaces (task and joint spaces) was considered.

Since the end-effector presents various performances due to a number of factors (control specification, joint flexibility, joint friction, etc.), a general methodology to reduce high torques and to increase manipulability was proposed. The corresponding equation considers the overall performance of the entire manipulator, as the determinant of the Jacobian is proportional to the product of its singular values, and the mechanical power index is obtained by a sum involving all the joints. The methodology also includes the manipulator positioning error analysis, through the definition of constraint functions in the optimization model.

The serial manipulator usually performs differently in different zones of the workspace. Considering also the cases for which different joint velocity and torque profiles may result different performance levels, a better position into the workspace (in the sense that the performance indexes are optimized) for performing a specified task was determined.

The optimization problem was based in a global search heuristic, improving the robustness of the process with respect to the initial guess.

Numerical experiments showed that the task repositioning successfully increases the manipulability and reduces the total mechanical power, simultaneously. As changes in the task positioning may result different associated manipulability indexes and different joint velocity and torque profiles, the methodology searches for an optimal way of using the robot possibilities (as described by the performance indexes) by finding an optimal position into the workspace to perform the task.

In some practical cases, the manipulated objects cannot be displaced because of physical constraints such as weight or geometry. Even in such cases the proposed strategy is suitable to be applied by changing the position of the manipulator itself. The corresponding changes in the robot's base are given by the optimal profile by using the opposite sign of the optimal values, *i.e.*, ($-\beta$, $-\delta_x$, $-\delta_y$, $-\delta_z$). It is also possible to adjust the analysis to consider different objectives, as the total torque instead of the mechanical power, for example. The steps to be followed remain the same.

In summary, the main contributions of this paper are the simultaneous analysis of both the required mechanical power and the manipulability (dynamics and kinematics aspects), together with a global deterministic heuristic to achieve the optimal task placement.

As the proposed methodology is suitable for serial manipulators of arbitrary complexity, the authors believe that the present methodology represents a contribution concerning general optimal robot path planning and design.

9. Acknowledgements

The authors are thankful to the agencies FAPEMIG and CNPq (INCT-EIE) for partially funding this work.

10. References

[1] S. F. P. Saramago and V. Steffen Jr., "Optimization of the Trajectory Planning of Robot Manipulators Taking Into-Account the Dynamics of the System," *Mechanism and Machine Theory*, Vol. 33, 1998, pp. 883-894.

[2] R. R. D. Santos, V. Steffen Jr. and S. F. P. Saramago, "Robot Path Planning: Avoiding Obstacles," *In: 18th International Congress of Mechanical Engineering*, Ouro Preto-MG, Brazil, 2005.

[3] S. S. Chiddarwar and N. R. Babu, "Offline Decoupled Path Planning Approach for Effective Coordination of Multiple Robots," *Robotica*, Vol. 28, No. 4, 2010, pp. 477-491.

[4] R. R. D. Santos, V. Steffen Jr. and S. F. P. Saramago, "Optimal Path Planning and Task Adjustment for Cooperative Flexible Manipulators," *ABCM Symposium Series in Mechatronics, Associação Brasileira de Engenharia e Ciências Mecânicas, ABCM*, Vol. 3, 2008, pp. 236-245.

[5] O. Von Stryk and M. Schlemmer, "Optimal Control of the Industrial Robot Manutec r3," *Computational Optimal Control, International Series of Numerical Mathematics*, Vol. 115, 1994, pp. 367-382.

[6] R. R. D. Santos, V. Steffen Jr. and S. F. P. Saramago, "Robot Path Planning in a Constrained Workspace by Using Optimal Control Techniques," *In: III European Conference on Computational Mechanics*, Lisbon, Portugal, 2006, pp. 159-177.

[7] J. E. Bobrow, S. Dubowsky and J. S. Gibson, "Time-Optimal Control of Robotic Manipulators along Specified Paths," *The Intemational Journal of Robotics Research*, Vol. 4, 1995, pp. 3-17.

[8] J. E. Slotine and H. S. Yang, "Improving the Efficiency of Time-Optimal Path-Following," *IEEE Transaction on Robotics and Automation*, Vol. 5, No. 1, 1989, pp. 118-124.

[9] D. Constantinescu and E. A. Croft, "Smooth and Time-Optimal Trajectory Planning for Industrial Manipulators along Specified Paths," *Journal of Robotic Systems*, Vol. 17, 2000, pp. 233-249.

[10] H. Seraji, "Reachability Analysis for Base Placement in Mobile Manipulator," *Journal of Robotic Systems*, Vol. 12, 1995, pp. 29-43.

[11] S. Zeghloul and J. A. Pamanes, "Multi-Criteria Optimal Placement of Robots in Constrained Environments," *Robotica*, Vol. 11, No. 2, 1993, pp. 105-110.

[12] J. T. Feddema, "Kinematically Optimal Robot Placement for Minimum Time Coordinated Motion," *Proceedings of the IEEE International Conference of Robotics and Automation*, Minneapolis, 1996, pp. 3395-3400.

[13] F. C. Park, "Distance Metrics on the Rigid-Body Motions with Applications to Mechanicsm Design," *ASME Journal of Mechanical Design*, Vol. 117, 1995, pp. 48-54.

[14] G. S. Chirikjian and S. Zhou, "Metrics on Motion and Deformation of Solid Models," *ASME Journal of Mechanical Design*, Vol. 120, 1998, pp. 252-261.

[15] J. M. R. Martinez and J. Duffy, "On the Metrics of Rigid Body Displacement for Infinite and Finite Bodies," *ASME Journal of Mechanical Design*, Vol. 117, No. 1, 1995, pp. 41-47.

[16] B. Tabarah, B. Benhabib, R. Fenton and R. Cohen, "Cycle-Time Optimization for Single-Arm and Two-Arm Robots Performing Continuous Path Operation," 21*st Biennial Mechanisms Conference*, Chicago IL, 1990, pp. 401-406.

[17] Y. Zhang, and K. Li, "Bi-Criteria Velocity Minimization of Robot Manipulators Using LVI-Based Primal-Dual Neural Network and Illustrated Via PUMA560 Robot Arm," *Robotica*, Vol. 28, No. 4, 2010, pp. 525-537.

[18] K. Abdel-Malek and W. Yu, "On the Placement of Serial Manipulator," *Proceedings of DETC00 2000 ASME Design Engineering Technical Conferences*, Baltimore, MD, 2000, pp. 1-8.

[19] Y. Chen and A. A. Desrochers, "Structure of Minimum-Time Control Law for Robotic Manipulators with Constrained Paths," *IEEE International Conference on*

Robot and Automat, Scottsdale, Az, USA, 1989, pp. 971-976.

[20] Z. Shiller and H. H. Lu, "Computation of Path Constrained Time Optimal Motions with Dynamic Singularities," *ASME Journal of Dynamic Systems, Measurement, and Control*, Vol. 114, 1992, pp. 34-40.

[21] E. K. Xidias, P. T. Zacharia and N. A. Aspragathos, "Time-Optimal Task Scheduling for Articulated Manipulators in Environments Cluttered with Obstacles," *Robotica*, Vol. 28, No. 3, 2010, pp. 427-440.

[22] K. S. Fu, R. C. Gonzales and C. S. G. Lee, "Robo-Tics: Control, Sensing, Vision and Intelligence," McGraw-Hill, New York, 1987.

[23] T. Yoshikawa, "Manipulability of Robot Mechanisms," *The International Jouranl Robotics Research*, Vol. 4, 1985, pp. 3-9.

[24] M. J. Richard and C. M. Gosselin, "A Survey of Simulation Programs for the Analysis of Mechanical Systems," *Mathematics and Computers in Simulation*, Vol. 35, No. 2, 1993, pp. 103-121.

[25] V. Mata, F. Benimeli, N. Farhat and A. Valera, "Dynamic Parameter Identification in Industrial Robots Considering Physical Feasibility," *Journal of Advanced Robotics*, Vol. 19, No. 1, pp. 101-120.

[26] V. Mata, S. Provenzano, F. Valero and J. I. Cuadrado, "Serial Robot Dynamics Algorithms for Moderately Large Numbers of Joints," *Mechanism and Machine Theory*, Vol. 37, No. 8, pp. 739-755.

[27] J. J. Craig, "Introduction to Robotics: Mechanics & Control," 2nd Edition, Reading, MA: Addison-Wesley, 1989.

[28] R. R. D. Santos, V. Steffen Jr. and S. F. P. Saramago, "Solving the Inverse Kinematics Problem through Performance Index Optimization," *In: XXVI Iberian Latin-American Congress on Computational Methods in Engineering*, Guarapari-ES, 1985.

[29] H. Eschenauer, J. Koski and A. Osyczka, "Multicriteria Design Optimization," Springer-Verlag, Berlin, 1990

[30] G. N. Vanderplaats, "Numerical Optimization Techniques for Engineering Design," 3rd Edition, VR & D Inc., 2001.

[31] A. V. Levy and S. Gomez, "The Tunneling Method Applied to Global Optimization. Numerical Optimization," (Ed. P. T. Boggs) R. H. Byrd and R. B. Schnabel, Eds., *Society for Industrial and Applied Mathematics*, 1985, pp. 213-244.

[32] A. V. Levy and A. Montalvo, "The Tunneling Algorithm for the Global Minimization of Functions," *The SIAM Journal on Scientific and Statistical Computing*, Vol. **6**, No. 1, 1985, pp. 15-29.

[33] D. G. Luenberger, "Linear and Nonlinear Programming," 2nd Edition, Addison-Wesley, USA, 1984.

[34] G. N. Vanderplaats, "An Efficient Feasible Direction Algorithm for Design Synthesis," *American Institute of Aeronautics and Astronautics Journal*, Vol. 22, No. 11, 1984, pp. 1633-1640.

[35] G. N. Vanderplaats, "DOT-Design Optimization Tools Program," Vanderplaats Research & Development, Inc., Colorado Springs, 1995.

Intelligent Information Management, **2010, 2, 526-551**
doi:10.4236/iim.2010.29062 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Filters and Ultrafilters as Approximate Solutions in the Attainability Problems with Constraints of Asymptotic Character

Alexander Chentsov
Institute of Mathematics and Mechanics UrB RAS
E-mail: *chentsov@imm.uran.ru*
Received April 1, 2010; *revised June* 2, 2010; *accepted July* 15, 2010

Abstract

Abstract problems about attainability in topological spaces are considered. Some nonsequential version of the Warga approximate solutions is investigated: we use filters and ultrafilters of measurable spaces. Attraction sets are constructed. AMS (MOS) subject classification. 46A, 49 K 40.

Keywords: Control problems, Ultrafilters, Topology

1. Introduction

This investigation is devoted to questions connected with attainability under constraints; these constraints can be perturbed. Under these perturbations, jumps of the attained quality can arise. If perturbation is reduced to a weakening of the initial standard constraints, then we obtain some payoff in a result. Therefore, behavior limiting with respect to the validity of constraints can be very interesting. But, the investigation of possibilities of the above-mentioned behavior is difficult. The corresponding "straight" methods are connected with constructions of asymptotic analysis. Very fruitful approach is connected with the extension of the corresponding problem. For example, in theory of control can be used different variants of generalized controls formalizable in the corresponding class of measures very often. In this connection, we note the known investigations of J. Warga (see [1]). We recall the notions of precise, generalized, and approximate controls (see [1]). In connection with this approach, we recall the investigations of R.V. Gamkrelidze [2]. For problems of impulse control, we note the original approach of N.N. Krasovskii (see [3]) connected with the employment of distributions. If is useful to recall some asymptotic constructions in mathematical programming (see [4,5]). We note remarks in [4,5] connected with the possible employment of nonsequential approximate (in the Warga terminology) solutions-nets.

The above-mentioned (and many other) investigations concern extremal problems. But, very important analogs are known for different quality problems. We recall the fundamental theorem about an alternative in differential games established by N.N. Krasovskii and A.I. Subbotin [6]. In the corresponding constructions, elements of extensions are used very active. Moreover, approximate motions were used. The concrete connection of generalized and approximate elements of the corresponding constructions was realized by the rule of the extremal displacement of N.N. Krasovskii.

In general, the problem of the combination of generalized and approximate elements in problems with constraints is very important. Namely, generalized elements (in particular, generalized controls) can be used for the representation of objects arising by the limit passage in the class of approximate elements (approximate solutions). These limit objects can be consider as attraction elements. Very often these elements suppose a sequential realizetion (see [1]). But, in other cases, attraction ele- ments should be defined by more general procedures.

So, we can consider variants of generalized representtation of asymptotic objects. This approach is developed by J. Warga in theory of control.

Similar problems can arise in distinct sections of mathematics. For example, adherent points of the filter base in topological space can be considered as attraction elements. Of course, here nonsequential variants of the limit passage are required very often.

In the following, the attainability problem with constraints of asymptotic character is considered.

Fix two nonempty sets E and $\mathbf{H}$, and an operator $\mathbf{h}$

from E into **H**. Elements of E are considered as solutions (sometimes controls) and elements of **H** play the role of estimates. We consider **h** as the aim mapping. If we have the set $E_o, E_o \subset E,$ of admissible (in traditional sense) solutions, then $\mathbf{h}^1(E_o) = \{\mathbf{h}(x) : x \in E_o\}$ play the role of an attainability domain in the estimate sp- ace. But, we can use another constraint: instead of E_o, a nonempty family Σ of subsets of E is given. In this case, we can use sequences $(x_i)_{i=1}^{\infty}$ in E with a special property in the capacity of approximate solutions. Nam- ely, we require that the sequence $(x_i)_{i=1}^{\infty}$ has the follo- wing property: for any $\tilde{E}_o \in \Sigma$ the inclusion $x_j \in \tilde{E}_o$ takes place from a certain index (*i.e.* for $j \geq \tilde{j}_o$, where $\tilde{j}_o$ is a fixed index depending on $\tilde{E}_o$). For such solutions we obtain the sequences $\left(\mathbf{h}(x_i)\right)_{i=1}^{\infty}$ in **H**. If **H** is equipped with a topology **t**, then we can consider the limits of such sequences $\left(\mathbf{h}(x_i)\right)_{i=1}^{\infty}$ as attraction elements (AE) in (**H**,**t**) Of course, our AE are "sequential": we use the limit passage in the class of sequences. This approach can be very limiting. The last statement is connected both with our family Σ and with topology **t**. The corresponding examples are known: see [7,8]. In many cases, the more general variants of the limit passage are required. Of course, we can consider nets (x_α) in E and, as a corollary, the corresponding nets $(\mathbf{h}(x_\alpha))$ in **H**. In addition, the basic requirement of admissibility it should be preserved: for any $\tilde{E}_o \in \Sigma$ the inclusion $x_\alpha \in \tilde{E}_o$ is valid starting from a certain index. With the employment of such nets, we can realize new AE; this effect takes place in many examples.

But, the representation of the "totality" of above-mentioned (Σ-admissible) nets as a set is connected with difficulties. Really, any net in the set E is defined by a mapping from a nonempty directed set (DS) $\mathcal{D}$ into E. the concrete choice of $\mathcal{D}$ is arbitrary ($\mathcal{D}$ is a nonempty set). Therefore we have the very large "totality" of nets with the point of view of traditional Zermelo axiomatics. But, this situation can be corrected by the employment of filters of E it is possible to introduce the set of all Σ-admissible filters of the set E In addition, the Σ-admissibility of a filter $\mathbf{F}$ is defined by the requirement $\Sigma \subset \mathbf{F}$. So, we can consider nonseq- uential approximate solutions (analogs of sequential approximate solutions of Warga) as filters $\mathbf{F}$ of E with the property $\Sigma \subset \mathbf{F}$. Moreover, we can be restricted to the employment of only ultrafilters (maximal filters) with the above-mentioned property. In two last cases, we obtain two variants of the set of admissible nonsequential approximate solutions defined in correspondence with Zermelo axiomatics. In our investigation, such point of view is postulated. And what is more, we give the basic attention to the consideration of ultrafilters. Here, the important property of compactness arises. Namely, the corresponding space of ultrafilters is equipped with a compact topology. This permits to consider ultrafilters as generalized elements (GE) too (we keep in mind the above-mentioned classification of Warga).

The basic difficulty is connected with realizability: the existence of free ultrafilters (for which effects of an extension are realized) is established only with the employment of axiom of choice. Roughly speaking, free ultrafilters are "invisible". This property is connected with ultrafilters of the family of all subsets of the corresponding "unit". But, we can to consider ultrafilters of measurable spaces with algebras and semialgebras of sets. We note that some measurable spaces admitting the representation of all such ultrafilters are known (see, for example, [9,§7.6]; in addition, the unessential transformation with the employment of finitely additive (0,1)-measures is used).

2. General Notions and Designations

We use the standard set-theoretical symbolics including quantors and propositional connectives; as usually, $\exists!$ replaces the expression "there exists and unique", $:=$ is the equality by definition. In the following, for any two objects x and y, $\{x;y\}$ is the unordered pair of x and y (see [10]). Then, $\{x\} := \{x;x\}$ is singleton containing an object x. Of course, for any objects x and y the object $(x,y) := \{\{x\};\ \{x;y\}\}$ is the ordered pair of objects x and y; here, we follow to [10]. By $\varnothing$ we denote the empty set. By a family we call a set all elements of which are sets.

By $P(X)$ we denote the family of all subsets of a set X; then, $P'(X) := P(X) \setminus \{\varnothing\}$ are the family of all non- empty subsets of X. Of course, for any set A, in the fo- rm of $P'(P(A))$ and $P'(P'(A))$, we have the family of all nonempty subfamilies of $P(A)$ and $P'(A)$ respectively.

If X is a set, then we denote by $Fin(X)$ the family of all finite sets of $P'(X)$; then $(FIN)[X] := Fin(X) \cup \{\varnothing\}$ is the family of all finite subsets of X.

For any sets A and B, we denote by B^A the set of all mappings from A into B. If A and B are sets, $f \in B^A$, and $C \in P(A)$, then

$$f^1(C) := \{f(x) : x \in C\} \in P(B)$$

(the image of C under the operation f) and $(f|C) \in B^C$ is the usual C-restriction of $f : (f|C)(y) := f(y)$ $\forall y \in C$. In the following, $\mathbb{N} := \{1;2;\ldots\}$ and $\mathbb{R}$ is the real line; $\mathbb{N} \subset \mathbb{R}$ Of course, we use the natural order $\leq$ of $\mathbb{R}$. If $n \in \mathbb{N}$, then

$$\overline{1,n} := \{i \in \mathbb{N} | i \le n\}.$$

Transformations of families. For any nonempty family $\mathbf{A}$ and a set B, we suppose that

$$\mathbf{A}|_B := \{A \cap B : A \in \mathbf{A}\} \in P'(P(B)).$$

If X and Y are sets and $f \in Y^X$, then we suppose that

$$\begin{aligned} &(f^1[\mathbf{X}] := \{f^1(A) : A \in \mathbf{X}\} \\ &\forall \mathbf{X} \in P'(P(X)))\& \\ &\&(f^{-1}[\mathbf{Y}] := \{f^{-1}(B) : B \in \mathbf{Y}\} \\ &\forall \mathbf{Y} \in P'(P(Y))); \end{aligned} \tag{2.1}$$

of course, in (2.1) nonempty families are defined.

If Σ is a family, then we suppose that

$$\{\cup\}(\Sigma) := \left\{\bigcup_{H \in \mathbb{H}} H : \mathbb{H} \in P(\Sigma)\right\}$$

(we keep in mind that $P(\Sigma)$ is a nonempty set $(\varnothing \in P(\Sigma))$ and, for $\tilde{R} \in P(\Sigma)$, $\tilde{R}$ is a family) and

$$\{\cap\}(\Sigma) := \left\{\bigcap_{H \in \mathbb{H}} H : \mathbb{H} \in P'(\Sigma)\right\}$$

(of course, for $\mathbb{H} \in P'(\Sigma)$, $\mathbb{H}$ is a nonempty family); mor- eover

$$\{\cap\}_{\mathbf{f}}(\Sigma) := \left\{\bigcap_{H \in \mathcal{K}} H : \mathcal{K} \in Fin(\Sigma)\right\}.$$

So, for any nonempty family Σ, we obtain that

$$\left(\{\cup\}(\Sigma) \in P'\left(P\left(\bigcup_{E \in \Sigma} E\right)\right) : \Sigma \subset \{\cup\}(\Sigma)\right)\&$$

$$\&\left(\{\cap\}(\Sigma) \in P'\left(P\left(\bigcup_{E \in \Sigma} E\right)\right) : \Sigma \subset \{\cap\}(\Sigma)\right)\&$$

$$\&\left(\{\cap\}_{\mathbf{f}}(\Sigma) \in P'\left(P\left(\bigcup_{E \in \Sigma} E\right)\right) : \Sigma \subset \{\cap\}_{\mathbf{f}}(\Sigma)\right);$$

of course, $\{\cap\}_{\mathbf{f}}(\Sigma) \subset \{\cap\}(\Sigma)$.

Special families. Let I be a set. Then, we suppose that

$$\begin{aligned} \pi[I] := \{&\mathbf{L} \in P'(P(I)) | (\varnothing \in \mathbf{L})\&(I \in \mathbf{L})\& \\ &\&(A \cap B \in \mathbf{L}\ \forall A \in \mathbf{L}\ \forall B \in \mathbf{L})\}; \end{aligned} \tag{2.2}$$

elements of (2.2) are called π-systems with "zero" and "unit". Moreover,

$$\begin{aligned} (\mathrm{LAT})[I] := \{&\mathbf{L} \in P'(P(I)) | (\varnothing \in \mathbf{L})\& \\ &\&(\forall A \in \mathbf{L}\ \forall B \in \mathbf{L}\ (A \cup B \in \mathbf{L})\& \\ &\&(A \cap B \in \mathbf{L}))\}; \end{aligned} \tag{2.3}$$

elements of (2.3) are lattices of subsets of I (with "zero"). Finally,

$$(\mathrm{LAT})_o[I] := \{\mathbf{L} \in (LAT)[I] | I \in \mathbf{L}\} \in P(\pi[I]). \tag{2.4}$$

Of course, in (2.4) lattices of sets with "zero" and "unit" are introduced. We note that

$$\mathbf{L} \cup \{I\} \in (\mathrm{LAT})_o[I]\ \ \forall \mathbf{L} \in (\mathrm{LAT})[I].$$

Of course,

$$\begin{aligned} (top)[I] &:= \left\{\tau \in \pi[I] | \bigcup_{G \in \mathbf{G}} G \in \tau\ \forall \mathbf{G} \in P'(\tau)\right\} \\ &= \left\{\tau \in \pi[I] | \bigcup_{G \in \mathbf{G}} G \in \tau\ \forall \mathbf{G} \in P(\tau)\right\} \end{aligned} \tag{2.5}$$

is the set of all topologies of I. If $\tau \in (top)[I]$, then the pair (I, τ) is a topological space (TS);

$$\begin{aligned} (\mathrm{clos})[I] := \{&F \in P'(P(I)) | (\varnothing \in F)\& \\ &\&(I \in F)\&(A \cup B \in F\ \forall A \in F\ \forall B \in F)\& \\ &\&(\bigcap_{H \in \tilde{H}} H \in F\ \forall \tilde{H} \in P'(F))\}; \end{aligned} \tag{2.6}$$

in (2.6) we have families dual with respect to topologies. It is obvious that

$$\begin{aligned} &((\mathrm{top})[I] \subset (\mathrm{LAT})_o[I])\& \\ &((\mathrm{clos})[I] \subset (\mathrm{LAT})_o[I]). \end{aligned} \tag{2.7}$$

We suppose that $\mathbf{C}_I : P'(P(I)) \to P'(P(I))$ is the mapping for which

$$\mathbf{C}_I(\tilde{H}) := \{I \setminus H : H \in \tilde{H}\}\ \ \forall \tilde{H} \in P'(P(I)). \tag{2.8}$$

From (2.5) – (2.8), we obtain the following propert- ies:

$$\begin{aligned} &(\mathbf{C}_I(\mathbf{C}_I(\mathcal{H})) = \mathcal{H} \forall \mathcal{H} \in P'(P(I)))\& \\ &\&(\mathbf{C}_I(\tau) \in (\mathrm{clos})[I] \forall \tau \in (\mathrm{top})[I])\& \\ &\&(\mathbf{C}_I(\mathcal{F}) \in (\mathrm{top})[I] \forall \mathcal{F} \in (\mathrm{clos})[I]). \end{aligned} \tag{2.9}$$

We note that $P(I) \in (\mathrm{top})[I] \cap (\mathrm{clos})[I]$; in addition,

$$\mathbf{C}_I(P(I)) = P(I).$$

Of course, in (2.9), we have (in particular) the natural duality used in general topology. Let

$$(c-\mathrm{top})[I] := \{\tau \in (\mathrm{top})[I] | \forall \xi \in P'(\tau)$$

$$\left(I = \bigcup_{G \in \xi} G\right) \Rightarrow \left(\exists \mathcal{K} \in \mathrm{Fin}(\xi) : I = \bigcup_{G \in \mathcal{K}} G\right)\}.$$

(the set of all compact topologies of I) Now, we introduce in consideration algebras of sets. Namely,

$$\begin{aligned} (\mathrm{alg})[I] := &\{A \in \pi[I] | I \setminus L \in A\ \forall L \in A\} \\ &\subset (\mathrm{LAT})_o[I]. \end{aligned} \tag{2.10}$$

In connection with (2.10), we note that

$$\{L \in \lambda \,|\, I \setminus L \in \lambda\} \in (\mathrm{alg})[I] \ \forall \lambda \in (\mathrm{LAT})_o[I].$$

If $\mathcal{L} \in (\mathrm{alg})[I]$, then $(I, \mathcal{L})$ is a measurable space with an algebra of sets.

If $\mathcal{L} \in \pi[I], n \in \mathbb{N}$ and $A \in P(I)$, then by $\Delta_n(A, \mathcal{L})$ we denote the set of all mappings

$$(L_i)_{i \in \overline{1,n}} : \overline{1,n} \longrightarrow \mathcal{L}$$

for each of which:

1) $A = \bigcup_{i=1}^{n} L_i$;

2) $L_{i_1} \cap L_{i_2} = \varnothing \forall i_1 \in \overline{1,n} \forall i_2 \in \overline{1,n} \setminus \{i_1\}$.

Then

$$\Pi[I] := \{\lambda \in \pi[I] \,|\, \forall L \in \lambda \ \exists n \in \mathbb{N} : \Delta_n(I \setminus L, \lambda) \neq \varnothing\} \tag{2.11}$$

is the set of all semialgebras of subsets of *I*. Of course,

$$(\mathrm{alg})[I] = \{\lambda \in \Pi[I] | I \setminus L \in \lambda \ \forall L \in \lambda\};$$

see (2.10). If we have a semialgebra of subsets of *I*, then algebra generated by the initial semialgebra is realized very simply: for any $\mathcal{L} \in \Pi[I]$,

$$\mathbf{a}_I^o(\mathcal{L}) := \{A \in P(I) | \exists n \in \mathbb{N} : \Delta_n(A, \mathcal{L}) \neq \varnothing\} \in (\mathrm{alg})[I]$$

has the properties: 1) $\mathcal{L} \subset \mathbf{a}_I^o(\mathcal{L})$; 2) $\forall \mathcal{A} \in (\mathrm{alg})[I]$

$$(\mathcal{L} \subset \mathcal{A}) \Rightarrow (\mathbf{a}_I^o(\mathcal{L}) \subset \mathcal{A}).$$

Now, we introduce some notions important for con- struc- tions of general topology. Namely, we consider topologi- cal bases of two types:

$$(\mathrm{op}-\mathrm{BAS})[I] := \Big\{\beta \in P\big(P(I)\big) \Big| \left(I = \bigcup_{B \in \beta} B\right) \& \big(\forall B_1 \in \beta \ \forall B_2 \in \beta \ \forall x \in B_1 \cap B_2 \ \exists B_3 \in \beta : (x \in B_3) \& (B_3 \subset B_1 \cap B_2)\big)\Big\}, \tag{2.12}$$

$$(\mathrm{cl}-\mathrm{BAS})[I] := \Big\{\beta \in P'(P(I)) | (I \in \beta) \& \left(\bigcap_{B \in \beta} B = \varnothing\right) \& \big(\forall B_1 \in \beta \ \forall B_2 \in \beta \ \forall x \in I \setminus (B_1 \cup B_2) \ \exists B_3 \in \beta : (B_1 \cup B_2 \subset B_3) \& (x \notin B_3)\big)\Big\}. \tag{2.13}$$

Of course,

$$(\mathrm{op}-\mathrm{BAS})[I] = \{\beta \in P(P(I)) | \{\cup\}(\beta) \in (\mathrm{top})[I]\}.$$

In connection with (2.12), we suppose that

$$(\mathrm{op}-\mathrm{BAS})_\varnothing[I] := \{\mathcal{B} \in (\mathrm{op}-\mathrm{BAS})[I] \,|\, \varnothing \in \mathcal{B}\},$$

$$(\mathrm{op}-\mathrm{BAS})_\varnothing[I] \subset P\big(P(I)\big);$$

$$\tilde{\mathcal{B}} \cup \{\varnothing\} \in (\mathrm{op}-\mathrm{BAS})_\varnothing[I] \ \forall \tilde{\mathcal{B}} \in (\mathrm{op}-\mathrm{BAS})[I].$$

Moreover, the following obvious property is valid:

$$\{\cup\}(\mathcal{B}) = \{\cup\}(\mathcal{B} \cup \{\varnothing\}) \ \forall \mathcal{B} \in (\mathrm{op}-\mathrm{BAS})[I].$$

We note the natural connection of open and closed bases:

$$\big(\mathbf{C}_I(\mathcal{B}) \in (\mathrm{op}-\mathrm{BAS})_\varnothing[I] \ \forall \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[I]\big) \& \& \big(\mathbf{C}_I(\beta) \in (\mathrm{cl}-\mathrm{BAS})[I] \ \forall \beta \in (\mathrm{op}-\mathrm{BAS})_\varnothing[I]\big). \tag{2.14}$$

Along with (2.14), we note the following important property:

$$\{\cap\}(\mathcal{B}) \in (\mathrm{clos})[I] \ \forall \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[I]. \tag{2.15}$$

From (2.9) and (2.15), we obtain the obvious state- ment:

$$\mathbf{C}_I\big(\{\cap\}(\mathcal{B})\big) \in (\mathrm{top})[I] \ \forall \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[I]. \tag{2.16}$$

So, closed bases can be used (see (2.16)) for topolo- gies constructing. We note the following obvious property (here we use (2.14) and (2.16)):

$$\mathbf{C}_I\big(\{\cap\}(\mathcal{B})\big) = \{\cup\}\big(\mathbf{C}_I(\mathcal{B})\big) \forall \mathcal{B} \in (\mathrm{cl}-\mathrm{BAS})[I]. \tag{2.17}$$

Of course, in (2.17), we use the usual duality property connected with (2.14) – (2.16).

Some additions. In the following, we suppose that

$$(\mathcal{D}-\mathrm{top})[I] := \{\tau \in (\mathrm{top})[I] | \{x\} \in \mathbf{C}_I[\tau] \forall x \in I\}; \tag{2.18}$$

if $\tau \in (\mathcal{D}-\mathrm{top})[I]$, then TS (I, τ) is called T_1-space. We use (2.18) under investigation of properties of topo- logies on ultrafilter spaces.

Finally, we suppose that $(\mathrm{LAT})^o[I] := \{\lambda \in (\mathrm{LAT})_o[I] | \{x\} \in \lambda \ \forall x \in I\}$. So, we introduce “continuous” lat- tices.

3. Nets and Filters as Approximate Solutions under Constraints of Asymptotic Character

In this section, we fix a nonempty set $\mathbb{E}$ considered (in particular) as the space of usual solutions. We consider families $\Sigma \in P'\big(P(\mathbb{E})\big)$ as constraints of asymptotic character. Of course, in this case, we use asymptotic version of solutions. The simplest variant is realized by the employment of sequences in $\mathbb{E}$: in the set $\mathbb{E}^{\mathbb{N}}$ the set of Σ -admissible sequences (see Section 1) is selected. It is logical to generalize this approach: we keep in mind the employment of nets. Later, we introduce so- me definitions connected with the Moore-Smith convergence. But, before we consider the filter convergence.

We denote by $\beta[\mathbb{E}]$ (by $\beta_o[\mathbb{E}]$) the set of all families $B \in P^{'}\left(P(\mathbb{E})\right)$ (families $B \in P^{'}\left(P^{'}(\mathbb{E})\right)$) for which

$$\forall B_1 \in \mathcal{B}\ \forall B_2 \in \mathcal{B}\ \exists B_3 \in \mathcal{B} : B_3 \subset B_1 \cap B_2;$$

$\beta_o[\mathbb{E}] \subset \beta[\mathbb{E}]$. Then, $\beta_o[\mathbb{E}]$ is the set of all filter bases on $\mathbb{E}$. By $\hat{F}[\mathbb{E}]$ we denote the set of all filters on $\mathbb{E}$:

$$\widehat{F}[\mathbb{E}] := \Big\{ \widetilde{F} \in P^{'}\left(P^{'}(\mathbb{E})\right) | \Big(A \cap B \in \widetilde{F}\ \forall A \in \widetilde{F}\ \forall B \in \widetilde{F} \Big) \&\ \&, \Big(\left\{ H \in P(\mathbb{E}) | F \subset H \right\} \subset \widetilde{F}\ \ \forall F \in \widetilde{F} \Big) \Big\}. \tag{3.1}$$

Using (3.1), we introduce the set $\widehat{F}_{\mathrm{u}}[\mathbb{E}]$ of all ultrafilters on $\mathbb{E}$:

$$\widehat{F}_{\mathrm{u}}[\mathbb{E}] := \Big\{ \mathcal{U} \in \widehat{F}[\mathbb{E}] | \forall \mathcal{F} \in \widehat{F}[E]\ \big((\mathcal{U} \subset \mathcal{F}) \Rightarrow (\mathcal{U} = \mathcal{F}) \big) \Big\}. \tag{3.2}$$

In connection with (3.1) and (3.2), see in particular [11, ch. I]. In addition,

$$(\mathbb{E} - \mathbf{fi})[\zeta] := \left\{ H \in P(\mathbb{E}) | \exists B \in \zeta : B \subset H \right\} \in \widehat{F}[\mathbb{E}]\ \ \forall \zeta \in \beta_o[\mathbb{E}]. \tag{3.3}$$

By (3.3) we define the filter on $\mathbb{E}$ generated by a base of $\beta_o[\mathbb{E}]$.

If $\Sigma \in P^{'}\left(P(\mathbb{E})\right)$, then by $\widehat{F_0}[\mathbb{E}|\Sigma]$ (by $\widehat{F}^{o}_{\mathbf{u}}[\mathbb{E}|\Sigma]$) we denote the set of all filters $\mathcal{F} \in \widehat{F}[\mathbb{E}]$ (ultrafilters $\mathcal{F} \in \widehat{F}_{\mathbf{u}}[\mathbb{E}]$) such that $\Sigma \subset \mathcal{F}$. Then, for any filter $\mathcal{F}_* \in \widehat{F}[\mathbb{E}]$, we have $\widehat{F}^{o}_{\mathbf{u}}[\mathbb{E}|\mathcal{F}_*] \in P^{'}(\widehat{F}_{\mathbf{u}}[E])$ and what is more $\mathcal{F}_*$ is the intersection of all ultrafilters $\mathcal{U} \in \widehat{F}^{o}_{\mathbf{u}}[\mathbb{E}|\mathcal{F}_*]$; see [11].

If a family $\Sigma \in P^{'}\left(P(E)\right)$ is considered as the constra int of asymptotic character, then ultrafilters $\mathcal{U} \in \widehat{F}^{o}_{\mathbf{u}}[E|\Sigma]$ are considered as (nonsequential) approximate solutions; of course, filters $\mathcal{F} \in \widehat{F}_o[\mathbb{E}|\Sigma]$ can be considered in this capacity also. But, ultrafilters have better properties; therefore, now we are restricted to employment of ultrafilters as approximate solutions.

The filter of neighborhoods. If $\tau \in (\mathrm{top})[\mathbb{E}]$ and $x \in \mathbb{E}$, then

$$N^{o}_{\tau}(x) := \left\{ G \in \tau | x \in G \right\} \in \beta_o[\mathbb{E}]$$

and $N_\tau(x) := (\mathbb{E} - \mathbf{fi})[N^{o}_{\tau}(x)]$; of course, $N_\tau(x) \in \widehat{F}[\mathbb{E}]$ in correspondence with (3.3). We were introduce the filter of neighborhoods of x in the sense of [11,ch.I]. In the following,

$$\mathrm{cl}(A,\tau) := \left\{ x \in \mathbb{E} | A \cap H \neq \varnothing\ \ \forall H \in N_\tau(x) \right\}\ \ \forall \tau \in (\mathrm{top})[\mathbb{E}]\ \ \forall A \in P(\mathbb{E}).$$

So, we introduce the closure operation in a TS. Moreover, we suppose that

$$(x - \mathrm{bas})[\tau] := \{ \beta \in P\left(N_\tau(x)\right) | \ \forall A \in N_\tau(x)\ \ \exists B \in \beta : B \subset A \}\ \ \forall \tau \in (\mathrm{top})[\mathbb{E}]\ \ \forall x \in \mathbb{E}. \tag{3.4}$$

The filter convergence. We follow to [11]. Suppose that $\forall \tau \in (\mathrm{top})[\mathbb{E}]\ \forall \mathcal{B} \in \beta_o[\mathbb{E}]\ \forall x \in \mathbb{E}$

$$(\mathcal{B} \overset{\tau}{\Longrightarrow} x) \overset{def}{\Longleftrightarrow} \left(N_\tau(x) \subset (\mathbb{E} - \mathbf{fi})[\mathcal{B}] \right). \tag{3.5}$$

In addition, $\widehat{F}[\mathbb{E}] \subset \beta_o[\mathbb{E}]$; see (3.1). Therefore, we can use (3.5) in the case of $\mathcal{B} = \mathcal{F}$, where $\mathcal{F} \in \widehat{F}[\mathbb{E}]$; we note that $(\mathbb{E} - \mathbf{fi})[\mathcal{F}] = \mathcal{F}$. Then, by (3.5) $\forall \tau \in (\mathrm{top})[\mathbb{E}]\ \forall \mathcal{F} \in \widehat{F}[\mathbb{E}]\ \forall x \in E$

$$(\mathcal{F} \overset{\tau}{\Longrightarrow} x) \Longleftrightarrow \left(N_\tau(x) \subset \mathcal{F} \right). \tag{3.6}$$

Of course, it is possible to use the variant of (3.6) corresponding to the case $\mathcal{F} = \mathcal{U}$, where $\mathcal{U} \in \widehat{F}_{\mathrm{u}}[E]$.

Nets and the Moore-Smith convergence. On the basis of (3.6), we can to introduce the standard Moore-Smith convergence of nets. We call a net in the set $\mathbb{E}$ arbitrary triplet $(D,\preceq,f)$, where $(D,\preceq)$ is a nonempty DS and $f \in \mathbb{E}^D$. If $(D,\preceq,f)$ is a net in the set $\mathbb{E}$ then

$$(\mathbb{E} - \mathrm{ass})[D;\preceq;f] := \{ V \in P(\mathbb{E}) |\ \exists d \in D\ \ \forall \delta \in D\ \ \big((d \preceq \delta) \Rightarrow (f(\delta) \in V) \big) \} \in \widehat{F}[\mathbb{E}]; \tag{3.7}$$

we obtain the filter of $\mathbb{E}$ associated with $(D,\preceq,f)$. Now, for any topology $\tau \in (\mathrm{top})[\mathbb{E}]$, a net $(D,\preceq,f)$ in the set $\mathbb{E}$ and $x \in \mathbb{E}$, we suppose that

$$\Big((D,\preceq,f) \overset{\tau}{\longrightarrow} x \Big) \overset{def}{\Longleftrightarrow} \Big((\mathbb{E} - \mathrm{ass})[D;\preceq;f] \overset{\tau}{\Rightarrow} x \Big). \tag{3.8}$$

From (3.6) and (3.7), we obtain that (3.8) is the "us- ual" Moore-Smith convergence (see [12]). Of course, any sequence $\mathbf{x} := (x_i)_{i \in \mathbb{N}} \in \mathbb{E}^{\mathbb{N}}$ generates the net $(\mathbb{N},\leq,\mathbf{x})$, where $\leq$ is the usual order of $\mathbb{N}$.

If $\mathcal{E} \in P^{'}\left(P(E)\right)$, then a net $(D,\preceq,f)$ in E is called Σ -admissible if $\mathcal{E} \subset (\mathbb{E} - \mathrm{ass})[D;\preceq;f]$. In this case, Σ can be considered as a constraint of asymptotic character and $(D,\preceq,f)$ plays the role of nonsequential (generally speaking) approximate solution.

In conclusion, we note that

$$(\mathbb{E}-\text{ult})[x]:=\{H\in P(\mathbb{E})\,|\,x\in H\}\in \widehat{F}_{\mathbf{u}}[\mathbb{E}]\ \forall x\in\mathbb{E}. \quad (3.9)$$

In (3.9), trivial ultrafilters are defined.

4. Attraction Sets

In this section, we construct nonsequential (generally speaking) attraction sets (AS) using different variants of the representation of approximate solutions. Since nets are similar to sequences very essential, we begin our consideration with the representation (of AS) using nets.

For brevity, in this section, we fix following two nonempty sets: X and Y. In addition, under $f\in Y^X$ and $\mathcal{B}\in\beta_o[X]$

$$f^1[\mathcal{B}]\in\beta_o[Y]; \quad (4.1)$$

of course, in (4.1), we can use a filter or ultrafilter instead of $\mathcal{B}$. In addition, the important property takes place: if $f\in Y^X$ and $\mathcal{B}\in\beta_o[X]$, then

$$\big((X-\mathbf{fi})[\mathcal{B}]\in\widehat{F}_{\mathbf{u}}[X]\big)\Rightarrow\big((Y-\mathbf{fi})\big[f^1[\mathcal{B}]\big]\in\widehat{F}_{\mathbf{u}}[Y]\big). \quad (4.2)$$

So, by (4.2) image of an ultrafilter base is an ultrafilter base. Of course, the image of an ultrafilter is an ultrafilter base also.

Introduce AS: if $f\in Y^X,\tau\in(\text{top})[Y]$ and $\widetilde{X}\in P'\big(P(X)\big)$, then by $(\mathbf{as})[X;Y;\tau;f;\widetilde{X}]$ we denote the set of all $y\in Y$ for each of which there exists a net $(D,\preceq,g)$ in the set X such that

$$\big(\widetilde{X}\subset(X-\text{ass})[D;\preceq;g]\big)\,\&\,\Big((D,\preceq,f\circ g)\xrightarrow{\tau}y\Big); \quad (4.3)$$

we consider $(\mathbf{as})[X;Y;\tau;f;X]$ as AS. In this definition, we use nets. But, for any filter $\mathcal{F}\in\widehat{F}[X]$ there exists a net $(D,\preceq,g)$ in the set X for which $\mathcal{F}=(X-\text{ass})[D;\preceq;g]$ (see [13]).

Proposition 4.1. For any $f\in Y^X,\tau\in(\text{top})[Y]$ and $\widetilde{X}\in P'\big(P(X)\big)$

$$(\mathbf{as})[X;Y;\tau;f;\widetilde{X}]=\Big\{y\in Y\,|\,\exists\mathcal{F}\in\widehat{F}_o[X\,|\,\widetilde{X}]:f^1[\mathcal{F}]\overset{\tau}{\Longrightarrow}y\Big\}. \quad (4.4)$$

Proof. Fix $f\in Y^X,\tau\in(\text{top})[Y]$, and $\widetilde{X}\in P'\big(P(X)\big)$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are the sets on the left and right sides of (4.4) respectively. Let $y^*\in\mathbf{A}$. Then $y^*\in Y$ and, for a net $(D,\preceq,g)$ in X, the relation (4.3) is valid under $y=y^*$. Then, by (4.3)

$$\mathcal{F}^*:=(X-\text{ass})[D;\preceq;g]\in\widehat{F}_o[X\,|\,\widetilde{X}]. \quad (4.5)$$

Moreover, by (3.8) and (4.3) $(Y-\text{ass})[D;\preceq;f\circ g]\overset{\tau}{\Rightarrow}y^*$ So, by (3.6)

$$N_\tau(y^*)\subset(Y-\text{ass})[D;\preceq;f\circ g]. \quad (4.6)$$

Let $H^*\in N_\tau(y^*)$. Then by (3.7) and (4.6), for some $d^*\in D$, the following property is valid: $\forall d\in D$

$$(d^*\preceq d)\Rightarrow\big(f\big(g(d)\big)\in H^*\big). \quad (4.7)$$

In addition, $D^*:=\big\{d\in D\,|\,d^*\preceq d\big\}\in P'(D)$ and by (3.7) and (4.5)

$$g^1(D^*)\in\mathcal{F}^*.$$

As a corollary, $f^1\big(g^1(D^*)\big)=(f\circ g)^1(D^*)\in f^1[\mathcal{F}^*]$ But, by (4.7) $f^1\big(g^1(D^*)\big)\subset H^*$. By (3.3) $H^*\in(Y-\mathbf{fi})\big[f^1[\mathcal{F}^*]\big]$. Since the choice of H^* was arbitrary, the inclusion $N_\tau(y^*)\subset(Y-\mathbf{fi})\big[f^1[\mathcal{F}^*]\big]$ is established. By (3.5)

$$f^1[\mathcal{F}^*]\overset{\tau}{\Rightarrow}y^*. \quad (4.8)$$

By (4.5) and (4.8) $y^*\in\mathbf{B}$. The inclusion $\mathbf{A}\subset\mathbf{B}$ is established.

Let $y^o\in\mathbf{B}$. Then, for $y^o\in Y$, we have a filter $\mathcal{F}^o\in\widehat{F}_o[X\,|\,\widetilde{X}]$ such that

$$f^1[\mathcal{F}^o]\overset{\tau}{\Rightarrow}y^o. \quad (4.9)$$

Choose a net $(\mathbb{D},\sqsubseteq,\varphi)$ in X for which $\mathcal{F}^o=(X-\text{ass})[\mathbb{D};\sqsubseteq;\varphi]$. By (4.1) $f^1[\mathcal{F}^o]\in\beta_o[Y]$ and, as a corollary, by (3.5) and (4.9)

$$N_\tau(y^o)\subset(Y-\mathbf{fi})\big[f^1[\mathcal{F}^o]\big]. \quad (4.10)$$

Then by (3.3) and (4.10) we obtain that $\forall H\in N_\tau(y^o)\ \exists B\in f^1[\mathcal{F}^o]:B\subset H$. Using (2.1) we have the property: $\forall H\in N_\tau(y^o)\ \exists F\in\mathcal{F}^o:f^1(F)\subset H$. Choose arbitrary $H^o\in N_\tau(y^o)$; then, for some $\widetilde{F}^o\in\mathcal{F}^o$ the inclusion $f^1(\widetilde{F}^o)\subset H^o$ is valid. By (3.7) and the choice of $(\mathbb{D},\sqsubseteq,\varphi)$, for some $d^o\in\mathbb{D}$, the following property is realized: $\forall\delta\in\mathbb{D}$

$$(d^o\sqsubseteq\delta)\Rightarrow(\varphi(\delta)\in\widetilde{F}^o).$$

By the choice of $\widetilde{F}^o$ we obtain that $\forall\delta\in\mathbb{D}$

$$(d^o\sqsubseteq\delta)\Rightarrow((f\circ\varphi)(\delta)\in H^o).$$

Then, $H^o\in(Y-\text{ass})[\mathbb{D};\sqsubseteq;f\circ\varphi]$. So, the important inclusion

$$N_\tau(y^o)\subset(Y-\text{ass})[\mathbb{D};\sqsubseteq;f\circ\varphi]$$

is valid. Then $(Y-\text{ass})[\mathbb{D};\sqsubseteq;f\circ\varphi]\overset{\tau}{\Rightarrow}y^o$ (see (3.6)). By (3.8)

$$(\mathbb{D},\sqsubseteq,f\circ\varphi)\xrightarrow{\tau}y^o. \quad (4.11)$$

Moreover, by the choice of $\mathcal{F}^o$ and $(\mathbb{D},\sqsubseteq,\varphi)$ the inclusion

$$\widetilde{X} \subset (X-\text{ass})[\mathbb{D};\sqsubseteq;\varphi]$$

is valid. From (4.11), we have the inclusion $y^o \in \mathbf{A}$. So, $\mathbf{B} \subset \mathbf{A}$ and, as a corollary, $\mathbf{A} = \mathbf{B}$.

Proposition 4.2. For any $f \in Y^X, \tau \in (\text{top})[Y]$, and $\widetilde{X} \in P'(P(X))$

$$(\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = \{y \in Y \mid \exists \mathcal{U} \in \widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}]:\ f^1[\mathcal{U}] \overset{\tau}{\Rightarrow} y\}. \quad (4.12)$$

Proof. We denote respectively by $\mathbf{F}$ an $\mathbf{U}$ the sets on the left and right sides of (4.12). Since $\widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}] \subset \widehat{F}_o[X \mid \widetilde{X}]$, we have the obvious inclusion $\mathbf{U} \subset \mathbf{F}$ (see Proposition 4.1). Let $y_o \in \mathbf{F}$. Then by Proposition 4.1 $f^1[\mathcal{F}] \overset{\tau}{\Rightarrow} y_o$ for some $\mathcal{F} \in \widehat{F}_o[X \mid \widetilde{X}]$. Then $\mathcal{F} \in \widehat{F}[X]$ and $\widetilde{X} \subset \mathcal{F}$. We recall (see Section 3) that $\widehat{F}_{\mathbf{u}}^o[X \mid \mathcal{F}] \in P'(\widehat{F}_{\mathbf{u}}[X])$. Choose arbitrary $\mathfrak{U} \in \widehat{F}_{\mathbf{u}}^o[X \mid \mathcal{F}]$. Then $\mathfrak{U} \in \widehat{F}_{\mathbf{u}}[X]$ and $\widetilde{X} \subset \mathcal{F} \subset \mathfrak{U}$. Therefore, $\mathfrak{U} \in \widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}]$. Moreover, by (2.1) $f^1[\mathcal{F}] \subset f^1[\mathfrak{U}]$ and, as a corollary,

$$(Y-\mathbf{fi})[f^1[\mathcal{F}]] \subset (Y-\mathbf{fi})[f^1[\mathfrak{U}]] \quad (4.13)$$

(we recall that by (4.1) $f^1[\mathcal{F}] \in \beta_o[Y]$ and $f^1[\mathfrak{U}] \in \beta_o[Y]$). By the choice of F we have the inclusion

$$N_\tau(y_o) \subset (Y-\mathbf{fi})\left[f^1[\mathcal{F}]\right]$$

(see (3.5)). Then by (4.13) $N_\tau(y_o) \subset (Y-\mathbf{fi})\left[f^1[\mathfrak{U}]\right]$ and, as a corollary (see (3.5)),

$$f^1[\mathfrak{U}] \overset{\tau}{\Rightarrow} y_o.$$

Then, $y_o \in \mathbf{U}$. The inclusion $\mathbf{F} \subset \mathbf{U}$ is established.

Recall that, for any family $\widetilde{X} \in P'(P(X))$, $\{\cap\}_{\mathbf{f}}(\widetilde{X}) \in P'(P(X))$ and $\widetilde{X} \subset \{\cap\}_{\mathbf{f}}(\widetilde{X})$. We note the following obvious.

Proposition 4.3. For any $\widetilde{X} \in P'(P(X))$, the equality $\widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}] = \widehat{F}_{\mathbf{u}}^o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$ is valid.

Proof. Recall that $\widetilde{X} \subset \{\cap\}_{\mathbf{f}}(\widetilde{X})$. Therefore, $\widehat{F}_{\mathbf{u}}^o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})] \subset \widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}]$; on the other hand, from (3.1), we obtain that $\mathcal{F} = \{\cap\}_{\mathbf{f}}(\mathcal{F})\ \forall \mathcal{F} \in \widehat{F}[X]$. Then, for an ultrafilter $\mathcal{U} \in \widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}]$,

$$\{\cap\}_{\mathbf{f}}(\widetilde{X}) \subset \{\cap\}_{\mathbf{f}}(\mathcal{U}) = \mathcal{U}$$

and, as a corollary, $\mathcal{U} \in \widehat{F}_{\mathbf{u}}^o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$. So, since the choice of $\mathcal{U}$ was arbitrary, $\widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}] \subset \widehat{F}_{\mathbf{u}}^o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$ and, as a corollary, $\widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}] = \widehat{F}_{\mathbf{u}}^o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$.

Corollary 4.1. If $f \in Y^X, \tau \in (\text{top})[Y]$, and $\widetilde{X} \in P'(P(X))$, then

$$(\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = (\mathbf{as})[X;Y;\tau;f;\{\cap\}_{\mathbf{f}}(\widetilde{X})].$$

The corresponding proof is realized by the immediate combination of Propositions 4.2 and 4.3. We note that, by definitions of Section 2

$$\{\cap\}_{\mathbf{f}}(\widetilde{X}) \in \beta[X]\ \forall \widetilde{X} \in P'(P(X)). \quad (4.14)$$

In connection with (4.14), we note the following general property. Namely, $\forall f \in Y^X\ \forall \tau \in (\text{top})[Y]\ \forall \mathcal{B} \in \beta[X]$

$$(\mathbf{as})[X;Y;\tau;f;\mathcal{B}] = \bigcap_{B' \in \mathcal{B}} \text{cl}(f^1(\text{B}),\tau). \quad (4.15)$$

Then, by (4.14), (4.15), and Corollary 4.1

$$(\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = \bigcap_{B \in \{\cap\}_{\mathbf{f}}(\widetilde{X})} \text{cl}\left(f^1(B),\tau\right)$$
$$\forall f \in Y^X\ \forall \tau \in (\text{top})[Y]\ \forall \widetilde{X} \in P'(P(X)). \quad (4.16)$$

In connection with (4.16), we note that $\forall \widetilde{X} \in P'(P(X))$

$$\left(\varnothing \in \{\cap\}_{\mathbf{f}}(\widetilde{X})\right) \Rightarrow \left(\widehat{F}_{\mathbf{u}}^o[X \mid \widetilde{X}] = \varnothing\right). \quad (4.17)$$

Remark 4.1. By analogy with Proposition 4.3 we have that

$$\widehat{F}_o[X \mid \widetilde{X}] = \widehat{F}_0\left[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})\right]\ \forall \widetilde{X} \in P'(P(X)).$$

Really, fix $\widetilde{X} \in P'(P(X))$. Then $\widetilde{X} \subset \{\cap\}_{\mathbf{f}}(\widetilde{X})$. Therefore, $\widehat{F}_o\left[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})\right] \subset \widehat{F}_o[X \mid \widetilde{X}]$. Let $\widehat{F} \in \widehat{F}_o[X \mid \widetilde{X}]$. Then, $\mathcal{F} \in \widehat{F}[X]$ and $\widetilde{X} \subset \mathcal{F}$. But, from (3.1), we have the equality $\mathcal{F} = \{\cap\}_{\mathbf{f}}(\mathcal{F})$, where by the choice of $\mathcal{F}\ \{\cap\}_{\mathbf{f}}(\widetilde{X}) \subset \{\cap\}_{\mathbf{f}}(\mathcal{F})$. So, $\{\cap\}_{\mathbf{f}}(\widetilde{X}) \subset \mathcal{F}$ and, as a corollary, $\mathcal{F} \in \mathfrak{F}_o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$. The inclusion $\mathfrak{F}_o[X \mid \widetilde{X}] \subset \mathfrak{F}_o\left[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})\right]$ is establish- ed. So, $\mathfrak{F}_o[X \mid \widetilde{X}] = \mathfrak{F}_o[X \mid \{\cap\}_{\mathbf{f}}(\widetilde{X})]$.

Returning to (4.17), we note that by Proposition 4.2 $\forall f \in Y^X\ \forall \widetilde{X} \in P'(P(X))$

$$\left(\varnothing \in \{\cap\}_{\mathbf{f}}(\widetilde{X})\right) \Rightarrow ((\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = \varnothing\ \ \forall \tau \in (\text{top})[Y]). \quad (4.18)$$

Remark 4.2. We have that, for the case $\varnothing \notin \{\cap\}_{\mathbf{f}}(\widetilde{X})$ it is possible that

$$\exists \tau \in (\text{top})[Y] : (\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = \varnothing.$$

Indeed, consider the case $X = Y = \mathbb{R}$, $f(x) = x\ \forall x \in X$, $\tau = \tau_{\mathbb{R}}$ is the usual $|\cdot|$-topology of real line $\mathbb{R}$, and

$$\widetilde{X} = \{[c,\infty[:\ c \in \mathbb{R}\}.$$

Then, $\widetilde{X} \in \beta_o[X]$ and $\varnothing \notin \{\cap\}_{\mathbf{f}}(\widetilde{X})$. But, by (4.15)

$$(\mathbf{as})[X;Y;\tau;f;\widetilde{X}] = \bigcap_{c \in \mathbb{R}} [c,\infty[= \varnothing.$$

It is obvious the following.

Proposition 4.4. If $f \in Y^X, \tau \in (c-\text{top})[Y]$, and $\mathcal{B} \in \beta_o[X]$, then
$$(\mathbf{as})[X;Y;\tau;f;\mathcal{B}] \neq \varnothing.$$

Proof. The corresponding proof follows from known statements of general topology (see [11]). But, we consider this proof for a completeness of the account. In our case, we have (4.15). In addition,
$$\mathcal{T} := \left\{\text{cl}\left(f^1(H),\tau\right) : H \in \mathcal{B}\right\} \quad (4.19)$$
is nonempty family of sets closed in the compact topological space (TS) (Y,τ). Moreover, $\mathcal{T} \in \beta_o[Y]$ (we use known properties of the closure operation and the image operation). Since $\varnothing \notin \mathcal{B}$, we obtain that $\varnothing \notin \mathcal{T}$. In addition, $\mathcal{T} \in \beta[Y]$. Therefore, by [9] we have the following property: if $n \in \mathbb{N}$ and
$$(T_i)_{i\in\overline{1,n}} : \overline{1,n} \longrightarrow \mathcal{T},$$
then $\exists \widetilde{T} \in \mathcal{T} : \widetilde{T} \subset \bigcap_{i=1}^{n} T_i$. As a corollary, $\mathcal{T}$ is the non empty centered system of closed sets in a compact TS. Then, the intersection of all sets of $\mathcal{T}$ is not empty. By (4.19)
$$\bigcap_{H\in\mathcal{B}} \text{cl}\left(f^1(H),\tau\right) = \bigcap_{S\in\mathcal{T}} S \neq \varnothing.$$
Using (4.15), we obtain the required statement about the nonemptyness of attraction set.

Corollary 4.2. If $f \in Y^X$ and $\widetilde{X} \in P'\left(P(X)\right)$, then
$$\left(\varnothing \notin \{\cap\}_{\mathbf{f}}(\widetilde{X})\right) \Rightarrow \left((\mathbf{as})[X;Y;\tau;f;\widetilde{X}] \neq \varnothing \right.$$
$$\left.\forall \tau \in (c-\text{top})[Y]\right).$$

Proof. Let $\varnothing \notin \{\cap\}_{\mathbf{f}}(\mathcal{X})$. Choose arbitrary topology $\tau \in (c-\text{top})[Y]$. By (4.14) $\{\cap\}_{\mathbf{f}}(\mathcal{X}) \in \beta[X]$. Moreover, $\mathcal{X} \subset \{\cap\}_{\mathbf{f}}(\mathcal{X})$. Therefore, $\varnothing \notin \mathcal{X}$. Then, $\{\cap\}_{\mathbf{f}}(\mathcal{X}) \in \beta_o\ [X]$ and by Proposition 4.4
$$(\mathbf{as})[X;Y;\tau;f;\{\cap\}_{\mathbf{f}}(\mathcal{X})] \neq \varnothing.$$
Using Corollary 4.1, we obtain that $(\mathbf{as})[X;Y;\tau;f;\mathcal{X}] \neq \varnothing$.

In the following, we use the continuity notion. In this connection, suppose that
$$C(X,\tau_1,Y,\tau_2) := \left\{f \in Y^X \mid f^{-1}(G) \in \tau_1 \ \ \forall G \in \tau_2\right\}$$
$$\forall \tau_1 \in (\text{top})[X] \forall \tau_2 \in (\text{top})[Y]. \quad (4.20)$$
So, continuous functions are defined. In the following, we use bijections, open and closed mappings, and homeomorphisms. Let
$$(\text{bi})[X;Y] := \left\{f \in Y^X \mid \left(f^1(X) = Y\right) \& \right.$$
$$\& \left(\forall x_1 \in X \ \ \forall x_2 \in X \left(\left(f(x_1) = f(x_2)\right) \Rightarrow\right.\right.$$
$$\left.\left.\left.\Rightarrow (x_1 = x_2)\right)\right)\right\}. \quad (4.21)$$
In (4.21), the set of all bijections from X onto Y is defined. If $\tau_1 \in (\text{top})[X]$ and $\tau_2 \in (\text{top})[Y]$, then
$$C_{\text{op}}(X,\tau_1,Y,\tau_2) := \left\{f \in C(X,\tau_1,Y,\tau_2) \mid \right.$$
$$\left. f^1(G) \in \tau_2 \ \ \forall G \in \tau_1\right\}, \quad (4.22)$$
$$C_{\text{cl}}(X,\tau_1,Y,\tau_2) := \left\{f \in C(X,\tau_1,Y,\tau_2) \mid \right.$$
$$\left. f^1(F) \in \mathbf{C}_Y[\tau_2] \ \forall F \in \mathbf{C}_X[\tau_1]\right\}. \quad (4.23)$$
In (4.22) (in (4.23)), we consider open (closed) mappings. In addition,
$$(\text{Hom})[X;\tau_1;Y;\tau_2] := C_{\text{op}}(X,\tau_1,Y,\tau_2)$$
$$\cap (\text{bi})[X;Y] =$$
$$= C_{\text{cl}}(X,\tau_1,Y,\tau_2) \cap (\text{bi})[X;Y]$$
$$\forall \tau_1 \in (\text{top})[X] \ \forall \tau_2 \in (\text{top})[Y]. \quad (4.24)$$
So, in (4.24), the set of homeomorphisms is defined.

5. Some Properties of Ultrafilters of Measurable Spaces

In this section, we fix a nonempty set $\mathbf{E}$. We consider the very general measurable space $(\mathbf{E},\mathcal{L})$, where $\mathcal{L} \in \pi[\mathbf{E}]$ is fixed also. According to necessity, we will be supplement the corresponding suppositions with respect to $\mathcal{L}$. We suppose that $\mathbb{F}^*(\mathcal{L})$ is the set of all families $\mathcal{F} \in P'(\mathcal{L})$ such that
$$(\varnothing \notin \mathcal{F}) \& (A \cap B \in \mathcal{F} \ \ \forall A \in \mathcal{F} \ \ \forall B \in \mathcal{F}) \&$$
$$\& (\forall \widetilde{F} \in \mathcal{F} \ \ \forall \widetilde{L} \in \mathcal{L} \ \ (\widetilde{F} \subset L) \Rightarrow (\widetilde{L} \in \mathcal{F})).$$

Elements of the set $\mathbb{F}^*(\mathcal{L})$ are filters of $\mathcal{L}$. In addition,
$$\mathbb{F}_o^*(\mathcal{L}) := \left\{\mathcal{U} \in \mathbb{F}^*(\mathcal{L}) \mid \forall \mathcal{F} \in \mathbb{F}^*(\mathcal{L})\right.$$
$$\left.\left((\mathcal{U} \subset \mathcal{F}) \Rightarrow (\mathcal{U} = \mathcal{F})\right)\right\} \quad (5.1)$$
is the set of all ultrafilters of L. Recall that (see [16])
$$\forall \mathcal{F} \in \mathbb{F}^*(\mathcal{L}) \ \exists \mathcal{U} \in \mathbb{F}_o^*(\mathcal{L}) : \mathcal{F} \subset \mathcal{U}. \quad (5.2)$$
In the following, (5.2) plays the very important role.

We introduce the mapping $\Phi_{\mathcal{L}} : \mathcal{L} \to P\left(\mathbb{F}_o^*(\mathcal{L})\right)$ by the following rule:
$$\Phi_{\mathcal{L}}(L) := \left\{\mathcal{U} \in \mathbb{F}_o^*(\mathcal{L}) \mid L \in \mathcal{U}\right\} \ \forall L \in \mathcal{L}. \quad (5.3)$$
We note that $\{\mathbf{E}\} \in \mathbb{F}^*(\mathcal{L})$ and by (5.2) $\mathbb{F}_o^*(\mathcal{L}) \neq \varnothing$. In addition, we recall that (see Section 2)
$$(\mathbb{UF})[\mathbf{E};\mathcal{L}] := \left\{\Phi_{\mathcal{L}}(L) : \ L \in \mathcal{L}\right\} \in \pi[\mathbb{F}_o^*(\mathcal{L})]; \quad (5.4)$$
by (5.4) the pair $(\mathbb{F}_o^*(\mathcal{L}), (\mathbb{UF})[\mathbf{E};\mathcal{L}])$ is a nonempty multiplicative space. We note some simplest general properties. We obtain that

$$(\Sigma-\mathrm{set})[\mathbf{E}]:=\{A\in P(\mathbf{E})\mid A\cap S\neq\varnothing\ \forall S\in\Sigma\}$$
$$\in P\big(P'(\mathbf{E})\big)\ \forall\Sigma\in P'\big(P(E)\big).$$

We note that

$$\mathcal{B}|_A\in\beta_o[A]\ \forall\mathcal{B}\in\beta_o[\mathbf{E}]\ \forall A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}].$$

In addition, for $A\in P'(\mathbf{E})$ the inclusion $\beta_o[A]\subset\beta_o[\mathbf{E}]$ takes place. Therefore,

$$\mathcal{B}|_A\in\beta_o[\mathbf{E}]\ \forall\mathcal{B}\in\beta_o[\mathbf{E}]\ \forall A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}]. \quad (5.5)$$

With the employment of (5.5), we obtain that, for any $\mathcal{B}\in\beta_o[\mathbf{E}]$ and $A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}]$

$$(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A]\in\widehat{F}[\mathbf{E}]:\big((\mathbf{E}-\mathbf{fi})[\mathcal{B}]\subset(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A]\big)\&\big(A\in(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A]\big). \quad (5.6)$$

Now, we return to the space $(\mathbf{E},L)$. Suppose that

$$\beta^o_{\mathcal{L}}[\mathbf{E}]:=\{\mathcal{B}\in P'(\mathcal{L})\mid(\varnothing\notin\mathcal{B})\&\big(\forall B_1\in\mathcal{B}\ \forall B_2\in\mathcal{B}\ \exists B_3\in\mathcal{B}:\ B_3\subset B_1\cap B_2\big)\} \quad (5.7)$$

(the set of filter bases of $\mathcal{L}$); $\mathbb{F}^*(\mathcal{L})\subset\beta^o_{\mathcal{L}}[\mathbf{E}]$ and

$$\beta^o_{\mathcal{L}}[\mathbf{E}]=\beta_o[\mathbf{E}]\cap P(\mathcal{L})=\{\mathcal{B}\in\beta_o[\mathbf{E}]\mid\mathcal{B}\subset\mathcal{L}\}.$$

We note the obvious property: $\mathcal{F}\cap\mathcal{L}\in\mathbb{F}^*(\mathcal{L})\ \forall\mathcal{F}\in\widehat{F}[\mathbf{E}]$. In addition,

$$(\mathbf{E}-\mathbf{fi})[\mathcal{B}\mid\mathcal{L}]:=(\mathbf{E}-\mathbf{fi})[\mathcal{B}]\cap\mathcal{L}$$
$$=\{\tilde{L}\in\mathcal{L}\mid\exists B\in\mathcal{B}:B\subset\tilde{L}\}\in\mathbb{F}^*(\mathcal{L})\ \forall\mathcal{B}\in\beta^o_{\mathcal{L}}[\mathbf{E}].$$

Using (5.5) and the obvious inclusion $\beta^o_{\mathcal{L}}[\mathbf{E}]\subset\beta_o[\mathbf{E}]$, under $\mathcal{B}\in\beta^o_{\mathcal{L}}[\mathbf{E}]$ and $A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}$, we obtain, that $\mathcal{B}|_A\in\beta^o_{\mathcal{L}}[\mathbf{E}]$. We note that, under $\mathcal{B}\in\beta^o_{\mathcal{L}}[\mathbf{E}]$ and $A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}$, the filter

$$(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A\mid\mathcal{L}\in\mathbb{F}^*(\mathcal{L})$$

has the following properties

$$\big((\mathbf{E}-\mathbf{fi})[\mathcal{B}\mid\mathcal{L}]\subset(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A\mid\mathcal{L}]\big)\&\big(A\in(\mathbf{E}-\mathbf{fi})[\mathcal{B}|_A\mid\mathcal{L}]\big). \quad (5.8)$$

Of course, $(\mathbf{E}-\mathbf{fi})[\mathcal{F}\mid\mathcal{L}]=\mathcal{F}\ \forall\mathcal{F}\in\mathbb{F}^*(\mathcal{L})$. We can use this property in (5.8): for any $\mathcal{F}\in\mathbb{F}^*(\mathcal{L})$ and $A\in(\mathcal{F}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}$, the filter $(\mathbf{E}-\mathbf{fi})[\mathcal{F}|_A\mid\mathcal{L}]\in\mathbb{F}^*(\mathcal{L})$ has the properties

$$\big(\mathcal{F}\subset(\mathbf{E}-\mathbf{fi})[\mathcal{F}|_A\mid\mathcal{L}]\big)\&\big(A\in(\mathbf{E}-\mathbf{fi})[\mathcal{F}|_A\mid\mathcal{L}]\big). \quad (5.9)$$

In connection with (5.9), we recall the very general property: if $\mathcal{B}\in\beta^o_{\mathcal{L}}[\mathbf{E}]$ and $A\in(\mathcal{B}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}$, then $\mathcal{B}|_A\in\beta^o_{\mathcal{L}}[\mathbf{E}]$. Using the maximality property, we obtain that

$$(\mathcal{U}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}=\mathcal{U}\ \forall\mathcal{U}\in\mathbb{F}^*_o(\mathcal{L}).$$

And what is more, $\mathbb{F}^*_o(\mathcal{L})=\{\mathcal{F}\in\mathbb{F}^*(\mathcal{L})\mid\mathcal{F}=(\mathcal{F}-\mathrm{set})[\mathbf{E}]\cap\mathcal{L}\}$.

Of course, the above-mentioned properties are valid for

$$\mathcal{L}\in(\mathrm{LAT})_o[\mathbf{E}]. \quad (5.10)$$

The following reasoning is similar to the construction of [13,§3.6] connected with Wallman extension; in addition, later until the end of this section, we suppose that (5.10) is valid (so, we fix a lattice with "zero" and "unit").

So, if $\mathcal{U}\in\mathbb{F}^*_o(\mathcal{L}),A\in\mathcal{L}$, and $B\in\mathcal{L}$, then (under condition (5.10))

$$(A\cup B\in\mathcal{U})\Rightarrow\big((A\in\mathcal{U})\vee(B\in\mathcal{U})\big) \quad (5.11)$$

The property (5.11) is basic. As a corollary, $\forall\mathcal{U}\in\mathbb{F}^*_o(\mathcal{L})\ \forall A\in\mathcal{L}\ \forall B\in\mathcal{L}$

$$(A\cup B=\mathbf{E})\Rightarrow\big((A\in\mathcal{U})\vee(B\in\mathcal{U})\big) \quad (5.12)$$

We note that by (5.11) the following property is valid:

$$\Phi_{\mathcal{L}}(A\cup B)=\Phi_{\mathcal{L}}(A)\cup\Phi_{\mathcal{L}}(B)\ \forall A\in\mathcal{L}\ \forall B\in\mathcal{L}.$$

As a corollary, we obtain the property

$$(\mathbb{UF})[\mathbf{E};\mathcal{L}]\in(\mathrm{LAT})_o\big[\mathbb{F}^*_o(\mathcal{L})\big] \quad (5.13)$$

(so, under (5.10), the statement (5.4) is amplified). In (5.13), we have the lattice of subsets of $\mathbb{F}^*_o(\mathcal{L})$. This important fact used below.

6. Topological Properties, 1

As in the previous section, now we fix a nonempty set $\mathbf{E}$ and a family $\mathcal{L}\in\pi[\mathbf{E}]$. We note the following obvious property:

$$\{\tilde{L}\in\mathcal{L}\mid\exists B\in\mathcal{B}:B\subset\tilde{L}\}\in\mathbb{F}^*(\mathcal{L})\ \forall\mathcal{B}\in\beta_o[\mathbf{E}].$$

From definitions of the previous section, the following known property follows: $\forall\mathcal{U}_1\in\mathbb{F}^*_o(\mathcal{L})\ \forall\mathcal{U}_2\in\mathbb{F}^*_o(\mathcal{L})$

$$(\mathcal{U}_1\neq\mathcal{U}_2)\Rightarrow(\exists A\in\mathcal{U}_1\ \exists B\in\mathcal{U}_2:A\cap B=\varnothing). \quad (6.1)$$

Moreover, we note that $\forall U\in F^*_o(L)$

$$\{U\}=\bigcap_{\Lambda\in\mathcal{U}}\{\mathcal{F}\in\mathbb{F}^*_o(\mathcal{L})\mid\Lambda\in\mathcal{F}\}=\bigcap_{L\in\mathcal{U}}\Phi_{\mathcal{L}}(L). \quad (6.2)$$

Moreover, we note that $\pi[\mathbb{F}^*_o(\mathcal{L})]\subset(\mathrm{op}-\mathrm{BAS})[\mathbb{F}^*_o(\mathcal{L})]$. Therefore, by (5.4)

$$(\mathbb{UF})[\mathbf{E};\mathcal{L}]\in(\mathrm{op}-\mathrm{BAS})[\mathbb{F}^*_o(\mathcal{L})]. \quad (6.3)$$

As a corollary, we obtain (see Section 2) that

$$\mathbf{T}^*_{\mathcal{L}}[\mathbf{E}]:=\{\cup\}\big((\mathbb{UF})[\mathbf{E};\mathcal{L}]\big)=\{\mathbb{G}:\mathbb{G}\in P\big(\mathbb{F}^*_o(\mathcal{L})\big)\mid\forall\mathcal{U}\in\mathbb{G}\ \exists L\in\mathcal{U}:\Phi_{\mathcal{L}}(L)\subset\mathbb{G}\}\in(\mathrm{top})[\mathbb{F}^*_o(\mathcal{L})]. \quad 6.4)$$

We recall the very known definition of Hausdorff topology; namely, we introduce the set of such topologies: if M is a set, then

$$(\mathrm{top})_o[M]: = \{\tau \in (\mathrm{top})[M] | \forall m_1 \in M \ \forall m_2 \in M \setminus \{m_1\} \ \exists H_1 \in N_\tau(m_1) \ \exists H_2 \in N_\tau(m_2) : H_1 \cap H_2 = \varnothing\}.$$

For any set M we suppose that

$$(c-\mathrm{top})_o[M] := (c-\mathrm{top})[M] \cap (\mathrm{top})_o[M].$$

If $\tau \in (c-\mathrm{top})_o[M]$, then TS (M, τ) is called a compactum. Then, the obvious statement follows from the ultrafilter properties (see (5.3), (6.1)):

$$\mathbf{T}^*_{\mathcal{L}}[\mathbf{E}] \in (\mathrm{top})_o[\mathbb{F}^*_o(\mathcal{L})]. \tag{6.5}$$

So, by (6.5) $(\mathbb{F}^*_o(\mathcal{L}), \mathbf{T}^*_{\mathcal{L}}[\mathbf{E}])$ is a Hausdorff TS. Of course, we can use the previous statements of this section in the case of $\mathcal{L} \in (\mathrm{LAT})_o[\mathbf{E}]$, obtaining the Hausdorff topology (6.5). But, in the above-mentioned case, another construction of TS is very interesting. This construction is similar to Wallman extension (see [13,§3.6]). Moreover, in this connection, we note the fundamental investigation [14], where topological representations in the class of ideals are considered. We give the basic attention to the filter consideration in connection with construction of Section 3 concerning with the realization of AS. In this connection, we note that $P(\mathbf{E}) \in \pi[\mathbf{E}]$ and the sets $\mathbb{F}^*(P(\mathbf{E}))$ and $\mathbb{F}^*_o(P(\mathbf{E}))$ are defined. From (3.1) and definitions of Section 5, we have the equality $\mathbb{F}^*(P(\mathbf{E})) = \widehat{F}[\mathbf{E}]$. Moreover, from (3.2) and the above-mentioned definitions of Section 5, the equality

$$\mathbb{F}^*_o(P(\mathbf{E})) = \widehat{F}_{\mathbf{u}}[\mathbf{E}] \tag{6.6}$$

follows. By these properties (see (6.6)) the constructions of Section 3 obtain interpretation in terms of filters and ultrafilters of measurable spaces.

Now, we note one simple property; in addition, we use the inclusion chain $\mathbb{F}^*_o(\mathcal{L}) \subset \mathbb{F}^*(\mathcal{L}) \subset \beta_o[\mathbf{E}]$. So, by (3.3)

$$(\mathbf{E}-\mathbf{fi})[\mathcal{F}] \in \widehat{F}[\mathbf{E}] \ \forall \mathcal{F} \in \mathbb{F}^*(\mathcal{L}).$$

In particular, we have the following property:

$$(\mathbf{E}-\mathbf{fi})[\mathcal{U}] \in \widehat{F}[\mathbf{E}] \ \forall \mathcal{U} \in \mathbb{F}^*_o(\mathcal{L}). \tag{6.7}$$

We note one general simple property; namely, in general case of $\mathcal{L} \in \pi[\mathbf{E}]$

$$\forall \mathcal{U} \in \mathbb{F}^*_o(\mathcal{L}) \ \exists \widetilde{\mathcal{U}} \in \widehat{F}_{\mathbf{u}}[\mathbf{E}] : \mathcal{U} = \widetilde{\mathcal{U}} \cap \mathcal{L}. \tag{6.8}$$

Remark 6.1. We note that (6.8) is a variant of Proposition 2.4.1 of monograph [16]. Consider the corresponding proof. Fix $\mathcal{U} \in \mathbb{F}^*_o(\mathcal{L})$. Then by (6.7)

$$\mathcal{V} := (\mathbf{E}-\mathbf{fi})[\mathcal{U}] = \{H \in P(\mathbf{E}) | \ \exists B \in \mathcal{U} : B \subset H\} \in \widehat{F}[\mathbf{E}]. \tag{6.9}$$

From (6.9), we obtain (see Section 3) that $\widehat{F}^o_{\mathbf{u}}[\mathbf{E} | \mathcal{V}] \in P'(\widehat{F}_{\mathbf{u}}[E])$. Let

$$\mathcal{W} \in \widehat{F}^o_{\mathbf{u}}[\mathbf{E} | \mathcal{V}].$$

Then, $\mathcal{W} \in \widehat{F}_{\mathbf{u}}[\mathbf{E}]$ and $\mathcal{V} \subset \mathcal{W}$. In addition (see Section 5), $\mathcal{W} \cap \mathcal{L} \in \mathbb{F}^*(\mathcal{L})$. Let $\widetilde{U} \in \mathcal{U}$. Then, $\widetilde{U} \in \mathcal{L}$ and, in particular, $\widetilde{U} \in P(\mathbf{E})$. By (6.9) $\widetilde{U} \in \mathcal{V}$ and, as a corollary, $\widetilde{U} \in \mathcal{W}$. Then, $\widetilde{U} \in \mathcal{W} \cap \mathcal{L}$. So, the inclusion $\mathcal{U} \subset \mathcal{W} \cap \mathcal{L}$ is established; we obtain that

$$\mathcal{W} \cap \mathcal{L} \in \mathbb{F}^*(\mathcal{L}) : \mathcal{U} \subset \mathcal{W} \cap \mathcal{L}. \tag{6.10}$$

From (5.1) and (6.10), we have the equality $\mathcal{U} = \mathcal{W} \cap \mathcal{L}$. So,

$$\mathcal{W} \in \widehat{F}_{\mathbf{u}}[E] : \mathcal{U} = \mathcal{W} \cap \mathcal{L}.$$

Since the choice of $\mathcal{U}$ was arbitrary, the property (6.8) is established.

7. Topological Properties, 2

In this and following sections, we fix a nonempty set E and a lattice $\mathcal{L} \in (\mathrm{LAT})_o[E]$. We consider the question about constructing a compact T_1-space with "unit" $\mathbb{F}^*_o(\mathcal{L})$. This space is similar to Wallman extension for a T_1-space. But, we not use axioms of topology and operate lattice constructions (here, a natural analogy with constructions of [14] takes place). Later we use the following simple statement.

Proposition 7.1. $(\mathbb{UF})[E; \mathcal{L}] \in (\mathrm{cl}-\mathrm{BAS})[\mathbb{F}^*_o(\mathcal{L})]$.

Proof. We use (5.13). In particular, $\varnothing \in (\mathbb{UF})[E; \mathcal{L}]$. As a corollary,

$$\bigcap_{\zeta \in (\mathbb{UF})[E;\mathcal{L}]} \zeta = \varnothing. \tag{7.1}$$

Moreover, $\mathbb{F}^*_o(\mathcal{L}) = \Phi_{\mathcal{L}}(E) \in (\mathbb{UF})[E; \mathcal{L}]$ (see (5.4)). So, $(\mathbb{UF})[E; \mathcal{L}]$ is a family with "zero" and "unit". Moreover, by (5.13)

$$B_1 \cup B_2 \in (\mathbb{UF})[E; \mathcal{L}] \ \forall B_1 \in (\mathbb{UF})[E; \mathcal{L}] \ \forall B_2 \in (\mathbb{UF})[E; \mathcal{L}].$$

Therefore, by (2.13) the required statement is realized.

By (2.15) and Proposition 7.1 we have the following construction:

$$\{\cap\}((\mathbb{UF})[E; \mathcal{L}]) \in (\mathrm{clos})[\mathbb{F}^*_o(\mathcal{L})]. \tag{7.2}$$

Proposition 7.2. The following compactness property is valid:

$$\mathbf{C}_{\mathbb{F}^*_o(\mathcal{L})}(\{\cap\}((\mathbb{UF})[E; \mathcal{L}])) \in (c-\mathrm{top})[\mathbb{F}^*_o(\mathcal{L})]. \tag{7.3}$$

Proof. For brevity, we suppose that

$$\mathbf{U}:=\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big) \tag{7.4}$$

and $\theta:=\mathbf{C}_{\mathbb{F}_o^*(\mathcal{L})}[\mathbf{U}]$. Of course, by (2.9) $\theta\in(\mathrm{top})[\mathbb{F}_o^*(\mathcal{L})]$. Moreover, under $S\in\mathbf{U}$, the family

$$\mathbf{u}[S]:=\{T\in(\mathbb{UF})[E;\mathcal{L}]\,|\,S\subset T\}\in P'\big((\mathbb{UF})[E;\mathcal{L}]\big)$$

has the following obvious property

$$S=\bigcap_{T\in\mathbf{u}[S]}T. \tag{7.5}$$

We have the equality $\mathbf{U}=\mathbf{C}_{\mathbb{F}_o^*(\mathcal{L})}[\theta]$. So, $\mathbf{U}$ is the family of all subsets of $\mathbb{F}_o^*(\mathcal{L})$ closed in the TS

$$(\mathbb{F}_o^*(\mathcal{L}),\theta). \tag{7.6}$$

Let η be arbitrary nonempty centered subfamily of $\mathbf{U}$ (for any $m\in\mathbb{N}$ and $(T_i)_{i\in\overline{1,m}}\in\eta^m$ the intersection of all sets $T_i, i\in\overline{1,m}$, is not empty). If $H\in\eta$, then the family

$$\mathbb{D}_H:=\{\widetilde{L}\in\mathcal{L}\,|\,\Phi_{\mathcal{L}}(\widetilde{L})\in\mathbf{u}[H]\}\in P'(\mathcal{L}) \tag{7.7}$$

has the property: $\mathbb{D}_H\subset\mathcal{U}\ \forall\mathcal{U}\in H$. Of course,

$$\mathbf{L}:=\bigcup_{H\in\eta}\mathbb{D}_H\in P'(\mathcal{L})$$

is centered. Indeed, choose $n\in\mathbb{N}$ and $(\Lambda_i)_{i\in\overline{1,n}}\in\mathbf{L}^n$. Let $(\widetilde{H}_i)_{i\in\overline{1,n}}\in\eta^n$ be a procession with the property:

$$\Lambda_j\in\mathbb{D}_{\widetilde{H}_j}\quad\forall j\in\overline{1,n}.$$

Then, in particular, $(\Lambda_i)_{i\in\overline{1,n}}\in\mathcal{L}^n$. In addition, by (7.7) $\Phi_{\mathcal{L}}(\Lambda_j)\in\mathbf{u}[\widetilde{H}_j]\ \forall j\in\overline{1,n}$. Of course,

$$\bigcap_{i=1}^n\widetilde{H}_i\subset\bigcap_{i=1}^n\Phi_{\mathcal{L}}(\Lambda_i).$$

Since the intersection of all sets $\widetilde{H}_i, i\in\overline{1,n}$, is not empty (we use the centrality of η), we choose an ultrafilter

$$\widetilde{\mathcal{U}}\in\bigcap_{i=1}^n\widetilde{H}_i.$$

Then, $\Lambda_j\in\widetilde{\mathcal{U}}$ under $j\in\overline{1,n}$. By axioms of a filter (see Section 5) we obtain that

$$\bigcap_{i=1}^n\Lambda_i\neq\varnothing.$$

Since $\mathcal{L}$ is closed with respect to finite intersections, we obtain that

$$\{\cap\}_{\mathbf{f}}(\mathbf{L})\in P'(\mathcal{L}):(\varnothing\notin\{\cap\}_{\mathbf{f}}(\mathbf{L}))\,\&\,\big(\mathbf{L}\subset\{\cap\}_{\mathbf{f}}(\mathbf{L})\big). \tag{7.8}$$

Moreover, (7.8) is supplemented by the following obvious property; namely,

$$\forall B_1\in\{\cap\}_{\mathbf{f}}(\mathbf{L})\ \forall B_2\in\{\cap\}_{\mathbf{f}}(\mathbf{L})\ \exists B_3\in\{\cap\}_{\mathbf{f}}(\mathbf{L}):$$
$$B_3\subset B_1\cap B_2.$$

From (5.7), we obtain that $\{\cap\}_{\mathbf{f}}(\mathbf{L})\in\beta_{\mathcal{L}}^o[E]$. As a corollary,

$$\mathcal{V}:=(E-\mathbf{fi})[\{\cap\}_{\mathbf{f}}(\mathbf{L})|\mathcal{L}]\in\mathbb{F}^*(\mathcal{L});$$

in addition, by (7.8) $\mathbf{L}\subset\{\cap\}_{\mathbf{f}}(\mathbf{L})\subset\mathcal{V}$. Finally, we use (5.2). Let $\mathcal{W}\in\mathbb{F}_o^*(\mathcal{L})$ be an ultrafilter for which $\mathcal{V}\subset\mathcal{W}$. Then, $\mathbf{L}\subset\mathcal{W}$. So,

$$\mathcal{W}\in\mathbb{F}_o^*(\mathcal{L}):\mathbf{L}\subset\mathcal{W}. \tag{7.9}$$

Let $\mathbb{M}\in\eta$. Then, $\mathbb{D}_{\mathbb{M}}\in P'(\mathbf{L})$ and the equality

$$\mathbb{M}=\bigcap_{T\in\mathbf{u}[\mathbb{M}]}T \tag{7.10}$$

is valid (see (7.5)). Choose arbitrary $\Sigma\in\mathbf{u}[\mathbb{M}]$. Then, $\Sigma\in(\mathbb{UF})[E;\mathcal{L}]$ and $\mathbb{M}\subset\Sigma$. Using (5.4), we choose $D\in\mathcal{L}$ for which $\Sigma=\Phi_{\mathcal{L}}(D)$. Then

$$D\in\mathcal{L}:\Phi_{\mathcal{L}}(D)\in\mathbf{u}[\mathbb{M}].$$

By (7.7) $D\in\mathbb{D}_{\mathbb{M}}$ and, in particular, $D\in\mathbf{L}$. By (7.9) $D\in\mathcal{W}$ and, as a corollary, $\mathcal{W}\in\Phi_{\mathcal{L}}(D)$; see (5.3). So, $\mathcal{W}\in\Sigma$. Since the choice of Σ was arbitrary, we obtain that $\mathcal{W}\in B\ \forall B\in\mathbf{u}[\mathbb{M}]$. By (7.10) $\mathcal{W}\in\mathbb{M}$. So, we have the property:

$$\mathcal{W}\in H\quad\forall H\in\eta.$$

Then, the intersection of all sets of η is not empty. Since the choice of η was arbitrary, it is established that any nonempty centered family of closed (in TS (7.6)) sets has the nonempty intersection. So, TS (7.6) is compact (see [11-13]).

Using Proposition 7.2, by $\mathbf{T}_{\mathcal{L}}^o[E]$ we denote the topology (7.3); so,

$$\mathbf{T}_{\mathcal{L}}^o[E]:=\mathbf{C}_{\mathbb{F}_o^*(\mathcal{L})}\Big(\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big)\Big)\in(c-\mathrm{top})[\mathbb{F}_o^*(\mathcal{L})]. \tag{7.11}$$

We have the nonempty compact TS

$$(\mathbb{F}_o^*(\mathcal{L}),\mathbf{T}_{\mathcal{L}}^o[E]). \tag{7.12}$$

Proposition 7.3. If $\mathcal{U}\in\mathbb{F}_o^*(\mathcal{L})$, then

$$\{\mathcal{U}\}\in\{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big).$$

The corresponding proof follows from (6.2); of course, we use (5.4) also. From (2.18), (7.11), and Proposition 7.3, we obtain the following property:

$$\mathbf{T}_{\mathcal{L}}^o[E]\in(c-\mathrm{top})[\mathbb{F}_o^*(\mathcal{L})]\cap(\mathcal{D}-\mathrm{top})[\mathbb{F}_o^*(\mathcal{L})]. \tag{7.13}$$

So, by (7.13) we obtain that (7.12) is a nonempty compact T_1-space.

In conclusion of the given section, we note several properties. First, we recall that

$$\mathbb{F}_o^*(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\widetilde{L})=\{\mathcal{U}\in\mathbb{F}_o^*(\mathcal{L})\,|\,\widetilde{L}\notin\mathcal{U}\}\quad\forall\widetilde{L}\in\mathcal{L}. \tag{7.14}$$

In addition, from (7.11), the obvious representation follows:

$$\mathbf{T}^o_{\mathcal{L}}[E]=\Big\{G\in P(\mathbb{F}^*_o(\mathcal{L}))\,|\,\forall\mathcal{U}\in G$$
$$\exists\tilde{L}\in\mathcal{L}\setminus\mathcal{U}\ \ \forall\mathcal{V}\in\mathbb{F}^*_o(\mathcal{L})\qquad(7.15)$$
$$\Big((\tilde{L}\notin\mathcal{V})\Rightarrow(\mathcal{V}\in G)\Big)\Big\}.$$

With the employment of (7.15) the following statement is established.

Proposition 7.4. If $\mathcal{U}\in\mathbb{F}^*_o(\mathcal{L})$, then the family

$$\varphi_{\mathcal{U}}:=\Big\{\mathbb{F}^*_o(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\tilde{L}):\tilde{L}\in\mathcal{L}\setminus\mathcal{U}\Big\}$$

is a local base of TS (7.12) at $\mathcal{U}$:

$$\Big(\varphi_{\mathcal{U}}\subset N_{\mathbf{T}^o_{\mathcal{L}}[E]}(\mathcal{U})\Big)\&\Big(\forall H\in N_{\mathbf{T}^o_{\mathcal{L}}[E]}(\mathcal{U})$$
$$\exists B\in\varphi_{\mathcal{U}}:B\subset H\Big).$$

The proof is obvious. So, by (3.4) and Proposition 7.4

$$\Big\{\mathbb{F}^*_o(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\tilde{L}):\tilde{L}\in\mathcal{L}\setminus\mathcal{U}\Big\}\in(\mathcal{U}-\mathrm{bas})\Big[\mathbf{T}^o_{\mathcal{L}}[E]\Big]$$
$$\forall\mathcal{U}\in\mathbb{F}_o{}^*(\mathcal{L}).$$

We note that, from definitions, the following property is valid:

$$\mathbb{F}^*_o(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\tilde{L})\in\mathbf{T}^o_{\mathcal{L}}[E]\ \ \forall\tilde{L}\in\mathcal{L}.\qquad(7.16)$$

8. The Density Properties

In this section, we continue the investigation of TS (7. 12). Of course, we preserve the suppositions of Section 7 with respect to E and $\mathcal{L}$. But, in this section, we postulate that $\{x\}\in\mathcal{L}\ \ \forall x\in E.$ So, in this section

$$\mathcal{L}\in(\mathrm{LAT})^o[E]\qquad(8.1)$$

unless otherwise stipulated. So, $\mathcal{L}\in(\mathrm{LAT})_o[E]$ and $\{x\}\in\mathcal{L}\ \ \forall x\in E$. Therefore, with regard (3.9) and (8.1), we obtain that

$$(E-\mathrm{ult})[x]\cap\mathcal{L}=\big\{\Lambda\in\mathcal{L}\,|\,x\in\Lambda\big\}\in\mathbb{F}^*_o(\mathcal{L})\ \ \forall x\in E.\ (8.2)$$

Of course, for any $x\in E,$ the inclusion $\{x\}\in(E-\mathrm{ult})[x]\cap\mathcal{L}$ is valid.

Proposition 8. 1.

$$\mathbb{F}^*_o(\mathcal{L})=\mathrm{cl}\Big(\big\{(E-\mathrm{ult})[x]\cap L:x\in E\big\},\mathbf{T}^o_L[E]\Big).$$

Proof. Let $\mathcal{U}\in\mathbb{F}^*_o(\mathcal{L})$ and $\mathbb{H}\in N_{\mathbf{T}^o_{\mathcal{L}}[E]}(\mathcal{U}).$ We use Propposition 7.4. Namely, we choose a set $\tilde{L}\in\mathcal{L}\setminus\mathcal{U}$ for which

$$\mathbb{F}^*_o(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\tilde{L})\subset\mathbb{H}.\qquad(8.3)$$

Since $E\in\mathcal{U}$ by axioms of a filter (see Section 5), we obtain that $\tilde{L}\neq E.$ in addition, $\tilde{L}\in\mathcal{L}$ and by (2.4) and (8.1) $\tilde{L}\subset E.$ So, $E\setminus\tilde{L}\neq\varnothing.$ Choose arbitrary point $e\in E\setminus\tilde{L}$ and consider the ultrafilter

$$\Sigma:=(E-\mathrm{ult})[e]\cap\mathcal{L}\in\mathbb{F}^*_o(\mathcal{L});\qquad(8.4)$$

see (8.2). In addition, $\{e\}\in\Sigma.$ As a corollary, by definitions of Section 5

$$\Big(\tilde{L}\in\Sigma\Big)\Rightarrow\Big(\tilde{L}\cap\{e\}\neq\varnothing\Big).\qquad(8.5)$$

But, $\tilde{L}\cap\{e\}=\varnothing$ by the choice of e. Therefore, by (8. 5) $\tilde{L}\notin\Sigma.$ From (5.3) we have the property $\Sigma\notin\Phi_{\mathcal{L}}(\tilde{L}).$ As a corollary, by (8.4)

$$\Sigma\in\mathbb{F}^*_o(\mathcal{L})\setminus\Phi_{\mathcal{L}}(\tilde{L}).\qquad(8.6)$$

From (8.3) and (8.6), we obtain that $\Sigma\in\mathbb{H}.$ By (8.4)

$$\big\{(E-\mathrm{ult})[x]\cap\mathcal{L}:x\in E\big\}\cap\mathbb{H}\neq\varnothing.\qquad(8.7)$$

Since the choice of $\mathbb{H}$ was arbitrary,

$$\mathcal{U}\in\mathrm{cl}\Big(\big\{(E-\mathrm{ult})[x]\cap\mathcal{L}:x\in E\big\},\mathbf{T}^o_{\mathcal{L}}[E]\Big).$$

Since the choice of $\mathcal{U}$ was arbitrary, the inclusion

$$\mathbb{F}^*_o(\mathcal{L})\subset\mathrm{cl}\Big(\big\{(E-\mathrm{ult})[x]\cap\mathcal{L}:x\in E\big\},\mathbf{T}^o_{\mathcal{L}}[E]\Big)$$

is established. The inverse inclusion is obvious (see (7. 11)).

So, we obtain that trivial ultrafilters (8.2) realize an everywhere dense set in the TS (7.12).

Returning to (7.11), we note one obvious property connected with (7.16). Namely, by (2.14) and Proposition 7.1, in general case of $\mathcal{L}\in(\mathrm{LAT})_o[E]$

$$\mathbf{C}_{\mathbb{F}^*_o(\mathcal{L})}\big[(\mathbb{UF})[E;\mathcal{L}]\big]\in(\mathrm{op}-\mathrm{BAS})_{\varnothing}\Big[\mathbb{F}^*_o(\mathcal{L})\Big]$$

and, in particular, $\mathbf{C}_{\mathbb{F}^*_o(\mathcal{L})}\big[(\mathbb{UF})[E;\mathcal{L}]\big]\in(\mathrm{op}-\mathrm{BAS})\Big[\mathbb{F}^*_o(\mathcal{L})\Big];$ then, for $\Lambda\in(\mathrm{LAT})_o[E]$

$$\{\cup\}\Big(\mathbf{C}_{\mathbb{F}^*_o(\Lambda)}\big[(\mathbb{UF})[E;\Lambda]\big]\Big)\in(\mathrm{top})\Big[\mathbb{F}^*_o(\Lambda)\Big].$$

And what is more by (2.17), (7.11), and Proposition 7.1, in general case of $\Lambda\in(\mathrm{LAT})_o[E]$

$$\begin{aligned}\mathbf{T}^o_{\Lambda}[E]&=\mathbf{C}_{\mathbb{F}^*_o(\Lambda)}\Big[\{\cap\}\big((\mathbb{UF})[E;\Lambda]\big)\Big]\\&=\{\cup\}\Big(\mathbf{C}_{\mathbb{F}^*_o(\Lambda)}\big[(\mathbb{UF})[E;\Lambda]\big]\Big);\end{aligned}\qquad(8.8)$$

so, by (8.8) $\mathbf{C}_{\mathbb{F}^*_o(\Lambda)}\big[(\mathbb{UF})[E;\Lambda]\big]$ is a base of topology (7.11). We recall that by (2.8) and (5.4), for general case of $\Lambda\in(\mathrm{LAT})_o[E]$

$$\begin{aligned}&\mathbf{C}_{\mathbb{F}^*_o(\Lambda)}\big[(\mathbb{UF})[E;\Lambda]\big]\\&=\Big\{\mathbb{F}^*_o(\Lambda)\setminus B:B\in(\mathbb{UF})[E;\Lambda]\Big\}\\&=\Big\{\mathbb{F}^*_o(\Lambda)\setminus\Phi_{\Lambda}(L):L\in\Lambda\Big\}.\end{aligned}\qquad(8.9)$$

Connection with Wallman extension. Let $\tau\in(\mathcal{D}-\mathrm{top})[E].$ Then, $\mathbf{C}_E[\tau]\in(\mathrm{clos})[E]$ and by (2.18) $\{x\}\in\mathbf{C}_E[\tau]\ \forall x\in E.$ Using (2.7), we obtain that $\mathbf{C}_E[\tau]\in$

$(\mathrm{LAT})_o[E]$. with the employment of the above-mentioned closedness of singletons, by the corresponding definition of Section 2 we obtain that

$$\mathbf{C}_E[\tau] \in (\mathrm{LAT})^o[E]. \tag{8.10}$$

Until the end of the present section, we suppose that

$$\mathcal{L} = \mathbf{C}_E[\tau]. \tag{8.11}$$

So, in our case, $(E, \mathcal{L})$ is the lattice of closed sets in T_1-space. Then, (7.12) is the corresponding Wallman compact space (see [13]). On the other hand, by (8.10) and (8.11) we obtain that this variant of $(E, \mathcal{L})$ corresponds to general statements of our section (for example, see (8.2) and Proposition 8.1). In this connection, we consider the mapping

$$x \mapsto (E-\mathrm{ult})[x] \cap \mathcal{L} : E \longrightarrow \mathbb{F}_o^*(\mathcal{L}); \qquad 8.12)$$

we denote the mapping (8.12) by $\mathbf{f}$. So, $\mathbf{f} \in \mathbb{F}_o^*(\mathcal{L})^E$ and

$$\mathbf{f}(x) := (E-\mathrm{ult})[x] \cap \mathcal{L} \ \ \forall x \in E.$$

Consider some simple properties. First, we note that $\mathbf{f}$ is injective: $\forall x_1 \in E \ \forall x_2 \in E$

$$\big(\mathbf{f}(x_1) = \mathbf{f}(x_2)\big) \Rightarrow (x_1 = x_2). \tag{8.13}$$

Indeed, for $x_1 \in E$ and $x_2 \in E$ with the property $\mathbf{f}(x_1) = \mathbf{f}(x_2)$, by (3.9) we have that $\{x_1\} \in \mathbf{f}(x_2)$ and, as a corollary, $x_2 \in \{x_1\}$; so, $x_1 = x_2$.

Of course, $\mathbf{f}$ is a bijection from E onto the set

$$\mathbf{f}^1(E) = \{(E-\mathrm{ult})[x] \cap \mathcal{L} : x \in E\} \in P'\big(\mathbb{F}_o^*(L)\big). \tag{8.14}$$

If $\Lambda \in \mathcal{L}$ and $x \in E$, then $\big(\Lambda \in \mathbf{f}(x)\big) \Leftrightarrow (x \in \Lambda)$. As a corollary, we obtain that

$$\mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big) = \Lambda \ \ \forall \Lambda \in \mathcal{L}. \tag{8.15}$$

Remark 8.1. Of course, in (8.15), we use the representation (8.12). Fix $\Lambda \in \mathcal{L}$. Let $x_* \in \mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big)$. Then $x_* \in E$ and $\mathbf{f}(x_*) \in \Phi_{\mathcal{L}}(\Lambda)$. By (5.3) $\Lambda \in \mathbf{f}(x_*)$ and, as a corollary, $x_* \in \Lambda$. So,

$$\mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big) \subset \Lambda. \tag{8.16}$$

If $x^* \in \Lambda$, then $\Lambda \in \mathbf{f}(x^*)$; see (8.12). Therefore, by (5.3) $\mathbf{f}(x^*) \in \Phi_{\mathcal{L}}(\Lambda)$ and, as a corollary, $x^* \in \mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big)$. So, $\Lambda \subset \mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big)$. Therefore (see (8.16)) Λ and $\mathbf{f}^{-1}\big(\Phi_{\mathcal{L}}(\Lambda)\big)$ coincide.

From (5.4) and (8.15), we obtain that

$$\mathbf{f}^{-1}(B) \in \mathcal{L} \ \ \forall B \in (\mathbb{UF})[E;\mathcal{L}]. \tag{8.17}$$

Proposition 8.2. $\mathbf{f} \in C\big(E, \tau, \mathbb{F}_o^*(\mathcal{L}), \mathbf{T}_{\mathcal{L}}^o[E]\big)$.

Proof. We use the construction dual with respect to (4.20). Let $F \in \mathbf{C}_{\mathbb{F}_o^*(\mathcal{L})}\big[\mathbf{T}_{\mathcal{L}}^o[E]\big]$. Then, by (8.8) $F \in \{\cap\}\big((\mathbb{UF})[E;\mathcal{L}]\big)$. Therefore, for some $\mathbf{F} \in P'\big((\mathbb{UF})[E;\mathcal{L}]\big)$

$$F = \bigcap_{B \in \mathbf{F}} B.$$

As a result, we obtain that

$$\mathbf{f}^{-1}(F) = \bigcap_{B \in \mathbf{F}} \mathbf{f}^{-1}(B), \tag{8.18}$$

where $\mathbf{f}^{-1}(\tilde{B}) \in \mathcal{L} \ \forall \tilde{B} \in \mathbf{F}$; see (8.17). By (2.6), (2.9), (8.11), and (8.18) we have the property:

$$\lambda := \{\mathbf{f}^{-1}(B) : B \in \mathbf{F}\} \in P'(\mathbf{C}_E[\tau]) \ \text{ and}$$

$$\mathbf{f}^{-1}(F) = \bigcap_{\Lambda \in \lambda} \Lambda \in \mathbf{C}_E[\tau]. \tag{8.19}$$

Since the choice of F was arbitrary, from (8.19) we obtain the required continuity property (see [16, (2.5.2)]).

Corollary 8.1. $\mathbf{f} \in C\Big(E, \tau, \mathbf{f}^1(E), \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}\Big)$.

Proof. Recall that $\mathbf{f}(x) \in \mathbf{f}^1(E) \ \forall x \in E$. In addition, by (8.14)

$$\mathbf{f}^1(E) = \{\mathbf{f}(x) : x \in E\} \subset \mathbb{F}_o^*(\mathcal{L}).$$

Let $G \in \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}$ and $\Gamma \in \mathbf{T}_{\mathcal{L}}^o[E]$ realizes the equality $G = \mathbf{f}^1(E) \cap \Gamma$. By Proposition 8.2

$$\mathbf{f}^{-1}(\Gamma) \in \tau. \tag{8.20}$$

In addition, $\mathbf{f}^{-1}(G) \subset \mathbf{f}^{-1}(\Gamma)$ (indeed, $G \subset \Gamma$). Let $x_* \in \mathbf{f}^{-1}(\Gamma)$. Then, $x_* \in E$ and $\mathbf{f}(x_*) \in \Gamma$. But, $\mathbf{f}(x_*) \in \mathbf{f}^1(E)$ too. Then, $\mathbf{f}(x_*) \in \mathbf{f}^1(E) \cap \Gamma$. So, $\mathbf{f}(x_*) \in G$. Therefore, $x_* \in \mathbf{f}^{-1}(G)$. Since the choice of x_* was arbitrary, the inclusion

$$\mathbf{f}^{-1}(\Gamma) \subset \mathbf{f}^{-1}(G)$$

is established. So, $\mathbf{f}^{-1}(G) = \mathbf{f}^{-1}(\Gamma)$. By (8.20) $\mathbf{f}^{-1}(G) \in \tau$. Since the choice of G was arbitrary, the inclusion $\mathbf{f} \in C\Big(E, \tau, \mathbf{f}^1(E), \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}\Big)$ is established.

Recall that $\mathbf{f} \in (\mathrm{bi})[E; \mathbf{f}^1(E)]$ (see (4.21)).

Proposition 8.3. $\mathbf{f} \in C_{\mathrm{op}}\Big(E, \tau, \mathbf{f}^1(E), \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}\Big)$.

Proof. Let $G \in \tau$. Then $\mathbf{f}^1(G) = \{\mathbf{f}(x) : x \in G\}$ and $F := E \setminus G \in \mathbf{C}_E[\tau]$. By (8.11) $F \in \mathcal{L}$. In addition, by (5.3)

$$\Phi_{\mathcal{L}}(F) = \{\mathcal{U} \in \mathbb{F}_o^*(L) | F \in \mathcal{U}\}. \tag{8.21}$$

Of course, by (5.4) $\Phi_{\mathcal{L}}(F) \in (\mathbb{UF})[E;\mathcal{L}]$. Then

$$\mathbb{F}_o^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(F) \in \mathbf{C}_{\mathbb{F}_o^*(\mathcal{L})}\big[(\mathbb{UF})[E;\mathcal{L}]\big].$$

As a corollary, $\mathbb{F}_o^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(F) \in \mathbf{T}_{\mathcal{L}}^o[E]$. Therefore,

$$\mathbf{G} := \mathbf{f}^1(E) \cap \Big(\mathbb{F}_o^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(F)\Big) \in \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}. \tag{8.22}$$

Now, we compare $\mathbf{f}^1(G)$ and $\mathbf{G}$ (8.22). Let $\mathcal{V} \in \mathbf{f}^1(G)$. Then, for some $x_* \in G$,

$$\mathcal{V} = \mathbf{f}(x_*) = (E-\text{ult})[x_*] \cap \mathcal{L}. \qquad (8.23)$$

Of course, $G \in (E-\text{ult})[x_*]$. By (3.9) $F \notin (E-\text{ult})[x_*]$ (indeed, $G \cap F = \varnothing \notin (E-\text{ult})[x_*]$). By (8.23) $F \notin \mathcal{V}$ and, as a corollary, $\mathcal{V} \notin \Phi_{\mathcal{L}}(F)$; see (8.21). We obtain that

$$\mathcal{V} \in \mathbb{F}_o^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(F). \qquad (8.24)$$

Since $\mathbf{f}^1(G) \subset \mathbf{f}^1(E)$, we have the inclusion $\mathcal{V} \in \mathbf{f}^1(E)$. Using (8.22) and (8.24), we obtain that $\mathcal{V} \in \mathbf{G}$. The inclusion

$$\mathbf{f}^1(G) \subset \mathbf{G} \qquad (8.25)$$

is established. Choose arbitrary $\mathcal{W} \in \mathbf{G}$. then, by (8.22), for some $x^* \in E$, the equality $\mathcal{W} = \mathbf{f}(x^*)$ is valid. So,

$$\mathcal{W} = (E-\text{ult})[x^*] \cap \mathcal{L}. \qquad (8.26)$$

Moreover, $\mathcal{W} \in \mathbb{F}_o^*(\mathcal{L}) \setminus \Phi_{\mathcal{L}}(F)$. So, $\mathcal{W} \notin \Phi_{\mathcal{L}}(F)$. By (8.21) $F \notin \mathcal{W}$. Since $F \in \mathcal{L}$, by (8.26) $F \notin (E-\text{ult})[x^*]$. From (3.9), the property $x^* \notin F$ follows. Then, $x^* \in E \setminus F$. Therefore, $x^* \in G$. as a corollary, $\mathcal{W} = \mathbf{f}(x^*) \in \mathbf{f}^1(G)$. The inclusion $\mathbf{G} \subset \mathbf{f}^1(G)$ is established. Using (8.25), we obtain that $\mathbf{f}^1(G) = \mathbf{G}$. By (8.22)

$$\mathbf{f}^1(G) \in \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}.$$

Since the choice of G was arbitrary, by Corollary 8.1 and (4.22) we have the inclusion

$$\mathbf{f} \in C_{\text{op}}\left(E, \tau, \mathbf{f}^1(E), \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}\right).$$

By (4.24), (8.13), and Proposition 8.3 we obtain that

$$\mathbf{f} \in (\text{Hom})\left[E; \tau; \mathbf{f}^1(E); \mathbf{T}_{\mathcal{L}}^o[E]|_{\mathbf{f}^1(E)}\right]. \qquad (8.27)$$

So, we construct the concrete homeomorphic inclusion of T_1-space in the compact T_1-space (in this connection, we recall that by Proposition 8.1

$$\mathbb{F}_o^*(\mathcal{L}) = \text{cl}\left(\mathbf{f}^1(E), \mathbf{T}_{\mathcal{L}}^o[E]\right);$$

moreover, see (7.13)). So, we have the "usual" Wallman extension.

9. Ultrafilters of Measurable Space

In this section, we fix a nonempty set $\mathbf{I}$ and an algebra $\mathcal{A}$ of subsets of $\mathbf{I}$. So, in this section, $(\mathbf{I}, \mathcal{A})$ is a measurable space with an algebra of sets: $\mathcal{A} \in (\text{alg})[\mathbf{I}]$. Of course, we can to use constructions of Section 5; indeed, in particular, we have the inclusion $\mathcal{A} \in (\text{LAT})_o[\mathbf{I}]$; see (2.10). As a corollary, by (2.4) $\mathcal{A} \in \pi[\mathbf{I}]$. So, we use the sets $\mathbb{F}^*(\mathcal{A})$ and $\mathbb{F}_o^*(\mathcal{A})$ of Section 5; we use properties of these sets also. We note the known representation (see [15]):

$$\mathbb{F}_o^*(\mathcal{A}) = \left\{\mathcal{F} \in \mathbb{F}^*(\mathcal{A}) \mid \forall A \in \mathcal{A} \ (A \in \mathcal{F}) \vee (\mathbf{I} \setminus A \in \mathcal{F})\right\}. \qquad (9.1)$$

Now, we use (9.1) for investigation of TS (7.12) in the case $\mathcal{L} = \mathcal{A}$. First, we note the obvious corollary of (9.1):

$$\mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A}) = \Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A}) \ \forall \tilde{A} \in \mathcal{A}. \qquad (9.2)$$

Remark 9.1. Let $\tilde{A} \in \mathcal{A}$ is fixed. Choose arbitrary $\mathcal{U}_1 \in \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A})$. Then, by (7.14) $\tilde{A} \notin \mathcal{U}_1$. By (9.1) $\mathbf{I} \setminus \tilde{A} \in \mathcal{U}_1$, where $\mathbf{I} \setminus \tilde{A} \in \mathcal{A}$ by axioms of an algebra of sets. So, by (5.3) $\mathcal{U}_1 \in \Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A})$. The inclusion

$$\mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A}) \subset \Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A}) \qquad (9.3)$$

is established. Let $\mathcal{U}_2 \in \Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A})$. Then, by (5.3) $\mathcal{U}_2 \in \mathbb{F}_o^*(\mathcal{A})$ and $\mathbf{I} \setminus \tilde{A} \in \mathcal{U}_2$. By axioms of a filter

$$(\tilde{A} \in \mathcal{U}_2) \Rightarrow \left(\tilde{A} \cap (\mathbf{I} \setminus \tilde{A}) \neq \varnothing\right).$$

So, $\tilde{A} \notin \mathcal{U}_2$ and $\mathcal{U}_2 \notin \Phi_{\mathcal{A}}(\tilde{A})$. As a corollary, $\mathcal{U}_2 \in \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A})$. So, the inclusion

$$\Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A}) \subset \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A})$$

is established. Using (9.3), we obtain the required coincidence $\mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\tilde{A})$ and $\Phi_{\mathcal{A}}(\mathbf{I} \setminus \tilde{A})$.

Returning to (9.2) in general case, we note the following obvious

Proposition 9.1. $(\mathbb{UF})[\mathbf{I}; \mathcal{A}] = \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[(\mathbb{UF})[\mathbf{I}; \mathcal{A}]\right]$.

Proof. Let $B_o \in (\mathbb{UF})[\mathbf{I}; \mathcal{A}]$. Using (5.4), we choose $L_o \in \mathcal{A}$ such that $B_o = \Phi_{\mathcal{A}}(L_o)$. Then $\mathbf{I} \setminus L_o \in \mathcal{A}$ and by (9.2)

$$\mathbb{F}_o^*(\mathcal{A}) \setminus B_o = \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(L_o) = \Phi_{\mathcal{A}}(\mathbf{I} \setminus L_o). \qquad (9.4)$$

From (5.4), we have the obvious inclusion $\Phi_{\mathcal{A}}(\mathbf{I} \setminus L_o) \in (\mathbb{UF})[\mathbf{I}; \mathcal{A}]$. By (9.4)

$$\mathbb{F}_o^*(\mathcal{A}) \setminus B_o \in (\mathbb{UF})[\mathbf{I}; \mathcal{A}].$$

Therefore, we obtain the following property:

$$\begin{aligned} B_o &= \mathbb{F}_o^*(\mathcal{A}) \setminus (\mathbb{F}_o^*(\mathcal{A}) \setminus B_o) \\ &= \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(\mathbf{I} \setminus L_o) \in \mathbf{C}_{\mathbb{F}^*{}_o(\mathcal{A})}\left[(\mathbb{UF})[I; \mathcal{A}]\right]. \end{aligned}$$

The inclusion $(\mathbb{UF})[\mathbf{I}; \mathcal{A}] \subset \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[(\mathbb{UF})[\mathbf{I}; \mathcal{A}]\right]$ is established. Choose arbitrary

$$\Lambda \in \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[(\mathbb{UF})[\mathbf{I}; \mathcal{A}]\right]. \qquad (9.5)$$

Using (2.8), we choose $B^o \in (\mathbb{UF})[\mathbf{I}; \mathcal{A}]$ such that $\Lambda = \mathbb{F}_o^*(\mathcal{A}) \setminus B^o$. Let $L^o \in \mathcal{A}$ be the set for which $B^o = \Phi_{\mathcal{A}}(L^o)$; see (5.4). Then, by (9.2)

$$\Lambda = \mathbb{F}_o^*(\mathcal{A}) \setminus \Phi_{\mathcal{A}}(L^o) = \Phi_{\mathcal{A}}(\mathbf{I} \setminus L^o), \qquad (9.6)$$

where $\mathbf{I} \setminus L^o \in \mathcal{A}$. Since by (5.4) $\Phi_{\mathcal{A}}(I \setminus L^o) \in (\mathbb{UF})[\mathbf{I}; \mathcal{A}]$, from (9.6), we obtain that

$$\Lambda \in (\mathbb{UF})[\mathbf{I};\mathcal{A}].$$

Since the choice of Λ (9.5) was arbitrary, the inclusion

$$\mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[(\mathbb{UF})[\mathbf{I};\mathcal{A}]\right] \subset (\mathbb{UF})[\mathbf{I};\mathcal{A}]$$

is established. So, we obtain the required equality.

From (6.4), (8.8), and Proposition 9.1, the simple (but useful) statement follows.

Proposition 9.2. $\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] = \mathbf{T}_{\mathcal{A}}^o[\mathbf{I}]$.

So, for measurable spaces with algebras of sets, the topological representations of Sections 6 and 7, 8 realize the same topology. By (6.5), (7.13), and Proposition 9.2

$$\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] \in (c-\text{top})_o[\mathbb{F}_o^*(\mathcal{A})]. \quad (9.7)$$

So, we obtain a nonempty compactum. Recall that (see (7.11), Proposition 9.2)

$$\{\cap\}\left((\mathbb{UF})[\mathbf{I};\mathcal{A}]\right) = \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right] \quad (9.8)$$

is the family of all sets closed in the sense of topology (9.7). We note the following obvious property (see [15, ch.I])

$$(\mathbb{UF})[\mathbf{I};\mathcal{A}] \in (\text{alg})[\mathbb{F}_o^*(\mathcal{A})]. \quad (9.9)$$

Remark 9.2. We recall (5.4). Let $\Gamma \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]$. Using (5.4), we choose $\Lambda \in \mathcal{A}$ such that $\Gamma = \Phi_{\mathcal{A}}(\Lambda)$. Then, $\mathbf{I}\setminus\Lambda \in \mathcal{A}$ and by (9.2)

$$\mathbb{F}_o^*(\mathcal{A})\setminus\Gamma = \mathbb{F}_o^*(\mathcal{A})\setminus\Phi_{\mathcal{A}}(\Lambda) = \Phi_{\mathcal{A}}(\mathbf{I}\setminus\Lambda). \quad (9.10)$$

By (5.4) and (9.10) $\mathbb{F}_o^*(\mathcal{A})\setminus\Gamma \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]$. So, we establish that

$$\mathbb{F}_o^*(\mathcal{A})\setminus H \in (\mathbb{UF})[\mathbf{I};\mathcal{A}] \ \forall H \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]. \quad (9.11)$$

From (2.10), (5.4), and (9.11), the property (9.9) follows.

Proposition 9.3. $(\mathbb{UF})[\mathbf{I};\mathcal{A}] = \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] \cap \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right]$.

Proof. Recall that by statements of Section 2 and (9.8) the inclusion

$$(\mathbb{UF})[\mathbf{I};\mathcal{A}] \subset \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right]. \quad (9.12)$$

From (6.4), the inclusion $(\mathbb{UF})[\mathbf{I};\mathcal{A}] \subset \mathbf{T}_{\mathcal{A}}^*[E]$ follows too. So, by (9.12)

$$(\mathbb{UF})[\mathbf{I};\mathcal{A}] \subset \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] \cap \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right]. \quad (9.13)$$

Let $\Omega \in \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] \cap \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right]$. Since Ω is open, then by (6.4) we obtain that, for some family

$$\mathfrak{W} \in P\left((\mathbb{UF})[\mathbf{I};\mathcal{A}]\right), \quad (9.14)$$

the following equality is realized:

$$\Omega = \bigcup_{\widetilde{W}\in\mathfrak{W}} \widetilde{W}. \quad (9.15)$$

If $\mathfrak{W} = \varnothing$, then by (9.15) $\Omega = \varnothing$ and, as a corollary, $\Omega = \Phi_{\mathcal{A}}(\varnothing)$, where $\varnothing \in \mathcal{A}$. So, by (5.4) we obtain the implication

$$(\mathfrak{W} = \varnothing) \Rightarrow \left(\Omega \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]\right). \quad (9.16)$$

Let $\mathfrak{W} \neq \varnothing$. Then, $\mathfrak{W} \in P'\left((\mathbb{UF})[\mathbf{I};\mathcal{A}]\right)$. Since Ω is a closed subset of a compactum, we have the compactess property of Ω; then, by (9.14), for some $\mathbb{K} \in \text{Fin}(\mathfrak{W})$

$$\Omega = \bigcup_{\widetilde{W}\in\mathbb{K}} \widetilde{W}. \quad (9.17)$$

In particular, $\mathbb{K} \in \text{Fin}\left((\mathbb{UF})[\mathbf{I};\mathcal{A}]\right)$. We note that $(\mathbb{UF})[\mathbf{I};\mathcal{A}]$ is closed with respect to finite unions (indeed, by (9.9) $(\mathbb{UF})[\mathbf{I};\mathcal{A}]$ is an algebra of sets). Therefore, by (9.17) $\Omega \in (UF)[\mathbf{I};A]$ in the case $\mathfrak{W} \neq \varnothing$. So,

$$(\mathfrak{W} \neq \varnothing) \Rightarrow \left(\Omega \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]\right). \quad (9.18)$$

Using (9.16) and (9.18), we obtain that $\Omega \in (\mathbb{UF})[\mathbf{I};\mathcal{A}]$ in any possible cases. Since the choice of Ω was arbitrary, the inclusion

$$\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}] \cap \mathbf{C}_{\mathbb{F}_o^*(\mathcal{A})}\left[\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right] \subset (\mathbb{UF})[\mathbf{I};\mathcal{A}] \quad (9.19)$$

in established. From (9.13) and (9.18), the required statement follows.

So, $(\mathbb{UF})[\mathbf{I};\mathcal{A}]$ is the family of all open-closed sets in the nonempty compactum

$$\left(\mathbb{F}_o^*(\mathcal{A}), \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right) = \left(\mathbb{F}_o^*(\mathcal{A}), \mathbf{T}_{\mathcal{A}}^o[\mathbf{I}]\right). \quad (9.20)$$

In connection with the above-mentioned property of nonempty compactum (9.20), we recall [15, ch. I]. With the employment of (9.1), the following obvious property is established: in our case of measurable space with an algebra of sets

$$(\mathbf{I}-\text{ult})[x] \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A}) \ \forall x \in \mathbf{I}. \quad (9.21)$$

Remark 9.3. For a completeness, we consider the scheme of the proof of (9.21). For this, we note that by (3.9) and the corresponding definition of Section 5

$$(\mathbf{I}-\text{ult})[x] \cap \mathcal{L} \in \mathbb{F}^*(\mathcal{L}) \ \forall \mathcal{L} \in \pi[\mathbf{I}] \ \forall x \in \mathbf{I}. \quad (9.22)$$

In particular, by (9.22) $(\mathbf{I}-\text{ult})[x] \cap \mathcal{A} \in \mathbb{F}^*(\mathcal{A}) \ \forall x \in \mathbf{I}$. Fix $x_* \in \mathbf{I}$ and suppose that

$$\mathcal{F}_* := (\mathbf{I}-\text{ult})[x_*] \cap \mathcal{A};$$

of course, $\mathcal{F}_* \in \mathbb{F}^*(\mathcal{A})$. In addition, $\mathcal{A} \subset P(\mathbf{I})$. Then, $\forall \widetilde{A} \in \mathcal{A}$

$$(x_* \in \widetilde{A}) \vee (x_* \in \mathbf{I}\setminus\widetilde{A}). \quad (9.23)$$

Of course, by (3.9), for $\widetilde{A} \in \mathcal{A}$, we have the following obvious implications:

$$\left((x_* \in \widetilde{A}) \Rightarrow (\widetilde{A} \in \mathcal{F}_*)\right) \& \left((x_* \in \mathbf{I}\setminus\widetilde{A}) \Rightarrow (\mathbf{I}\setminus A \in \mathcal{F}_*)\right).$$

Then, by (9.23) $(\widetilde{A} \in \mathcal{F}_*) \vee (\mathbf{I}\setminus\widetilde{A} \in \mathcal{F}_*)$. Since the choice of $\widetilde{A}$ was arbitrary, by (9.1) $\mathcal{F}_* \in \mathbb{F}_o^*(\mathcal{A})$. So, (9.21) is established.

Using (9.21), we introduce the mapping

$$(\mathcal{A}-\text{ult})[\mathbf{I}] := \left((\mathbf{I}-\text{ult})[x] \cap \mathcal{A}\right)_{x\in\mathbf{I}} \in \mathbb{F}_o^*(\mathcal{A})^{\mathbf{I}}. \quad (9.24)$$

Of course, in (9.24) we have analog of the mapping $\mathbf{f}$ (8.12). But, in the given case, we realize the immersion of points of the initial set in the ultrafilter space under other conditions. We will use the specific character of measurable space with an algebra of sets. Now, we note the obvious property:

$$\left(\forall x \in \mathbf{I}\ \forall y \in \mathbf{I}\setminus\{x\}\ \exists \widehat{A} \in \mathcal{A} : (x \in \widehat{A})\,\&\,(y \notin \widehat{A})\right) \Rightarrow \left((\mathcal{A}-\text{ult})[\mathbf{I}] \in (\text{bi})\left[\mathbf{I};\ (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I})\right]\right). \quad (9.25)$$

In (9.25), the statement of the premise has the following sense: algebra $\mathcal{A}$ is distinguishing for points of $\mathbf{I}$.

If $\mathcal{J} \in P'(\mathcal{A})$, then by analogy with Section 4 we suppose that

$$\left(\mathbb{F}^*(\mathcal{A}\,|\,\mathcal{J}) := \left\{\mathcal{F} \in \mathbb{F}^*(\mathcal{A}) \,|\, \mathcal{J} \subset \mathcal{F}\right\}\right)\& \&\left(\mathbb{F}_o^*(\mathcal{A}\,|\,\mathcal{J}) := \left\{\mathcal{U} \in \mathbb{F}_o^*(\mathcal{A}) \,|\, \mathcal{J} \subset \mathcal{U}\right\}\right); \quad (9.26)$$

of course, $\mathbb{F}_o^*(\mathcal{A}|\mathcal{J}) \subset \mathbb{F}^*(\mathcal{A}|\mathcal{J})$ and moreover the following property is valid:

$$\forall \mathcal{F} \in \mathbb{F}^*(\mathcal{A}|\mathcal{J})\ \exists \mathcal{U} \in \mathbb{F}_o^*(\mathcal{A}|\mathcal{J}) : \mathcal{F} \subset \mathcal{U}. \quad (9.27)$$

Returning to (9.25), we note that

$$(\mathcal{A}-\text{ult})[\mathbf{I}]^{-1}\left(\mathbb{F}_o^*(\mathcal{A}|\mathcal{I})\right) = \bigcap_{A\in\mathcal{I}} A\ \ \forall \mathcal{I} \in P'(\mathcal{A}). \quad (9.28)$$

In (9.28), we can use $\mathcal{I}$ as constraints of asymptotic character. Of course, $\mathbf{F}_o^*(\mathcal{A}) \subset \mathbf{F}^*(\mathcal{A}) \subset \beta_{\mathcal{A}}^o[\mathbf{I}] \subset \beta_o[\mathbf{I}]$ (see Section 5). Then, by (3.3)

$$(\mathbf{I}-\mathbf{fi})[\mathcal{U}] = \{H \in P(\mathbf{I}) | \exists B \in \mathcal{U} : B \subset H\} \in \widehat{F}[\mathbf{I}]\ \ \forall \mathcal{U} \in \mathbb{F}_o^*(\mathcal{A}). \quad (9.29)$$

By analogy with (9.29) we note that $\mathbb{F}^*(\mathcal{A}) \subset \beta_o[\mathbf{I}]$ and $(\mathbf{I}-\mathbf{fi})[\mathcal{F}] \in \widehat{F}[\mathbf{I}]\ \forall \mathcal{F} \in \mathbb{F}^*(\mathcal{A})$. These properties permit realize an asymptotic analogs of solutions of the set (9.28). In this capacity, we can use elements of the sets $\mathbb{F}^*(\mathcal{A}\,|\mathcal{I})$ and $\mathbb{F}_o^*(\mathcal{A}\,|\mathcal{I})$, where $\mathcal{I} \in P'(\mathcal{A})$ is used as "asymptotic constraints". Of course, $\mathcal{A}$ bounds our possibilities: we can use only subfamilies of $\mathcal{A}$.

Proposition 9.4. $\mathbb{F}_o^*(\mathcal{A}) = \text{cl}\left((\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}), \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right)$.

Proof. Fix $\mathcal{F} \in \mathbb{F}_o^*(\mathcal{A})$. Let $\widehat{A} \in \mathcal{F}$. Then $\widehat{A} \in P'(\mathbf{I})$. So, $\widehat{A} \neq \varnothing$ and $\widehat{A} \subset \mathbf{I}$. Choose arbitrary $a \in \widehat{A}$. Then, by (9.24)

$$(\mathbf{I}-\text{ult})[a] \cap \mathcal{A} = (\mathcal{A}-\text{ult})[\mathbf{I}](a) \in (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}). \quad (9.30)$$

By the choice of a we have the inclusion $\widehat{A} \in (\mathbf{I}-\text{ult})[a]$. Since $\mathcal{F} \subset \mathcal{A}$, we obtain that $\widehat{A} \in \mathcal{A}$. Then, by (9.30) $\widehat{A} \in (\mathbf{I}-\text{ult})[a] \cap \mathcal{A}$. Since $(\mathbf{I}-\text{ult})[a] \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A})$, by (5.3)

$$(\mathbf{I}-\text{ult})[a] \cap \mathcal{A} \in \Phi_{\mathcal{A}}(\widehat{A}). \quad (9.31)$$

By (9.30) and (9.31) we obtain the following property

$$\Phi_{\mathcal{A}}(\widehat{A}) \cap (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}) \neq \varnothing.$$

Since the choice of $\widehat{A}$ was arbitrary, we have (see (8.3)) the statement

$$\Phi_{\mathcal{A}}(L) \cap (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}) \neq \varnothing\ \ \forall L \in \mathcal{F}. \quad (9.32)$$

Choose arbitrary $\Omega \in N_{\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]}(\mathcal{F})$. Then, for some $\Omega^o \in N^o_{\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]}(\mathcal{F})$, the inclusion $\Omega^o \subset \Omega$ is valid. Therefore, $\Omega^o \in \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]$ and $\mathcal{F} \in \Omega^o$. By (6.4), there exists $\Lambda \in \mathcal{F}$ such that

$$\Phi_{\mathcal{A}}(\Lambda) \subset \Omega^o. \quad (9.33)$$

From (9.32), the property $\Phi_{\mathcal{A}}(\Lambda) \cap (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}) \neq \varnothing$ is valid. By (9.33) we obtain that

$$\Omega \cap (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}) \neq \varnothing$$

(indeed, $\Phi_{\mathcal{A}}(\Lambda) \subset \Omega$). Since the choice of Ω was arbitrary,

$$S \cap (\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}) \neq \varnothing\ \ \forall S \in N_{\mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]}(\mathcal{F}).$$

Then, $\mathcal{F} \in \text{cl}\left((\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}), \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right)$. So, the inclusion

$$\mathbb{F}_o^*(\mathcal{A}) \subset \text{cl}\left((\mathcal{A}-\text{ult})[\mathbf{I}]^1(\mathbf{I}), \mathbf{T}_{\mathcal{A}}^*[\mathbf{I}]\right)$$

is established. The opposite inclusion is obvious.

We note that Proposition 9.4 is similar to Proposition 8.1. But, in the given section, the condition

$$\{x\} \in \mathcal{A}\ \ \forall x \in \mathbf{I} \quad (9.34)$$

is supposed not; in Section 8 (in particular, in Proposition 8.1), the condition similar to (9.34) is essential. So, Proposition 9.4 has the independent meaning.

10. Attraction Sets Under the Restriction in the Form of Algebra of Sets

In the following, we fix a nonempty set E, a TS $(\mathbf{H}, \tau)$, where $\mathbf{H} \neq \varnothing$, and a mapping $\mathbf{h} \in \mathbf{H}^E$. Elements $e \in E$ are considered as usual solutions and elements $y \in \mathbf{H}$ play the role of some estimates. The natural variant of an obtaining of y is realized in the form $y = \mathbf{h}(e)$, where $e \in E$. But, we admit the possibility of the limit realization of y. This is natural in questions of asymptotic analysis. In the last case, it is natural to use "asymptotic constraints" in the form of a nonempty subfamilies of $P(E)$. Then, we obtain constructions of Section 4 under $X = E$, $Y = \mathbf{H}$, and $f = \mathbf{h}$. But, we admit yet one possibility: along with "usual" AS, we use the sets

$$(\tau - \mathbb{AS})[\Sigma|\mathcal{A}] := \Big\{ y \in \mathbf{H} \mid \exists \mathcal{F} \in \mathbb{F}^*(\mathcal{A}|\Sigma) : \mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} y \Big\} \quad \forall \mathcal{A} \in (\mathrm{alg})[E] \ \forall \Sigma \in P'(\mathcal{A}). \tag{10.1}$$

Of course, we use remarks of the conclusion of the previous section.

Proposition 10.1. If $\mathcal{A} \in (\mathrm{alg})[E]$ and $\Sigma \in P'(\mathcal{A})$, then

$$(\tau - \mathbb{AS})[\Sigma|\mathcal{A}] = \Big\{ y \in \mathbf{H} | \exists \mathcal{U} \in \mathbb{F}_o^*(\mathcal{A}|\Sigma) : \mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} y \Big\}. \tag{10.2}$$

Proof. We use reasoning analogous to the proof of Proposition 4.2. We denote by Ω the set on the right side of (10.2). Since $\mathbb{F}_o^*(\mathcal{A}|\Sigma) \subset \mathbb{F}^*(\mathcal{A}|\Sigma)$ (see Section 9), by (10.1)

$$\Omega \subset (\tau - \mathbb{AS})[\Sigma|\mathcal{A}]. \tag{10.3}$$

Let $y_o \in (\tau - \mathbb{AS})[\Sigma|\mathcal{A}]$. Then, by (10.1) $y_o \in \mathbf{H}$ and, for some $\mathcal{F} \in \mathbb{F}^*(\mathcal{A}|\Sigma)$,

$$\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} y_o. \tag{10.4}$$

Recall that $\mathcal{F} \in \beta_o[E]$ (see Section 9). Therefore, by (4.1) $\mathbf{h}^1[\mathcal{F}] \in \beta_o[\mathbf{H}]$. Then, (10.4) denotes that

$$N_\tau(y_o) \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{F}]\Big] \tag{10.5}$$

(see (3.5)). In addition, by the choice of $\mathcal{F}$ we have the inclusion $\Sigma \subset \mathcal{F}$; see (9.26). By (9.27), for some $\mathfrak{U} \in \mathbb{F}_o^*(\mathcal{A}|\Sigma)$, the inclusion $\mathcal{F} \subset \mathfrak{U}$ is valid. Then,

$$\mathbf{h}^1[\mathcal{F}] \subset \mathbf{h}^1[\mathfrak{U}].$$

As a corollary, by(3.3) and (10.5)

$$N_\tau(y_o) \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{F}]\Big] \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathfrak{U}]\Big],$$

where $\mathbf{h}^1[\mathfrak{U}] \in \beta_o[\mathbf{H}]$ (see Section 9). Then, by (3.5)

$$\mathbf{h}^1[\mathfrak{U}] \overset{\tau}{\Rightarrow} y_o. \tag{10.6}$$

By definition of Ω we obtain that $y_o \in \Omega$. Since the choice of y_o was arbitrary, the inclusion

$$(\tau - \mathbb{AS})[\Sigma|\mathcal{A}] \subset \Omega \tag{10.7}$$

is established. Using (10.3) and (10.7), we obtain the required equality

$$(\tau - \mathbb{AS})[\Sigma|\mathcal{A}] = \Omega. \tag{10.8}$$

From the definition of Ω and (10.8), we obtain (10.2).

Recall that $P(E) \in (\mathrm{alg})[E]$ and therefore

$$(\tau - \mathbb{AS})\big[\Sigma|P(E)\big] \in P(\mathbf{H}) \ \forall \Sigma \in P'\big(P(E)\big).$$

By definitions of Section 3, (6.6), and (9.26) we obtain that

$$\widehat{F}^{\,o}_{\mathbf{u}}[E|\Sigma] = \mathbb{F}_o^*\big(P(E)|\Sigma\big) \ \forall \Sigma \in P'\big(P(E)\big). \tag{10.9}$$

From Propositions 4.2 and 10.1, we have (see (10.9)) the property:

$$(\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma] = (\tau - \mathbb{AS})\big[\Sigma|P(E)\big] \ \forall \Sigma \in P'\big(P(E)\big).$$

So, our new construction is coordinated with AS of Section 4. Moreover, under $\mathcal{A} \in (\mathrm{alg})[E]$, we can consider AS $(\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma]$ for $\Sigma \in P'(\mathcal{A})$.

Proposition 10.2. If $\mathcal{A} \in (\mathrm{alg})[E]$ and $\Sigma \in P'(\mathcal{A})$, then

$$(\tau - \mathbb{AS})[\Sigma|A] \subset (\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma]. \tag{10.10}$$

Proof. We use (6.8). Choose $y_* \in (\tau - \mathbb{AS})[\Sigma|\mathcal{A}]$. Then, $y_* \in \mathbf{H}$ and, for some $\mathcal{U}_* \in \mathbb{F}_o^*(\mathcal{A}|\Sigma)$, the convergence

$$\mathbf{h}^1[\mathcal{U}_*] \overset{\tau}{\Rightarrow} y_* \tag{10.11}$$

is valid. Then, $\mathcal{U}_* \in \mathbb{F}_o^*(\mathcal{A})$ and $\Sigma \subset \mathcal{U}_*$; see (9.26). By (6.8) for some $\mathcal{U}^* \in \widehat{F}^{\,o}_{\mathbf{u}}[E]$, the equality $\mathcal{U}_* = \mathcal{U}^* \cap \mathcal{A}$ is valid. Then, $\Sigma \subset \mathcal{U}^*$. As a corollary, $\mathcal{U}^* \in \widehat{F}^{\,o}_{\mathbf{u}}[E|\Sigma]$. Now, we return to (10.11). In addition, $\mathcal{U}_* \in \beta_o[E]$. Therefore, $\mathbf{h}^1[\mathcal{U}_*] \in \beta_o[\mathbf{H}]$ and by (3.3)

$$(\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{U}_*]\Big] \in \widehat{F}[\mathbf{H}].$$

From (3.5) and (10.11), we have the obvious inclusion

$$N_\tau(y_*) \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{U}_*]\Big]. \tag{10.12}$$

In addition, $\mathcal{U}^* \in \beta_o[E]$ and $\mathbf{h}^1[\mathcal{U}^*] \in \beta_o[\mathbf{H}]$; see (4. 1). Since $\mathcal{U}_* \subset \mathcal{U}^*$, the inclusion $\mathbf{h}^1[\mathcal{U}_*] \subset \mathbf{h}^1[\mathcal{U}^*]$ is valid. As a corollary, by (3.3)

$$(\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{U}_*]\Big] \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{U}^*]\Big].$$

Using (10.12), we obtain the basic inclusion

$$N_\tau(y_*) \subset (\mathbf{H} - \mathbf{fi})\Big[\mathbf{h}^1[\mathcal{U}^*]\Big]. \tag{10.13}$$

From (3.5) and (10.13), we obtain the following convergence

$$\mathbf{h}^1[\mathcal{U}^*] \overset{\tau}{\Rightarrow} y_*. \tag{10.14}$$

So, $\mathcal{U}^* \in \widehat{F}^{\,o}_{\mathbf{u}}[E|\Sigma]$ has the property (10.14). Then, by Proposition 4.2

$$y_* \in (\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma].$$

Since the choice of y_* was arbitrary, the required inclusion (10.10) is established.

So, by (10.1) and (10.2) some "partial" AS are defined. Of course, the case for which (10.10) is converted in a equality is very interesting. For investigation of this case, we consider auxiliary constructions. In the following, in this section, we fix $\mathcal{A} \in (\mathrm{alg})[E]$. So, $(E, \mathcal{A})$ is a measurable space with an algebra of sets. In this case, we can supplement the property (6.8). Namely,

$$\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A}) \ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]. \qquad (10.15)$$

Remark 10.1. We omit the sufficiently simple proof (10.15). Now, we are restricted to brief remarks. Namely, by ultrafilter $\mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$ we can realize a finitely additive (0,1)-measure μ on the family $P(E)$ supposing that $\mu(L):=1$ under $L \in \mathcal{U}$ and $\mu(\Lambda):=0$ under $\Lambda \in P(E) \setminus \mathcal{U}$. In connection with such possibility, we use [9,(7.6.17)] (moreover, see [9,(7.6.7)]). The natural narrowing v of μ on our algebra $\mathcal{A}$ is finitely additive (0,1)-measure on $\mathcal{A}$ (of course, $\nu = (\mu \,|\, \mathcal{A})$). Therefore, for some $\mathcal{V} \in \mathbb{F}_o^*(\mathcal{A})$, by [9,(7.6.17)] ν is defined by the rule

$$\big(\nu(A) = 1 \ \forall A \in \mathcal{V}\big) \& \big(\nu(\widetilde{A}) = 0 \ \forall \widetilde{A} \in \mathcal{A} \setminus \mathcal{V}\big). \qquad (10.16)$$

On the other hand, the family $\mathcal{U} \cap \mathcal{A}$ realizes ν by the obvious rule:

$$\big(\nu(\widehat{A}) = 1 \ \forall \widehat{A} \in \mathcal{U} \cap \mathcal{A}\big) \& \big(\nu(\widetilde{A}) = 0 \ \forall \widetilde{A} \in \mathcal{A} \setminus (\mathcal{U} \cap \mathcal{A})\big). \qquad (10.17)$$

From (10.16) and (10.17), the required equality $\mathcal{U} \cap \mathcal{A} = \mathcal{V}$ follows. Then, by the choice of $\mathcal{V}$ we have the inclusion $\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A})$.

Using (6.8) and (10.15), we obtain that

$$\mathbb{F}_o^*(\mathcal{A}) = \big\{\mathcal{U} \cap \mathcal{A} : \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]\big\}. \qquad (10.18)$$

By (10.18) we establish the natural connection of $\widehat{F}_{\mathbf{u}}[E]$ and $\mathbb{F}_o^*(\mathcal{A})$. Now, we consider some other auxiliary properties.

If $\mathcal{B} \in \beta_o[\mathbf{H}]$ and $z \in \mathbf{H}$, then we have the following equivalence

$$(\mathcal{B} \overset{\tau}{\Rightarrow} z) \Longleftrightarrow \big(N_\tau^o(z) \subset (\mathbf{H} - \mathbf{fi})[\mathcal{B}]\big). \qquad (10.19)$$

Of course, we can use instead of $\mathcal{B}$ the corresponding image of a filter base in E. Indeed, by (4.1) and (10.19) $\forall \mathcal{B} \in \beta_o[E] \ \forall z \in \mathbf{H}$

$$(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z) \Longleftrightarrow \big(N_\tau^o(z) \subset (\mathbf{H} - \mathbf{fi})\big[\mathbf{h}^1[\mathcal{B}]\big]\big). \qquad (10.20)$$

Moreover, in connection with (10.20), we note that $\forall \mathcal{B} \in \beta_o[E] \ \forall z \in \mathbf{H}$

$$(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z) \Longleftrightarrow \big(\mathbf{h}^{-1}\big[N_\tau^o(z)\big] \subset (E - \mathbf{fi})[\mathcal{B}]\big). \qquad (10.21)$$

Remark 10.2. Consider the proof of (10.21). Fix $\mathcal{B} \in \beta_o[E]$ and $z \in \mathbf{H}$. Let $\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z$. Then, by (10.20)

$$N_\tau^o(z) \subset (\mathbf{H} - \mathbf{fi})\big[\mathbf{h}^1[\mathcal{B}]\big].$$

Therefore, for any $G_* \in N_\tau^o(z)$, there exists $B_* \in \mathcal{B}$ such that $\mathbf{h}^1(B_*) \subset G_*$. As a corollary,

$$B_* \subset \mathbf{h}^{-1}\big(\mathbf{h}^1(B_*)\big) \subset \mathbf{h}^{-1}(G_*).$$

Then, $\mathbf{h}^{-1}(G_*) \in (E - \mathbf{fi})[\mathcal{B}]$. Since the choice of G_* was arbitrary,

$$\mathbf{h}^{-1}\big[N_\tau^o(z)\big] \subset (E - \mathbf{fi})[\mathcal{B}].$$

So, $\big(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z\big) \Rightarrow \big(\mathbf{h}^{-1}\big[N_\tau^o(z)\big] \subset (E - \mathbf{fi})[\mathcal{B}]\big)$. Let

$$\mathbf{h}^{-1}\big[N_\tau^o(z)\big] \subset (E - \mathbf{fi})[\mathcal{B}]. \qquad (10.22)$$

Choose arbitrary neighborhood $G^* \in N_\tau^o(z)$. Then, by (10.22) $\mathbf{h}^{-1}(G^*) \in (E - \mathbf{fi})[\mathcal{B}]$. Therefore, for some $B^* \in \mathcal{B}$, the inclusion $B^* \subset \mathbf{h}^{-1}(G^*)$ is valid. In addition, $\mathbf{h}^1(B^*) \in \mathbf{h}^1[\mathcal{B}]$ and

$$\mathbf{h}^1(B^*) \subset \mathbf{h}^1\big(\mathbf{h}^{-1}(G^*)\big) \subset G^*.$$

Then, $G^* \in (\mathbf{H} - \mathbf{fi})\big[\mathbf{h}^1[\mathcal{B}]\big]$. Therefore, $N_\tau^o(z) \subset (\mathbf{H} - \mathbf{fi})\big[\mathbf{h}^1[\mathcal{B}]\big]$ and by (10.20) $\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z$. So,

$$\big(\mathbf{h}^{-1}\big[N_\tau^o(z)\big] \subset (E - \mathbf{fi})[\mathcal{B}]\big) \Rightarrow \big(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z\big).$$

The proof of (10.21) is completed.

We note that, in (10.21), we can use instead of $\mathcal{B}$ arbitrary filter of $(E, \mathcal{A})$. In this connection, we recall that by constructions of Section 5, for any $\mathcal{F} \in \mathbb{F}^*(\mathcal{A})$, we obtain (in particular) that $\mathcal{F} \in \beta_o[E]$ and

$$(E - \mathbf{fi})[\mathcal{F}] \cap \mathcal{A} = (E - \mathbf{fi})[F \,|\, \mathcal{A}] = \mathcal{F}. \qquad (10.23)$$

Then, from (10.21) and (10.23), we have the following property: $\forall \mathcal{F} \in \mathbb{F}^*(\mathcal{A}) \ \forall z \in \mathbf{H}$

$$(\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} z) \Longleftrightarrow \big(N_\tau^o(z) \subset (\mathbf{H} - \mathbf{fi})\big[\mathbf{h}^1[\mathcal{F}]\big]\big). \qquad (10.24)$$

Of course, (10.24) is the particular case of (10.21); in (10.23), we have the useful addition. We note that $\forall \mathcal{B} \in \beta_o[\mathbf{H}] \ \forall z \in \mathbf{H} \ \forall Z \in (z - \mathrm{bas})[\tau]$

$$(\mathcal{B} \overset{\tau}{\Rightarrow} z) \Longleftrightarrow \big(Z \subset (\mathbf{H} - \mathbf{fi})[\mathcal{B}]\big). \qquad 10.25)$$

Remark 10.3. Fix $B \in \beta_o[\mathbf{H}]$, $z \in \mathbf{H}$, and $Z \in (z - \mathrm{bas})[\tau]$. Consider the proof of (10.25). By (3.4) and (3.5) we have the following implication

$$(\mathcal{B} \overset{\tau}{\Longrightarrow} z) \Rightarrow \big(Z \subset (\mathbf{H} - \mathbf{fi})[\mathcal{B}]\big). \qquad (10.26)$$

Let $Z \subset (\mathbf{H} - \mathbf{fi})[\mathcal{B}]$. Choose arbitrary $S \in N_\tau(z)$. Then, by (2.18), for some $\widetilde{Z} \in Z$, the inclusion $\widetilde{Z} \subset S$ is valid. Since $\widetilde{Z} \in (\mathbf{H} - \mathbf{fi})[\mathcal{B}]$, by filter axioms (see 3.1)) $S \in (\mathbf{H} - \mathbf{fi})[\mathcal{B}]$. So, the inclusion $N_\tau(z) \subset (\mathbf{H} - \mathbf{fi})[\mathcal{B}]$ is established. By (3.5) we have the convergence $\mathcal{B} \overset{\tau}{\Rightarrow} z$. So,

$$\left(Z \subset (\mathbf{H}-\mathbf{fi})[\mathcal{B}]\right) \Rightarrow (\mathcal{B} \overset{\tau}{\Rightarrow} z).$$

Now, with the employment of (10.26), we obtain (10.25).

We note the following obvious corollary of (10.25) (in this connection, we recall (10.21)): $\forall \mathcal{B} \in \beta_o[E]\ \forall z \in \mathbf{H}$

$$\left(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z\right) \Longleftrightarrow \left(\exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset (E-\mathbf{fi})[\mathcal{B}]\right). \quad (10.27)$$

Remark 10.4. Consider the proof of (10.27). We fix $\mathcal{B} \in \beta_o[E]$ and $z \in \mathbf{H}$. Since $N_\tau^o(z) \in (z-\mathrm{bas})[\tau]$ (see (3.4) and definitions of Section 3), by (10.21)

$$\left(\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z\right) \Rightarrow \left(\exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset (E-\mathbf{fi})[\mathcal{B}]\right). \quad (10.28)$$

Let the corollary of (10.28) is valid. Fix $\mathfrak{Z} \in (z-\mathrm{bas})[\tau]$ with the property

$$\mathbf{h}^{-1}[\mathfrak{Z}] \subset (E-\mathbf{fi})[\mathcal{B}]. \quad (10.29)$$

Let $\mathbb{G} \in N_\tau^o(z)$. Then, by (3.4), for some $\mathbb{B} \in \mathfrak{Z}$, the inclusion $\mathbb{B} \subset \mathbb{G}$ is valid, where $\mathbf{h}^{-1}(\mathbb{B}) \in \mathbf{h}^{-1}[\mathfrak{Z}]$. By (10.29) $\mathbf{h}^{-1}(\mathbb{B}) \in (E-\mathbf{fi})[\mathcal{B}]$ and $\mathbf{h}^{-1}(\mathbb{B}) \subset \mathbf{h}^{-1}(\mathbb{G})$. From (3.1) and (3.3), the inclusion $\mathbf{h}^{-1}(\mathbb{G}) \in (E-\mathbf{fi})[\mathcal{B}]$ follows. Since the choice of $\mathbb{G}$ was arbitrary, the inclusion

$$\mathbf{h}^{-1}\left[N_\tau^o(z)\right] \subset (E-\mathbf{fi})[\mathcal{B}]$$

is established. By (10.21) $\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z$. So, we obtain that

$$\left(\exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset (E-\mathbf{fi})[\mathcal{B}]\right) \Rightarrow (\mathbf{h}^1[\mathcal{B}] \overset{\tau}{\Rightarrow} z).$$

Using the last implication and (10.28), we obtain the required property (10.27).

Using (10.15), we obtain the obvious corollary of (10.27): $\forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]\ \forall z \in \mathbf{H}$

$$\left(\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} z\right) \Leftrightarrow \left(\exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset (E-\mathbf{fi})[\mathcal{U} \cap \mathcal{A}]\right). \quad (10.30)$$

Remark 10.5. Consider the proof of (10.30), fixing $\mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$ and $z \in \mathbf{H}$. Then, by (10.15) $\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A})$. In particular (see Section 9), $\mathcal{U} \cap \mathcal{A} \in \beta_o[E]$. Now, (10.30) follows from (10.27).

Condition 10.1. $\forall z \in \mathbf{H}\ \exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset \mathcal{A}$.

Remark 10.6. It is possible to consider Condition 10.1 as a weakened variant of the measurability of $\mathbf{h}$. The usual measurability of $\mathbf{h}$ is not natural since $\mathcal{A}$ is only algebra of sets.

Until the end of the present section, we suppose that Condition 10.1 is valid.

Proposition 10.3. If Condition 10.1 is fulfilled, then $(\tau-\mathbb{AS})[\Sigma|A] = (\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma]\ \forall \Sigma \in P'(\mathcal{A})$.

Proof. Let Condition 10.1 be fulfilled. Fix $\Sigma \in P'(\mathcal{A})$ and $z \in (\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma]$. Then, $z \in \mathbf{H}$ and, for some $\mathcal{U} \in \widehat{F}_{\mathbf{u}}^o[E|\Sigma]$

$$\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} z \quad (10.31)$$

(see Proposition 4.2). Then, by (10.21) and (10.31) we have the inclusion $\mathbf{h}^{-1}\left[N_\tau^o(z)\right] \subset \mathcal{U}$, since $(E-\mathbf{fi})[\mathcal{U}] = \mathcal{U}$ by (3.1). As a corollary,

$$\mathbf{h}^{-1}\left[N_\tau(z)\right] \subset \mathcal{U}$$

(indeed, for $H \in N_\tau(z)$, we can choose $G \in N_\tau^o(z)$ such that $G \subset H$; therefore, by (2.1) $\mathbf{h}^{-1}(G) \in \mathcal{U}$, $\mathbf{h}^{-1}(G) \subset \mathbf{h}^{-1}(H)$, and by (3.1) $\mathbf{h}^{-1}(H) \in \mathcal{U}$). By Condition 10.1 there exists $\widetilde{Z} \in (z-\mathrm{bas})[\tau]$ such that $\mathbf{h}^{-1}[\widetilde{Z}] \subset \mathcal{A}$. In addition,

$$\mathbf{h}^{-1}[\widetilde{Z}] \subset \mathbf{h}^{-1}\left[N_\tau(z)\right] \subset \mathcal{U}; \quad (10.32)$$

therefore, $\mathbf{h}^{-1}[\widetilde{Z}] \subset \mathcal{U} \cap \mathcal{A}$, where by (10.15) $\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A})$. We recall that $\mathcal{U} \cap \mathcal{A} \in \beta_o[E]$ (see Section 9) and

$$\mathbf{h}^{-1}[\widetilde{Z}] \subset \mathcal{U} \cap \mathcal{A} \subset (E-\mathbf{fi})[\mathcal{U} \cap \mathcal{A}]. \quad (10.33)$$

Of course, by (4.1) $\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \in \beta_o[\mathbf{H}]$. In addition, by (10.33)

$$\exists Z \in (z-\mathrm{bas})[\tau]: \mathbf{h}^{-1}[Z] \subset (E-\mathbf{fi})[\mathcal{U} \cap \mathcal{A}].$$

By (10.27) $\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} z$. Recall that $\Sigma \subset \mathcal{U}$. Since $\Sigma \in P'(\mathcal{A})$, we obtain that $\Sigma \subset \mathcal{U} \cap \mathcal{A}$. Therefore (see (9.26)),

$$\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A}|\Sigma): \mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} z.$$

By Proposition 10.1 $z \in (\tau-\mathbb{AS})[\Sigma|\mathcal{A}]$. Since the choice of z was arbitrary, we have the inclusion

$$(\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma] \subset (\tau-\mathbb{AS})[\Sigma|\mathcal{A}]. \quad (10.34)$$

Using (10.34) and Proposition 10.2, we obtain the equality

$$(\tau-\mathbb{AS})[\Sigma|\mathcal{A}] = (\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};E].$$

So, we can use (see Proposition 10.1 and Condition 10.1) ultrafilters of the space $(E,\mathcal{A})$ as nonsequential approximate solutions in the case, when a nonempty subfamily of $\mathcal{A}$ is used as the constraint of asymptotic character. This property is very useful in the cases of spaces $(E,\mathcal{A})$ for which the set $\mathbb{F}_o^*(\mathcal{A})$ is realized effectively. In addition, for a semialgebra $\mathcal{L} \in \Pi[E]$ with the property $\mathcal{A} = a_E^o(\mathcal{L})$ (see Section 2), we consider the passage

$$\mathbb{F}_o^*(\mathcal{L}) \longrightarrow \mathbb{F}_o^*(\mathcal{A})$$

as an unessential transformation (see [9,§7.6] and [16,§ 2.4]; here it is appropriate to use the natural connection of ultrafilters and finitely additive (0,1)-measures). Then, after unessential transformations, the examples of [9,§ 7.6] can be used in our scheme sufficiently constructively.

11. Ultrasolutions

First, we recall some statements of [17]. In addition, we fix a nonempty set E and a TS $(\mathbf{H},\tau)$, where $\mathbf{H}\neq\varnothing$. We consider the nonempty set $\widehat{F}_{\mathbf{u}}[E]$. Suppose that $\mathbf{h}\in\mathbf{H}^E$. Then, we suppose that

$$(\mathbf{h}-\mathrm{LIM})\left[\mathcal{U}\,|\,\tau\right]:=\left\{z\in\mathbf{H}\,|\,\mathbf{h}^1[\mathcal{U}]\overset{\tau}{\Rightarrow}z\right\}\forall\mathcal{U}\in\widehat{F}_{\mathbf{u}}[E].\quad(11.1)$$

So, we introduce the limit sets corresponding to ultrafilters of E. By analogy with Proposition 5.4 of [17] the following statement is established.

Proposition 11.1. If $\tau\in(c-\mathrm{top})[\mathbf{H}]$, then $(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\in P'\left(\mathrm{cl}(\mathbf{h}^1(E),\tau)\right)\ \forall\mathcal{U}\in\widehat{F}_{\mathbf{u}}[E]$.

Proof. Fix $\mathcal{U}\in\widehat{F}_{\mathbf{u}}[E]$. Then $\mathbf{h}^1[\mathcal{U}]\in\beta_o[\mathbf{H}]$ and by (4.2)

$$\mathcal{K}:=(\mathbf{H}-\mathbf{fi})[\mathbf{h}^1[\mathcal{U}]]\in\widehat{F}_{\mathbf{u}}[\mathbf{H}]\qquad(11.2)$$

(recall that $(E-\mathbf{fi})[\mathcal{U}]=\mathcal{U}$). Since $(\mathbf{H},\tau)$ is a compact TS, there exists $y\in\mathbf{H}$ such that $\mathcal{K}\overset{\tau}{\Rightarrow}y$; see [9, ch. I]. Then, by (3.6) $N_\tau(y)\subset\mathcal{K}$ or

$$N_\tau(y)\subset(\mathbf{H}-\mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right]\qquad(11.3)$$

(see (11.2)). By (3.5) and (11.3) $\mathbf{h}^1[\mathcal{U}]\overset{\tau}{\Rightarrow}y$. Then, by (11.1) $y\in(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]$. So,

$$(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\neq\varnothing.\qquad(11.4)$$

Let $z\in(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]$. Then $z\in\mathbf{H}$ and $\mathbf{h}^1[\mathcal{U}]\overset{\tau}{\Rightarrow}z$. By (3.5) and (11.2) $N_\tau(z)\subset\mathcal{K}$. In addition, by (11.2) $\mathbf{h}^1[\mathcal{U}]\subset\mathcal{K}$. Then, by (3.1)

$$A\cap B\neq\varnothing\ \ \forall A\in N_\tau(z)\ \ \forall B\in\mathbf{h}^1[\mathcal{U}].$$

By (2.1) we obtain that

$$\mathbf{h}^1(\widetilde{U})\cap H\neq\varnothing\ \ \forall\widetilde{U}\in\mathcal{U}\ \ \forall H\in N_\tau(z).$$

Since $\widetilde{U}\neq\varnothing$ and $\mathbf{h}^1(\widetilde{U})\subset\mathbf{h}^1(E)$ for $\widetilde{U}\in\mathcal{U}$, we have the property:

$$\mathbf{h}^1(E)\cap H\neq\varnothing\ \ \forall H\in N_\tau(z).$$

So, $z\in\mathrm{cl}\left(\mathbf{h}^1(E),\tau\right)$. The inclusion

$$(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\subset\mathrm{cl}\left(\mathbf{h}^1(E),\tau\right)$$

is established. Using (11.4), we obtain that $(\mathbf{h}-\mathrm{LIM})[\mathcal{U}|\tau]\in P'\left(\mathrm{cl}(\mathbf{h}^1(E),\tau)\right)$.

We note the following obvious property too: if $\tau\in(\mathrm{top})_o[\mathbf{H}],\mathcal{F}\in\widehat{F}[\mathbf{H}],y_1\in\mathbf{H}$, and $y_2\in\mathbf{H}$, then

$$\left((\mathcal{F}\overset{\tau}{\Rightarrow}y_1)\,\&\,(\mathcal{F}\overset{\tau}{\Rightarrow}y_2)\right)\Rightarrow(y_1=y_2).\quad(11.5)$$

Remark 11.1. Let the premise of (11.5) be fulfilled. Then, by (3.6)

$$\left(N_\tau(y_1)\subset\mathcal{F}\right)\&\left(N_\tau(y_2)\subset\mathcal{F}\right).\qquad(11.6)$$

Then, $y_1=y_2$. Indeed, suppose the contrary: $y_1\neq y_2$. Then, by (6.1), for some $H_1\in N_\tau(y_1)$ and $H_2\in N_\tau(y_2)$, the equality $H_1\cap H_2=\varnothing$ is valid. But, by (11.6) $H_1\in\mathcal{F}$ and $H_2\in\mathcal{F}$. Then, by (3.1) $H_1\cap H_2\neq\varnothing$. The obtained contradiction means that $y_1\neq y_2$ is impossible. So, $y_1=y_2$.

Proposition 11.2. If $\tau\in(c-\mathrm{top})_o[\mathbf{H}]$ and $\mathcal{U}\in\hat{F}_{\mathbf{u}}[E]$, then

$$\exists!\tilde{z}\in\mathbf{H}:(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]=\{\tilde{z}\}.$$

Proof. The corresponding proof is the obvious combination of (11.1), (11.5), and Proposition 11.1. Indeed, by Proposition 11.1 $(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\neq\varnothing$ and $(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\subset\mathbf{H}$. Let $y\in(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]$. Then, $y\in\mathbf{H}$ and

$$\mathbf{h}^1[\mathcal{U}]\overset{\tau}{\Rightarrow}y.\qquad(11.7)$$

Let $z\in(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]$. Then, $z\in\mathbf{H}$ and

$$\mathbf{h}^1[\mathcal{U}]\overset{\tau}{\Rightarrow}z.\qquad(11.8)$$

For $\mathcal{H}:=(\mathbf{H}-\mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right]\in\hat{F}_{\mathbf{u}}[\mathbf{H}]$, by (11.7) and (11.8)

$$\left(N_\tau(y)\subset\mathcal{H}\right)\&\left(N_\tau(z)\subset\mathcal{H}\right).$$

So, by (3.6) $\mathcal{H}\overset{\tau}{\Rightarrow}y$ and $\mathcal{H}\overset{\tau}{\Rightarrow}z$. From (11.5) the equality $y=z$ is valid. Then, $z\in\{y\}$. The inclusion

$$(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]\subset\{y\}\qquad(11.9)$$

is established. But, by the choice of y we have the inclusion $\{y\}\subset(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]$. Using (11.9), we obtain that

$$(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]=\{y\}.$$

The uniqueness of y is obvious.

From Proposition 11.2 the natural corollary follows: if $\tau\in(c-\mathrm{top})_o[\mathbf{H}]$, then

$$\exists!g\in\mathbf{H}^{\widehat{F}_{\mathbf{u}}[E]}:(\mathbf{h}-\mathrm{LIM})[\mathcal{U}\,|\,\tau]=\{g(\mathcal{U})\}\ \ \forall\mathcal{U}\in\widehat{F}_{\mathbf{u}}[E].\qquad(11.10)$$

In the following, we postulate that

$$\tau\in(c-\mathrm{top})_o[\mathbf{H}].\qquad(11.11)$$

Then (see (11.10) and (11.11)), we suppose that

$$H_o[\tau]\in\mathbf{H}^{\widehat{F}_{\mathbf{u}}[E]}\qquad(11.12)$$

is defined by the following rule: if $U\in\widehat{F}_{\mathbf{u}}[E]$, then

$H_o[\tau](\mathcal{U}) \in \mathbf{H}$ has the property:

$$(\mathbf{h} - \mathrm{LIM})[\mathcal{U} \mid \tau] = \{H_o[\tau](\mathcal{U})\}. \quad (11.13)$$

We note that by (11.13) and Proposition 11.1

$$H_o[\tau](\mathcal{U}) \in \mathrm{cl}\left(\mathbf{h}^1(E), \tau\right) \ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E] \quad (11.14)$$

So, by (11.12) and (11.14) $H_o[\tau] : \widehat{F}_{\mathbf{u}}[E] \to \mathrm{cl}\left(\mathbf{h}^1(E), \tau\right)$. In this connection, we note the following typical situation: under condition (11.11), $\mathbf{h}^1(E) \neq \mathbf{H}$ and $\mathbf{h}^1(E) \notin (\tau - \mathrm{comp})[\mathbf{H}]$. Of course, by (11.11)

$$\mathrm{cl}\left(\mathbf{h}^1(E), \tau\right) \in (\tau - \mathrm{comp})[\mathbf{H}].$$

Indeed, any closed set in a compact TS is compact too. Recall that

$$H_o[\tau] \in \mathrm{cl}\left(\mathbf{h}^1(E), \tau\right)^{\widehat{F}_{\mathbf{u}}[E]}. \quad (11.15)$$

Returning to (11.1) and (11.13) we note that

$$\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} H_o[\tau](\mathcal{U}) \ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]. \quad (11.16)$$

With the employment of (3.9), we introduce the natural immersion of E in $\widehat{F}_{\mathbf{u}}[E]$ supposing that

$$(E - \mathrm{ult})[\cdot] := \left((E - \mathrm{ult})[x]\right)_{x \in E} \in \widehat{F}_{\mathbf{u}}[E]^E. \quad (11.17)$$

In connection with (11.17), we note the following obvious equality:

$$\mathbf{h} = H_o[\tau] \circ (E - \mathrm{ult})[\cdot]. \quad (11.18)$$

Remark 11.2. Consider the proof of (11.18). Fix $x \in E$. By (3.9) we obtain that

$$\mathbf{h}(x) \in \mathbf{h}^1(S) \ \forall S \in (E - \mathrm{ult})[x]. \quad (11.19)$$

So, by (2.1) and (11.19) we obtain the following property:

$$\mathbf{h}(x) \in T \ \forall T \in \mathbf{h}^1\left[(E - \mathrm{ult})[x]\right]. \quad (11.20)$$

In addition, by (3.5), (3.9), and (11.16)

$$N_\tau\left(H_o[\tau]\left((E - \mathrm{ult})[x]\right)\right) \subset$$
$$\subset (\mathbb{H} - \mathbf{fi})\left[\mathbf{h}^1\left[(E - ult)[x]\right]\right].$$

So, $\forall \Gamma \in N_\tau\left(H_o[\tau]\left((E - \mathrm{ult})[x]\right)\right) \ \exists T \in \mathbf{h}^1\left[(E - \mathrm{ult})[x]\right] : T \subset \Gamma$. Then, by (11.20) $\mathbf{h}(x) \in \Gamma \ \forall \Gamma \in N_\tau\left(H_o[\tau]\left((E - \mathrm{ult})[x]\right)\right)$. Using the separability of τ (11.11), we obtain the equality chain

$$\mathbf{h}(x) = H_o[\tau]\left((E - \mathrm{ult})[x]\right)$$
$$= \left(H_o[\tau] \circ (E - \mathrm{ult})[\cdot]\right)(x).$$

Since the choice of x was arbitrary, we obtain that (11.18) is fulfilled.

Proposition 11.3. If $\mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$, then the following equality is valid:

$$\bigcap_{A \in \mathcal{U}} \mathrm{cl}\left(\mathbf{h}^1(A), \tau\right) = \{H_o[\tau](\mathcal{U})\}.$$

Proof. Let $u := H_o[\tau](\mathcal{U})$. Then $u \in \mathbf{H}$ and by (11.16)

$$\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} u.$$

Then, by (3.5) $N_\tau(u) \subset (\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right]$. Therefore, for any $T \in N_\tau(u)$, there exists $\widetilde{U} \in \mathcal{U}$ such that $\mathbf{h}^1(\widetilde{U}) \subset T$ (see (3.3) and (4.1)).

Let $A_* \in \mathcal{U}$. Then, $\mathbf{h}^1(A_*) \in P(\mathbf{H})$. If $S \in N_\tau(u)$, then, for some $U_S \in \mathcal{U}$, the inclusion $\mathbf{h}^1(U_S) \subset S$ is valid; moreover, $A_* \cap U_S \neq \varnothing$ and

$$\mathbf{h}^1(A_* \cap U_S) \subset \mathbf{h}^1(A_*) \cap \mathbf{h}^1(U_S) \subset \mathbf{h}^1(A_*) \cap S, \quad (11.21)$$

where $\mathbf{h}^1(A_* \cap U_S) \neq \varnothing$. So, $\mathbf{h}^1(A_*) \cap S \neq \varnothing$. Since the choice of S was arbitrary, we obtain that

$$\mathbf{h}^1(A_*) \cap H \neq \varnothing \ \forall H \in N_\tau(u).$$

Therefore, $u \in \mathrm{cl}\left(\mathbf{h}^1(A_*), \tau\right)$. Since the choice of A_* was arbitrary too, we have the inclusion $u \in \bigcap_{A \in \mathcal{U}} \mathrm{cl}\left(\mathbf{h}^1(A), \tau\right)$. Therefore,

$$\{u\} \subset \bigcap_{A \in \mathcal{U}} \mathrm{cl}\left(\mathbf{h}^1(A), \tau\right).$$

Choose arbitrary $q \in \bigcap_{A \in \mathcal{U}} \mathrm{cl}\left(\mathbf{h}^1(A), \tau\right)$. Then $q \in \mathbf{H}$ and

$$\widetilde{W} \cap \mathbf{h}^1(A) \neq \varnothing \ \forall A \in \mathcal{U} \ \forall \widetilde{W} \in N_\tau(q). \quad (11.22)$$

Then, for $W \in N_\tau(q)$, we obtain the property $W \cap T \neq \varnothing \ \forall T \in \mathbf{h}^1[\mathcal{U}]$. Using (3.3), we have the following statement:

$$W \cap M \neq \varnothing \ \forall M \in (\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right].$$

Therefore, $W \in \left((\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right] - \mathrm{set}\right)[\mathbf{H}]$ (see Section 5), where

$$(\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right] \in \widehat{F}_{\mathbf{u}}[\mathbf{H}] \quad (11.23)$$

(we use (4.2)). In addition, using (6.6), (11.23), and statements of Section 5, we obtain that

$$\left((\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right] - \mathrm{set}\right)[\mathbf{H}]$$
$$= \left((\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right] - \mathrm{set}\right)[\mathbf{H}] \cap P(\mathbf{H})$$
$$= (\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right].$$

Therefore, $W \in (\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right]$. Since the choice of W was arbitrary, we obtain that

$$N_\tau(q) \subset (\mathbf{H} - \mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right].$$

So, by (3.5) $\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} q$. Then, we have the following properties:

$$\left(\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} u\right) \& \left(\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Longrightarrow} q\right).$$

By (11.5) $u = q$. Then $q \in \{u\}$. Since the choice of q was arbitrary, we obtain that

$$\bigcap_{A \in \mathcal{U}} \mathrm{cl}\left(\mathbf{h}^1(A), \tau\right) \subset \{u\}.$$

The opposite inclusion was established previously. Therefore, $\{u\}$ and the intersection of all sets $\mathrm{cl}\left(\mathbf{h}^1(A), \tau\right)$, $A \in \mathcal{U}$, coincide.

From (4.15) and Proposition 11.3 we obtain that

$$(\mathbf{as})[E; \mathbf{H}; \tau; \mathbf{h}; \mathcal{U}] = \{H_o[\tau](\mathcal{U})\} \quad \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]. \quad (11.24)$$

So, ultrafilters of E realize very perfect constraints of asymptotic character.

12. Ultrafilters of Measurable Space with Algebra of Sets

In this section, we fix a nonempty set E, TS $(\mathbf{H}, \tau), \mathbf{H} \neq \varnothing$, and $\mathbf{h} \in \mathbf{H}^E$. Moreover, we fix $\mathcal{A} \in (\mathrm{alg})[E]$. Finally, we suppose that Condition 10.1 is fullfiled. Then, we have the statement of Proposition 3 and other statements of Section 10. We suppose that (11.11) is valid also. So, we have the mapping (11.12). In addition, we have the natural uniqueness of the filter limit: (11.5) is fulfilled. Now, we supplement (11.5). Namely, $\forall \mathcal{B} \in \beta_o[\mathbf{H}]\ \forall y_1 \in \mathbf{H}\ \forall y_2 \in \mathbf{H}$

$$\left((\mathcal{B} \overset{\tau}{\Rightarrow} y_1) \,\&\, (\mathcal{B} \overset{\tau}{\Rightarrow} y_2)\right) \Rightarrow (y_1 = y_2). \quad (12.1)$$

Remark 12.1. For the proof of (12.1), we fix $\mathcal{B} \in \beta_o[\mathbf{H}], y_1 \in \mathbf{H}$, and $y_2 \in \mathbf{H}$. Let the premise statement of (12.1) is valid: $\mathcal{B}$ converges to y_1 and y_2. Then, for $\mathcal{F} := (\mathbf{H} - \mathbf{fi})[\mathcal{B}] \in \mathfrak{F}_{\mathbf{u}}[\mathbf{H}]$ (see (3.3)), the inclusions

$$\left(N_\tau(y_1) \subset \mathcal{F}\right) \& \left(N_\tau(y_2) \subset \mathcal{F}\right)$$

are fulfilled. Therefore, by (3.6) the following two properties are valid:

$$(\mathcal{F} \overset{\tau}{\Rightarrow} y_1) \,\&\, (\mathcal{F} \overset{\tau}{\Rightarrow} y_2).$$

By (11.5) $y_1 = y_2$. So, (12.1) is established.

We recall (10.23) and (10.24): if $\mathcal{F} \in \mathbb{F}^*(\mathcal{A})$, then $\mathcal{F} \in \beta_o[E]$ and $\mathbf{h}^1[\mathcal{F}] \in \beta_o[\mathbf{H}]$ (see (4.1)). We use (10.15).

Proposition 12.1. $\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} H_o[\tau](\mathcal{U})\ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$.

Proof. Let $\mathcal{U} \in \widehat{F}[E]$ and $z := H_o[\tau](\mathcal{U})$. Then $z \in \mathbf{H}$ and by Condition 10.1, for some $Z \in (z - \mathrm{bas})[\tau]$, the inclusion $\mathbf{h}_\tau^{-1}[Z] \subset \mathcal{A}$ is fulfilled. In addition, by (11.16) $\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} z$. Since $\mathcal{U} \in \beta_o[E]$ and $\mathcal{U} = (E - \mathbf{fi})[\mathcal{U}]$, then

$$\mathbf{h}^{-1}\left[N_\tau^o(z)\right] \subset \mathcal{U}. \quad (12.2)$$

From (12.2) the inclusion $\mathbf{h}^{-1}\left[N_\tau(z)\right] \subset \mathcal{U}$ follows (namely, for any $S \in \mathbf{h}^{-1}\left[N_\tau(z)\right]$, there exists $T \in \mathbf{h}^{-1}\left[N_\tau^o(z)\right]$ such that $T \subset S$; then, $T \in \mathcal{U}$ by (12.2) and $S \in \mathcal{U}$ by axioms of a filter). In addition, $Z \subset N_\tau(z)$. Then,

$$\mathbf{h}^{-1}[Z] \subset \mathbf{h}^{-1}\left[N_\tau(z)\right] \subset \mathcal{U}$$

and (by the choice of Z) $\mathbf{h}^{-1}[Z] \subset \mathcal{U} \cap \mathcal{A} \subset (E - \mathbf{fi})[\mathcal{U} \cap \mathcal{A}]$. Since $Z \in (z - \mathrm{bas})[\tau]$, by (10.27)

$$\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} z.$$

By definition of z $\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} H_o[\tau](\mathcal{U})$.

Proposition 12.2. $\forall \mathcal{F} \in \mathbb{F}_o^*(\mathcal{A})\ \exists! z \in \mathbf{H} : \mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} z$.

Proof. Fix $\mathcal{F} \in \mathbb{F}_o^*(\mathcal{A})$. Using (10.18), we choose $\mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$ such that

$$\mathcal{F} = \mathcal{U} \cap \mathcal{A}. \quad (12.3)$$

Then, by (11.12) $H_o[\tau](\mathcal{U}) \in \mathbf{H}$ and by (12.3) and Proposition 12.1

$$\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} H_o[\tau](\mathcal{U}). \quad (12.4)$$

In addition, $\mathcal{F} \in \beta_o[E]$ and $\mathbf{h}^1[\mathcal{F}] \in \beta_o[\mathbf{H}]$. Therefore, by (12.1) and (12.4) $\forall y \in \mathbf{H}$

$$\left(\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} y\right) \Rightarrow \left(y = H_o[\tau](\mathcal{U})\right).$$

From Proposition 12.2, the obvious corollary follows; namely $\exists! g \in \mathbf{H}^{\mathbb{F}_o^*(\mathcal{A})}$:

$$\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} g(\mathcal{F})\ \forall \mathcal{F} \in \mathbb{F}_o^*(\mathcal{A}).$$

Now, we suppose that the mapping

$$\mathfrak{H}_A[\tau] : \mathbb{F}_o^*(\mathcal{A}) \longrightarrow \mathbf{H} \quad (12.5)$$

is defined by the following rule: if $\mathcal{F} \in \mathbb{F}_o^*(\mathcal{A})$, then

$$\mathbf{h}^1[\mathcal{F}] \overset{\tau}{\Rightarrow} \mathfrak{H}_A[\tau](\mathcal{F}). \quad (12.6)$$

From (10.15) and (12.5), the obvious property follows; namely, $\mathfrak{H}_A[\tau](\mathcal{U} \cap \mathcal{A}) \in \mathbf{H}\ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$.

Proposition 12. 3. $\mathfrak{H}_A[\tau](\mathcal{U} \cap \mathcal{A}) = H_o[\tau](\mathcal{U})\ \forall \mathcal{U} \in \widehat{F}_{\mathbf{u}}[E]$.

Proof. Fix $\mathcal{U} \in \widehat{F}_u\left[E\right]$ Then, by (11.12) $H_o[\tau](\mathcal{U}) \in \mathbf{H}$. By (10.15) we obtain that $\mathcal{U} \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A})$. In particular, $\mathcal{U} \cap \mathcal{A} \in \beta_o[E]$ and by (4.1) $\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \in \beta_o[\mathbf{H}]$. From Proposition 12.1, we have the following convergence

$$\mathbf{h}^1[\mathcal{U} \cap \mathcal{A}] \overset{\tau}{\Rightarrow} H_o[\tau](\mathcal{U}). \quad (12.7)$$

Using Proposition 12.2, (12.1), (12.6), and (12.7), we obtain that $\mathfrak{H}_A[\tau](\mathcal{U} \cap \mathcal{A}) = H_o[\tau](\mathcal{U})$.

Proposition 12.4. If $\mathcal{U} \in \mathbb{F}_o^*(\mathcal{A})$ and $U \in \mathcal{U}$, then $\mathfrak{H}_A[\tau](\mathcal{U}) \in \mathrm{cl}\left(\mathbf{h}^1(U), \tau\right)$.

Proof. Using (10.18), we choose $\mathcal{V} \in \widehat{F}_{\mathbf{u}}[E]$ such that $\mathcal{U} = \mathcal{V} \cap \mathcal{A}$. Then, by Proposition 11.3 we have the inclusion

$$H_o[\tau](\mathcal{V}) \in \mathrm{cl}\left(\mathbf{h}^1(\widetilde{U}), \tau\right) \quad (12.8)$$

(we use the obvious inclusion $\widetilde{U} \in \mathcal{V}$ realized by the choice of U). By Proposition 12.3

$$\mathfrak{H}_A[\tau](\mathcal{U}) = \mathfrak{H}_A[\tau](\mathcal{V} \cap \mathcal{A}) = H_o[\tau](\mathcal{V}).$$

From (12.8), the inclusion $\mathfrak{H}_A[\tau](\mathcal{U}) \in \mathrm{cl}\left(\mathbf{h}^1(U), \tau\right)$ follows.

We note that (see [11,12,13]) by (11.11) the space $(\mathbf{H}, \tau)$ is regular: if $x \in \mathbf{H}$, then

$$\exists \mathfrak{X} \in (x-\mathrm{bas})[\tau] : \mathfrak{X} \subset \mathbf{C}_{\mathbf{H}}(\tau). \quad (12.9)$$

Proposition 12.5. The mapping (12.5) is continuous:

$$\mathfrak{H}_A[\tau] \in C\left(\mathbb{F}_o^*(\mathcal{A}), \mathbf{T}_{\mathcal{A}}^*[E], \mathbf{H}, \tau\right). \quad (12.10)$$

Proof. Fix $\mathcal{U} \in \mathbb{F}_o^*(\mathcal{A})$. Then, by (12.5) $z := \mathfrak{H}_A[\tau](\mathcal{U}) \in \mathbf{H}$. In addition, by (12.6)

$$\mathbf{h}^1[\mathcal{U}] \overset{\tau}{\Rightarrow} z. \quad (12.11)$$

Of course, $\mathcal{U} \in \beta_o[E]$ and $\mathbf{h}^1[\mathcal{U}] \in \beta_o[\mathbf{H}]$ (see (4.1)). As a corollary, by (3.3)

$$\mathcal{H} := (\mathbf{H}-\mathbf{fi})\left[\mathbf{h}^1[\mathcal{U}]\right] \in \widehat{F}[\mathbf{H}]. \quad (12.12)$$

From (3.5), (12.11), and (12.12), we obtain the following inclusion:

$$N_\tau(z) \subset \mathcal{H}. \quad (12.13)$$

From (2.1), (3.3), and (12.13), we obtain that

$$\forall S \in N_\tau(z)\ \exists \widetilde{U} \in \mathcal{U} : \mathbf{h}^1(\widetilde{U}) \subset S. \quad (12.14)$$

Fix $\mathbf{N} \in N_\tau(z)$. Using (12.9), we choose $Z \in (z-\mathrm{bas})[\tau]$ such that $Z \subset \mathbf{C}_{\mathbf{H}}(\tau)$. Then, by (3.4), for some $\mathbf{F} \in Z$, the inclusion

$$\mathbf{F} \subset \mathbf{N} \quad (12.15)$$

is valid. Therefore, $\mathbf{F} \in \mathbf{C}_{\mathbf{H}}(\tau)$. Of course, $\mathbf{F} \in N_\tau(z)$. Therefore, by (12.14), for some $\mathbf{U} \in \mathcal{U}$

$$\mathbf{h}^1(\mathbf{U}) \subset \mathbf{F}. \quad (12.16)$$

In addition, $\Phi_{\mathcal{A}}(\mathbf{U}) \in (\mathbb{UF})[E;\mathcal{A}]$ (see (5.4)). By (6.4) $\Phi_{\mathcal{A}}(\mathbf{U}) \in \mathbf{T}_{\mathcal{A}}^*[E]$. In addition, by (5.3) $\mathcal{U} \in \Phi_{\mathcal{A}}(\mathbf{U})$. Therefore,

$$\Phi_{\mathcal{A}}(\mathbf{U}) \in N^o_{\mathbf{T}_{\mathcal{A}}^*[E]}(\mathcal{U}). \quad (12.17)$$

Choose arbitrary ultrafilter $\mathcal{V} \in \Phi_{\mathcal{A}}(\mathbf{U})$. Then, $\mathcal{V} \in \mathbb{F}_o^*(\mathcal{A})$ and $\mathbf{U} \in \mathcal{V}$; see (5.3). By Proposition 12.4

$$\mathfrak{H}_A[\tau](\mathcal{V}) \in \mathrm{cl}\left(\mathbf{h}^1(\mathbf{U}), \tau\right). \quad (12.18)$$

By the closedness of $\mathbf{F}$ and (12.16) $\mathrm{cl}\left(\mathbf{h}^1(\mathbf{U}), \tau\right) \subset \mathbf{F}$. So, from (12.18), we have the inclusion

$$\mathfrak{H}_A[\tau](\mathcal{V}) \in \mathbf{F}.$$

Using (12.15), we obtain that $\mathfrak{H}_A[\tau](\mathcal{V}) \in \mathbf{N}$. Since the choice of $\mathcal{V}$ was arbitrary, the inclusion

$$\mathfrak{H}_{\mathcal{A}}[\tau]^1\left(\Phi_{\mathcal{A}}(\mathbf{U})\right) \subset \mathbf{N} \quad (12.19)$$

is established. Since the choice of $\mathbf{N}$ was arbitrary too, from (12.17), we obtain that

$$\forall S \in N_\tau\left(\mathfrak{H}_A[\tau](\mathcal{U})\right)$$
$$\exists T \in N_{\mathbf{T}_{\mathcal{A}}^*[E]}(\mathcal{U}) : \mathfrak{H}_{\mathcal{A}}[\tau]^1(T) \subset S.$$

So, the mapping $\mathfrak{H}_A[\tau]$ is continuous at the point $\mathcal{U}$. Since the choice of $\mathcal{U}$ was arbitrary, the required inclusion (12.10) is established (see [16], (2.5.4)).

In connection with Proposition 12.5, we recall Proposition 9.2 and known statement about the possibility of an extension of continuous functions defined on the initial space; in this connection, see, for example, Theorem 3.6.21 of monograph [13]. For this approach, constructions of Section 8 are essential. Of course, under corresponding conditions, we can use the natural connection with the Wallman extension (see (8.27) and Proposition 9.2).

In this case, Proposition 12.5 can be “replaced” (in some sense) by statements similar to the above-mentioned Theorem 3.6.21 of [13] (of course, this approach requires a correction, since we consider ultrafilters of the measurable space). But, we use the “more straight” way with point of view of asymptotic analysis: we construct the required continuous mapping by the limit passage (see Proposition 12.5). We recall (9.24). Then, by (9.24) and (12.5) the mapping

$$\mathfrak{H}_A[\tau] \circ (\mathcal{A}-\mathrm{ult})[E] = \left(\mathfrak{H}_A[\tau]\left((E-ult)[x] \cap \mathcal{A}\right)\right)_{x \in E} \in \mathbf{H}^E \quad (12.20)$$

is defined; moreover, $\mathbf{h} \in \mathbf{H}^E$.

Proposition 12.6. The equality $\mathbf{h} = \mathfrak{H}_A[\tau] \circ (\mathcal{A}-\mathrm{ult})[E]$ is valid.

Proof. Fix $x \in E$. Then by (11.17) $(E-\mathrm{ult})[x] \in \widehat{F}_{\mathbf{u}}[E]$. In addition, by (11.18) the obvious equality follows:

$$\mathbf{h}(x) = H_o[\tau]\left((E-\mathrm{ult})[x]\right). \quad (12.21)$$

Moreover, by (9.24) we obtain that

$$(\mathcal{A}-\mathrm{ult})[E](x) = (E-\mathrm{ult})[x] \cap \mathcal{A} \in \mathbb{F}_o^*(\mathcal{A}). \quad (12.22)$$

Then, by Proposition 12.3 and (12.22) we have the equality chain

$$\mathfrak{H}_A[\tau]\left((\mathcal{A}-\mathrm{ult})[E](x)\right) = H_o[\tau]\left((E-\mathrm{ult})[x]\right) = \mathbf{h}(x).$$

So, $\left(\mathfrak{H}_A[\tau] \circ \left(\mathcal{A}-\mathrm{ult}[E]\right)\right)(x) = \mathfrak{H}_A[\tau]\left((\mathcal{A}-ult)[E](x)\right) = \mathbf{h}(x)$. Since the choice of x was arbitrary, $\mathbf{h} = \mathfrak{H}_A[\tau] \circ (\mathcal{A}-\mathrm{ult})[E]$.

Since (9.7) is valid, from Propositions 12.5 and 12.6, we have the important corollary connected with Proposition 5.2.1 of [9]:

$$(\mathbb{F}_o^*(\mathcal{A}), \mathbf{T}_{\mathcal{A}}^*[E], (\mathcal{A}-\mathrm{ult})[E], \mathfrak{H}_A[\tau]) \qquad (12.23)$$

is a compactificator, for which (in the considered case)

$$(\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma] = \mathfrak{H}_A[\tau]^1\left((\mathbf{as})[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E]; (\mathcal{A}-\mathrm{ult})[E];\Sigma]\right) \ \forall \Sigma \in P'\left(P(E)\right); \qquad (12.24)$$

in (12.24) we use Proposition 3.1 and Corollary 3.1 of [18]. In addition, $P'(\mathcal{A}) \subset P'\left(P(E)\right)$. Therefore, by (12. 24)

$$(\mathbf{as})[E;\mathbf{H};\tau;\mathbf{h};\Sigma] = \mathfrak{H}_A[\tau]^1\left((\mathbf{as})[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E]; (\mathcal{A}-\mathrm{ult})[E];\Sigma]\right) \ \forall \Sigma \in P'(\mathcal{A}). \qquad (12.25)$$

In (12.25), we have the important particular case. We consider this case in the following section.

13. Ultrafilters as Generalized Solutions

We suppose that $E, (\mathbf{H},\tau), \mathbf{h}$, and $\mathcal{A}$ satisfy to the conditions of Section 12. We postulate (11.11). Finally, we postulate Condition 10.1. Therefore, we can use constructions of the previous section. In particular, (12.25) is fulfilled (the more general property (12.24) is fulfilled too). In connection with (12.25), the obtaining of more simple representations of AS

$$(\mathbf{as})\left[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\right], \Sigma \in P'(\mathcal{A}), \qquad (13.1)$$

is important. For this goal, we use the natural construction of Theorem 8.1 in [17]. Namely, we have the following

Proposition 13. 1. If $\Sigma \in P'(\mathcal{A})$, then $(\mathbf{as})\left[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\right] = \mathbb{F}_o^*(\mathcal{A}\,|\,\Sigma)$.

Proof. Let $\mathcal{F} \in (\mathbf{as})\left[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\right]$. Then, by the corresponding definition of Section 4 (see (4.3)) $\mathcal{F} \in \mathbb{F}_o^*(\mathcal{A})$ and, for some net $(D, \preceq, f)$ in the set E,

$$\left(\Sigma \subset (E-\mathrm{ass})[D;\preceq;f]\right) \& \left((D,\preceq,(\mathcal{A}-\mathrm{ult})[E]\circ f) \xrightarrow{\mathbf{T}_{\mathcal{A}}^*[E]} \mathcal{F}\right). \qquad (13.2)$$

Fix $\widetilde{A} \in \Sigma$. Then, by (13.2) $\widetilde{A} \in (E-\mathrm{ass})[D;\preceq;f]$. Using (3.7), we choose $d_1 \in D$ such that $\forall \delta \in D$

$$(d_1 \preceq \delta) \Rightarrow \left(f(\delta) \in \widetilde{A}\right). \qquad (13.3)$$

Of course, $\widetilde{A} \in P(E)$. And what is more, $\widetilde{A} \in \mathcal{F}$. Indeed, let us assume the contrary:

$$\widetilde{A} \in \Sigma \setminus \mathcal{F}. \qquad (13.4)$$

Recall that $\Sigma \subset \mathcal{A}$. Therefore, $\widetilde{A} \in \mathcal{A}$. By (9.1) and (13.4) we have the inclusion $E \setminus \widetilde{A} \in \mathcal{F}$. Then, by (5.4) $\Phi_{\mathcal{A}}(E \setminus \widetilde{A}) \in (\mathbb{UF})[E;\mathcal{A}]$. In particular (see (6.4)),

$$\Phi_{\mathcal{A}}(E \setminus \widetilde{A}) \in \mathbf{T}_{\mathcal{A}}^*[E]. \qquad (13.5)$$

Moreover, by (5.3) $\mathcal{F} \in \Phi_{\mathcal{A}}(E \setminus \widetilde{A})$. Using (13.5), we obtain that

$$\Phi_{\mathcal{A}}(E \setminus \widetilde{A}) \in N^o_{\mathbf{T}_{\mathcal{A}}^*[E]}(\mathcal{F}), \qquad (13.6)$$

where $N^o_{\mathbf{T}_{\mathcal{A}}^*[E]}(\mathcal{F}) \subset N_{\mathbf{T}_{\mathcal{A}}^*[E]}(\mathcal{F})$. From (13.6) and the second statement of (13.2) we have the following property: there exists $d_2 \in D$ such that $\forall \delta \in D$

$$(d_2 \preceq \delta) \Rightarrow \left(\left((\mathcal{A}-\mathrm{ult})[E]\circ f\right)(\delta) \in \Phi_{\mathcal{A}}(E \setminus \widetilde{A})\right). \qquad (13.7)$$

By axioms of DS there exists $d_3 \in D$ for which $d_1 \preceq d_3$ and $d_2 \preceq d_3$. By (13.3) $f(d_3) \in \widetilde{A}$. Moreover, by (13.7)

$$\left((\mathcal{A}-\mathrm{ult})[E]\circ f\right)(d_3) \in \Phi_{\mathcal{A}}(E \setminus \widetilde{A}).$$

By (9.24) $\left((\mathcal{A}-\mathrm{ult})[E]\circ f\right)(d_3) = (E-\mathrm{ult})\left[f(d_3)\right] \cap \mathcal{A}$. Therefore,

$$(E-\mathrm{ult})\left[f(d_3)\right] \cap \mathcal{A} \in \Phi_{\mathcal{A}}(E \setminus \widetilde{A}).$$

From (5.3), the inclusion $E \setminus \widetilde{A} \in (E-\mathrm{ult})\left[f(d_3)\right] \cap \mathcal{A}$ follows; in particular, $E \setminus \widetilde{A} \in (E-\mathrm{ult})\left[f(d_3)\right]$. By (3.9) $f(d_3) \in E \setminus \widetilde{A}$. So,

$$\left(f(d_3) \in \widetilde{A}\right) \& \left(f(d_3) \in E \setminus \widetilde{A}\right).$$

We have the obvious contradiction. This contradiction means that (13.4) is impossible. So, $\widetilde{A} \in \mathcal{F}$. Since the choice of $\widetilde{A}$ was arbitrary, the inclusion $\Sigma \subset \mathcal{F}$ is established. Then (see (9.26)), $\mathcal{F} \in \mathbb{F}_o^*(\mathcal{A}\,|\,\Sigma)$. So, we obtain the inclusion

$$(\mathbf{as})\left[E;\mathbb{F}_o^*(\mathcal{A});\mathbf{T}_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\right] \subset \mathbb{F}_o^*(\mathcal{A}\,|\,\Sigma). \qquad (13.8)$$

Choose arbitrary $\mathcal{V} \in \mathbb{F}_o^*(\mathcal{A}\,|\,\Sigma)$. Then, by (9.26) $\mathcal{V} \in \mathbb{F}_o^*(\mathcal{A})$ and $\Sigma \subset \mathcal{V}$. By Proposition 9.4 and [9,(3.3.7)], for some net $(\mathbb{D}, \sqsubseteq, g)$ in the set E, the convergence

$$(\mathbb{D}, \sqsubseteq, (\mathcal{A}-\mathrm{ult})[E]\circ g) \xrightarrow{\mathbf{T}_{\mathcal{A}}^*[E]} \mathcal{V} \qquad (13.9)$$

is fulfilled. Now, we use axiom of choice. Fix $\Omega \in \Sigma$ Then, by the choice of $\mathcal{V}$ the inclusion $\Omega \in \mathcal{V}$ is fulfilled. Of course, by (5.4)

$$\Phi_{\mathcal{A}}(\Omega) \in (\mathbb{UF})[E;\mathcal{A}]; \qquad (13.10)$$

in addition, by (5.3) $\mathcal{V} \in \Phi_{\mathcal{A}}\left(\Omega\right)$. Since by (6.4) and (13.10) $\Phi_{\mathcal{A}}\left(\Omega\right) \in T_{\mathcal{A}}^*\left[E\right]$ we have the inclusion

$$\Phi_{\mathcal{A}}(\Omega) \in N^O_{T_{\mathcal{A}}^*[E]}(\mathcal{V}) \qquad (13.11).$$

From (13.9) and (13.11), we have the property: for some $d \in \mathbb{D}$ we obtain that $\forall \delta \in \mathbb{D}$.

$$\left(d \sqsubseteq \delta\right) \Rightarrow \left(\left((\mathcal{A}-\mathrm{ult})\left[E\right]\circ g\right)\left(\delta\right) \in \Phi_{\mathcal{A}}\left(\Omega\right)\right). \qquad (13.12)$$

From (9.24) and (13.12), we have the following property: $\forall \delta \in \mathbb{D}$

$$(d \sqsubseteq \delta) \Rightarrow \big((E-\mathrm{ult})\big[g(\delta)\big] \cap \mathcal{A} \in \Phi_{\mathcal{A}}(\Omega)\big). \quad (13.13)$$

By (5.3) and (13.13) we obtain that, for $\delta \in \mathbb{D}$ with the property $d \sqsubseteq \delta$ the inclusion $\Omega \in (E-\mathrm{ult})\big[g(\delta)\big] \cap \mathcal{A}$ is valid and, as a corollary, by (3.9) $g(\delta) \in \Omega$ So, $\Omega \in P(E)$ and

$$\exists d_1 \in \mathbb{D}\ \forall d_2 \in \mathbb{D}\ (d_1 \sqsubseteq d_2) \Rightarrow (g(d_2) \in \Omega)$$

Then, by (3.7) $\Omega \in (E-\mathrm{ass})[\mathbb{D};\sqsubseteq;g]$ Since the choice of Ω was arbitrary, the inclusion

$$\Sigma \subset (E-\mathrm{ass})[\mathbb{D};\sqsubseteq;g] \quad (13.14)$$

is established. So, by (13.9) and (13.14) we obtain that the net $(\mathbb{D};\sqsubseteq;g)$ in the set E has the following properties:

$$\big(\Sigma \subset (E-\mathrm{ass})\big)[\mathbb{D};\sqsubseteq;g]\,\&$$
$$\&\left(\big(\mathbb{D},\sqsubseteq,(\mathcal{A}-\mathrm{ult})[E]\circ g\big) \overset{T_{\mathcal{A}}^*[E]}{\to} \mathcal{V}\right)$$

By definition of Section 4 (see (4.3)) $\mathcal{V} \in (\boldsymbol{as})\big[E;\mathbb{F}_o^*(\mathcal{A});T_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\big]$ So, the inclusion

$$\mathbb{F}_o^*(\mathcal{A}|\Sigma) \subset (\boldsymbol{as})\big[E;\mathbb{F}_o^*(\mathcal{A});T_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\big]$$

is established. Using (13.8), we have the required equality

$$(\boldsymbol{as})\big[E;\mathbb{F}_o^*(\mathcal{A});T_{\mathcal{A}}^*[E];(\mathcal{A}-\mathrm{ult})[E];\Sigma\big] = \mathbb{F}_o^*(\mathcal{A}|\Sigma)$$

From (12.25) and Proposition 13.1, we have the following

Theorem 13.1. If $\Sigma \in P'(\mathcal{A})$ then, AS in $(\mathbf{H},\tau)$ with constraints of the asymptotic character defined by Σ is realized by the rule

$$(\boldsymbol{as})[E;\mathbf{H};\tau;h;\Sigma] = \mathfrak{H}_{\mathcal{A}}[\tau]^1\big(\mathbb{F}_o^*(\mathcal{A}|\Sigma)\big).$$

We note that, in Theorem 13.1, the set $\mathbb{F}_o^*(\mathcal{A}|\Sigma)$ plays the role of the set of admissible generalized solutions.

14. Some Remarks

In our investigation, one approach to the representation of AS and approximate solutions is considered. This very general approach requires the employment of constructions of nonsequential asymptotic analysis. This is connected both with the necessity of validity of "asymptotic constraints" and with the general type of the convergence in TS. We fix a nonempty set of usual solutions (the solution space), the estimate space, and an operator from the solution space into the estimate space. In the estimate space, a topology is given. Then, under very different constraints, we can realize in this space both usual attainable elements and AE. But, if usual attainable elements are defined comparatively simply (in the logical relation), then AE are constructed very difficult. For last goal, extensions of the initial space are used. In addition, the corresponding spaces of GE are constructed. Ultrafilters of the initial space can be used as GE. But, the realizability problems arise: free ultrafilters are "invisible". In addition, free ultrafilters realize limit attainable elements which nonrealizable in the usual sense. In this connection, we propose to use ultrafilters of (nonstandard) measurable space; we keep in mind spaces with an algebra of sets. But, it is possible to consider the more general constructions with the employment of ultrafilters. In our investigation, ultrafilters of lattices of sets are used. On this basis, the interesting connection with the Wallman extension in general topology arises.

It is possible that the proposed approach motivated by problems of asymptotic analysis can be useful in other constructions of contemporary mathematics.

15. References

[1] J. Warga, "Optimal Control of Differential and Functional Eguations," Academic Press, New York, 1972.

[2] R. V. Gamkrelidze, "Foundations of Optimal Control Theory," Izdat. Tbil. Univ., Tbilissi, Russian, 1977.

[3] N. N. Krasovskii, "The Theory of the Control of Motion," Nauka, Moscow, Russian, 1968.

[4] R. J. Duffin, "Linear Inequalites and Related Systems," *Annals of Mathematics Studies*, Vol. 38, 1956, pp. 157-170.

[5] E. G. Gol'stein, "Duality Theory in Mathematical Programming and Its Applications," Nauka, Moscow, Russian, 1971.

[6] N. N. Krasovskii and A. I. Subbotin, "Game-Theoretical Control Problems," Springer Verlag, Berlin, 1988.

[7] A. G. Chentsov, "Constructing Operations of the Limit Passage with the Emploument of Ultrafilters of Measurable Spaces," *Avtomatika i telemekhanika*, Vol. 6, No. 11, 2007, pp. 208-222.

[8] A. G. Chentsov and S. I. Tarasova, "Extensions of Abstract Analog of Unstable Control Problems," *Functional Differential Equations*, Vol. 16, No. 2, 2009, pp. 237-261.

[9] A. G. Chentsov, "Asymptotic Attainability," Kluwer, Dordrecht, 1997.

[10] K. Kuratowski and A. Mostowski, "Set Theory," North-Holland, Amsterdam, 1967.

[11] N. Bourbaki, "General Topology," Nauka, Moskow, Russian, 1968.

[12] J. L. Kelley, "General Topology," Van Nostrand, Prnce-

ton, NJ, 1957.

[13] R. Engelking, "General Topology," PWN, Warszawa, 1977.

[14] G. Cr*ä*tzer, "General Lattice Theory," Akademie-Verlag, Berlin, 1978.

[15] J. Neveu, "Bases Math Matiques Du Calcul Des Probabilit s," Masson, Paris, 1964.

[16] A. G. Chentsov and S. I. Morina, "Extensions and Relaxations," Kluwer Academic Publishers, Dordrecht/ Boston/London, 2002.

[17] A. G. Chentsov, "Some Constructions of Asymptotic Analisis Connected with Stone-*Č*ech Compactification," *Contemporary Mathematics and its Applications*, Vol. 26, 2004, pp. 119-150.

[18] A. G. Chentsov, "Extensions of Abstract Problems of Attainability: Nonsequential Version. Proceedings of the Steklov Institute of Mathematics," Suppl. 2, 2007, pp. S46-S82. Pleiades Publishing, Ltd., 2007.

Intelligent Information Management, **2010, 2, 552-558**
doi:10.4236/iim.2010.29063 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Identification and Calculation Method of the Financial Benefits of IT Projects for Better Financial Evaluation

Jingchun Feng, Fujie Zhang, Lei Li
Business School, Hohai University, Nanjing, China
E-mail: *feng.jingchun@163.com, zhangfujie526@yahoo.com.cn, lilei926@126.com*
Received June 4, 2010; *revised July* 12, 2010; *accepted August* 22, 2010

Abstract

To financially evaluate an IT projects is to assess its financial feasibility, while the financial benefits are the core parameters of financial evaluation of IT projects. Therefore, correctly identifying the financial benefits of IT projects is the precondition to ensure the validity of the financial evaluation. Essentially, IT projects can be divided into productive IT projects and supportive IT projects. The paper analyzes the importance of identifying the financial benefits and introduces the meaning, characteristics and classification of financial benefits of IT projects. On this basis, the paper identifies the financial benefits of both productive IT projects and supportive IT projects and emphasizes the formation and specific calculation method of the two types of IT projects. For productive IT projects and supportive IT projects, the calculation of financial benefits should employ different methods.

Keywords: Productive IT Projects, Supportive IT Projects, Financial Evaluation, Financial Benefits, Benefit Identification

1. Introduction

In the process of decision-making and evaluation of IT projects, financial evaluation is an important work to determine the financial feasibility of the project. It is not only one important part of economic evaluation of projects but also the important basis for investment decision [1]. Financial evaluation needs to determine the financial parameters, while the identification of financial costs and benefits of IT projects is an important source to obtain the basis data for financial evaluation and that is what that influences the quality of IT projects decision-making [2]. The duration of IT projects is hard to estimate; meanwhile, unlike materials, machines and equipment, the labor costs in the process of implementation can not be exactly estimated because IT projects has the characteristic of intensive intelligence. Therefore, the financial costs and benefits of IT projects usually can not be easily and directly determined. Correctly identifying the financial costs and benefits of IT projects is a very important work.

2. The Meaning, Characteristics and Classification of Financial Benefits of IT Projects

2.1. The Meaning of Financial Benefits of IT Projects

The financial benefits of IT projects refer to the incomes that are generated after the implementation of projects and that are closely related with project target, which mainly refer to the operating income. Subsidy income obtained by some IT projects should be listed in financial benefits. Financial benefits of IT projects generally mean the income directly obtained, not including external benefits due to project construction and operation.

2.2. The Characteristics of Financial Benefits of IT Projects

1) Long time span of financial benefits

Financial benefits of IT projects occur in the O & M (Operation& Maintenance) period from the beginning of the implementation to the last. The time span of IT projects is comparatively long. The length of O & M period is mainly decided by national regulations or reasonable useful life of the project [3]. When there are national regulations of given period of IT projects, regulations should be followed; while there are no such regulations, it should be determined according to reasonable useful life.

For a specific IT projects, its actual useful life may be very long. But When the IT projects O & M period exceeds a certain time, the present value of its financial benefits can nearly be ignored from the aspect of financial evaluation. The reason is as follows:

Suppose that the useful life of IT projects O & M period is t (t = 1, 2…, n), interest rate is *i*, the financial benefits generated in the year of t is FB. Calculate the PV (Present Value) of FB by the follow formula (1) [4]:

$$PV = \frac{FB}{(1+i)^t} \quad (1)$$

When *i* is certain, PV becomes close to zero as t increases. Generally speaking, the variable t often takes a value of less than 50 years. When O & M period lasts for 50 years, that is t = 50, consider *i* = 10%, then its discount rate is 1/(1 + 10%) 50 = 0.0085; thus the PV is so small that it can be eliminated. When *i* is certain, PV becomes smaller and smaller as t increases. When t is close to infinity, PV(FB) is close to zero.

2) More financial benefits generated by immaterial results and less financial benefits generated by material results

Results of IT projects are mainly existed by immateriality due to high technology and intellectual input of IT projects such as communication network and website construction [5,6], with little entity results. Therefore, financial benefits of IT projects are mainly from the sales of immaterial results.

2.3. The Classification of Financial Benefits of IT Projects

There are usually two purposes to set up IT projects. One is to establish a new information technology project without a specific organization in order to fulfill the process of electronic commerce and product development. The output can not be produced without the IT projects. This is a process from scratch and we call it productive IT projects. The other is to base the original organization on information, that is, an information application in an organization in order to support the original organization and make it run in a better and more efficient way. We call it supportive IT projects. Thus the financial benefits of IT projects can be accordingly divided into two types, one is of productive IT projects; the other is supportive IT projects.

3. Identification and Calculation Methods of Financial Benefits of Productive IT Projects

3.1. The Meaning of Productive IT Projects

Productive IT projects refers to a kind of projects established from scratch, taking one-time development and lifelong maintenance as its main way and possessing specific output for the purpose of the realization of electronic commerce, software product development and website construction in applications of computer software, hardware and communication network technology. There are four points in the definition above: firstly, productive IT project is a process from scratch and it doesn't have any organization as its basis. Secondly, productive IT project is a one-time activity and it needs to be maintained continuously from the construction to the end of project life cycle. Thirdly, the purpose of establishing a productive IT projects is to realize the development of software products and websites, therefore the products are inexistence and can not be obtained by other traditional projects before the construction of productive IT projects. Fourthly, productive IT project has specific outputs such as application software and websites with specific functions.

3.2. The Formation of Financial Benefits of Productive IT Projects

Financial benefits of productive IT projects are mainly revealed as operating income and subsidy income.

1) Operating income of productive IT projects

Operating income refers to the income from selling products or offering service and that is the main part not only in the cash inflow of cash flow statement but also in income statement [7]. Operating income is an important data in financial evaluation, the evaluation veracity of which can largely influence the estimation of project financial benefits. The operating income of IT projects mainly includes sales income and service income.

a) Sales income of productive IT projects. Sales income of productive IT projects refers to the cash inflow brought by selling the products of productive IT projects, mainly including advertising income, call income, value-added business income, registration income, rental income, installation income, sales income of client terminals, sales income of virtual currency, etc.

i) Advertising income. It refers to the income that productive IT projects especially website type IT projects charge to advertise for other companies through the page image and text links.

ii) Call income. Call income refers to the communication costs that productive IT projects charge to customers due to its functions like multi-communication, surfing, etc.. Generally speaking, call income includes call cost of fixed telephones and cell phones, broadband fee, GPRS flow charge, .etc.

iii) Value-added business income. It refers to the re-venue obtained by the operators by the provision of a higher level information needs than basic communication needs. The characteristic of value-added business is that it can provide better service and can satisfy the personalized demands of different customers. So far, the value-added business on telecommunication network mainly includes e-mail, videotext, EDI, fax storage & transmit and so on [8]. 168 audio phone is a typical value-added business.

iv) Registration income. It refers to the cost that charged by productive IT projects for providing the usage of network, software or service.

v) Rental income. It includes space rental income, bandwidth rental income and so on.

vi) Installation income. When a productive IT project connects its backbone network with family terminal systems, terminal systems of commercial or educational organization, mobile terminal systems, .etc at the first time, it will charge for the access. For example, telecommunications network installation fees.

vii) Sales income of client terminals. It refers to the cost that the users of productive IT projects pay for client software in order to achieve the log function so that it can exchange information with the server terminals.

viii) Sales income of virtual currency. Virtual currency is a kind of electrical data or symbols which are issued on internet by network service operators to purchase virtual commodities or services provided by main service providers or contracted service providers [9]. There are many famous virtual currencies such as Q coins of Tencent, point certificates of SNDA, U coins of SINA, etc.

b) IT projects Service income. IT projects service income include advertising production income, upgrade and maintenance service income of client terminals .etc.

i) Advertising production income. It exists mainly in website type IT projects. The website advertisements are mainly represented by page images and text links, so the website operators can obtain benefit from providing advertising production for advertisers.

ii) Upgrade and maintenance service income of client terminals. After the installation of client terminal software, it is necessary to maintain the software as the update and upgrade of the project itself.

2) Subsidy income of productive IT projects

In order to fully play the role of market mechanism and achieve the market-oriented operation of IT projects, related government departments will give certain financial subsidy or loan discount to IT projects with the public good characteristics. Then such IT projects can obtain corresponding subsidy income according to some regulations. The subsidy incomes of IT projects include some government subsidy related to income, that is, value-added tax collected first and refunded last, ration subsidy calculated according to subsidy ration regulated by the government such as sales volume or workload and delivered on schedule, subsidies of other forms belonging to financial support.

3.3. Calculation Methods of the Financial Benefits of Productive IT Projects

The financial benefits of productive IT projects are generally visible, which can be calculated by some routine methods as are shown in **Table 1**.

4. Identification and Calculation Methods of Financial Benefits of Supportive IT Projects

4.1. The Meaning of Supportive IT Projects

Supportive IT projects refers to the project that can help organizations reduce costs, improve efficiency and enhance competitiveness in management in order to be seasoned with the present economic environment by applying computer software& hardware and communication network technology.

The meaning of supportive IT projects consists of four aspects: firstly, supportive IT projects progresses relying on certain forms of organizations. That is to say, organizations are in existence before the implementation of such projects. Secondly, the purpose of supportive IT projects is to help the organization improve operation efficiency by means of information technology. Before supportive IT projects is implemented, the original organization may have such problem as of low operation efficiency but it can still ensure the ordinary quality of products; while when supportive IT projects is in implementation, its operation efficiency will be improved. Thirdly, supportive IT projects mainly acts on organization management, which means that it can not help the organization produce new products but can help improve the management such as simplifying work flow, reducing information distortion and improving service quality. Fourthly, supportive IT projects can not change the original characters of the organization, just supporting it.

4.2. The Identification of Financial Benefits of Supportive IT Projects

Supportive IT projects doesn't have any independent output, with its value relying on the contribution to the organization. Therefore, the financial benefits of supportive IT projects can only be reflected by the difference between costs and benefits of the organization before and after the implementation of IT projects. In essence, financial

Table 1. Calculation methods of financial benefits of productive IT projects.

Financial benefits of productive IT projects				Calculation methods
Operating income	Sales income	Advertising income		(1) Unit price of standard complexity per unit time* complexity coefficient* number of ads* time
		Call income	Call cost of fix telephone and cell phones	(1) Call cost per unit time* call duration (2) Call cost per year* number of years (3) Call cost per month* number of months
			Broadband fee	(1) Broadband cost per unit time* call duration (2) Broadband cost per year* number of years (3) Broadband cost per month* number of months
			GPRS flow charge	(1) GPRS cost per unit flow * number of flow (2) GPRS cost per year* number of years (3) GPRS cost per month* number of months
		Value-added business income		(1) Value-added business cost per unit time* time (2) Cost per unit specific value-added business* number of specific value-added business (3) Value-added business cost per year* number of years (4) Value-added business cost per month* number of months
		Registration income		(1) Registration cost per user* number of users
		Rental income	Space rental income	(1) Rental income per time per space* time* number of spaces (2) Space rental income per year* number of years (3) Space rental income per month* number of months
			Bandwidth rental income	(1) Bandwidth rental income per time* time (2) Bandwidth rental income per year* number of years (3) Bandwidth rental income per month* number of months
		Installation income		(1) Installation income per user* number of users
		Sales income of client terminals		(1) Sales income of client terminal per user* number of users
		Sales income of virtual currency		(1) Sale income per unit virtual currency * number of virtual currency
	Service income	Advertising production income		(1) Advertising production cost of standard complexity per unit time* complexity coefficient* number of ads* time
		Upgrade and maintenance service income of client terminals		(1) Upgrade and maintenance service income of client terminal per user* number of users
Subsidy income				Calculated according to related regulations

benefits of supportive IT projects are mainly revealed as savings of organizational costs and subsidy income according to the principle of before and after comparison in benefits and costs identification.

1) The savings of management costs.

The savings of management costs include savings of labor costs, savings of paper costs, savings of error operations, savings of communication costs and savings of information acquisition costs.

Savings of labor costs refer to the labor costs reduced with the improvement of office efficiency through the application of IT technology. Savings of paper costs refer to the paper costs reduced by using electronic ways instead of traditional paper-based communication through the application of IT technology. Savings of error operations refer to the error operating loss reduced by less manual inputs due to the application of barcode technology and other related technologies. Savings of communication costs refer to the communication costs reduced by using networks and e-mail systems instead of traditional means of telephones and faxes to connect the inner and outer organizations. Savings of information acquisition costs are brought out by the open network and shared information, so it is very cheap for organizations to acquire information by IT systems. That is to say, the implementation of IT projects can effectively reduce the information acquisition costs.

2) The savings of production costs

As a means of factor allocation in production process, IT projects can help improve the efficiency of factor allocation, reduce the costs and to some degree it can avoid or decrease the phenomenon of materiel shortages and production interruptions, consequently with higher production efficiency, lower production costs and more benefits.

3) The savings of sales costs

The savings of sales costs include savings of transaction costs, savings of marketing and promotion costs and savings of customer service costs.

Transaction costs are generated in the process of commodities distribution and wholesale. It is not the part of production costs, but it can reflect the market demand and supply. As the increase of product sales, transaction costs will increase [10-12].

Marketing and promotion costs are spent to publicize the products to customers in order to broaden the product sales in the business activities characterized by equivalent exchange. By using information technology, IT projects can publicize products faster and more efficient compared with traditional marketing and promotion methods. Therefore it can help improve the marketing efficiency and reduce costs.

The implementation of IT projects can make the communication between enterprises and customers more convenient. It can advance customer relationship, improve customer satisfactory and loyalty [13,14], thus reduce the customer service costs.

4) The savings of financial costs

The implementation of IT projects can accelerate the cash flow and can withdrawal account receivable in advance, which makes possible to reduce bad debt and financial costs.

5) The savings of procurement and inventory costs

The implementation of IT projects makes it possible to achieve Global sourcing, real-time procurement and bulk purchases. Enterprises can integrate all the procurement information by internet and invite biddings all over the world in order to choose preferred suppliers; they can make real-time procurement to minimize inventory through connecting production information, inventory information and procurement systems together; It is possible to fulfill the automatic and scientific management of inventory and procurement and meanwhile minimize the influences of human factors with a higher procurement efficiency. The procurement and inventory costs reduced are the savings.

6) Subsidy income of supportive IT projects

Subsidy income of supportive IT projects is almost the same as that of productive IT projects, which should be calculated according to related regulations.

4.3. The Formation of Financial Benefits of Supportive IT Projects

On the basis of the identification of the financial benefits of supportive IT projects, its formation can be listed in **Figure 1**.

4.4. Calculation Methods of the Financial Benefits of Supportive IT Projects

According to the difficult degree of financial benefits calculating, we divide supportive IT projects into two types. One is type I supportive IT projects whose financial benefits are easy to calculate through routine methods. The other is type II supportive IT projects whose financial benefits are hard to determine.

1) Calculation methods of the financial benefits of type I supportive IT projects

The financial benefits of type I supportive IT projects can usually reveal in a short time, therefore we can use the costs difference between before and after to calculate its savings.

One typical example is the GPS information system project of logistics companies. The adoption of GPS can help reduce the number of traffic accidents of transport vehicles as well as their no-load mileage. What's more, drivers can better find the best routine, which makes it even possible to decrease the transport costs. Therefore, its financial benefits mainly rely on the savings as follows: one is the costs reduced with decreased traffic accidents, which can be calculated by statistic data of the

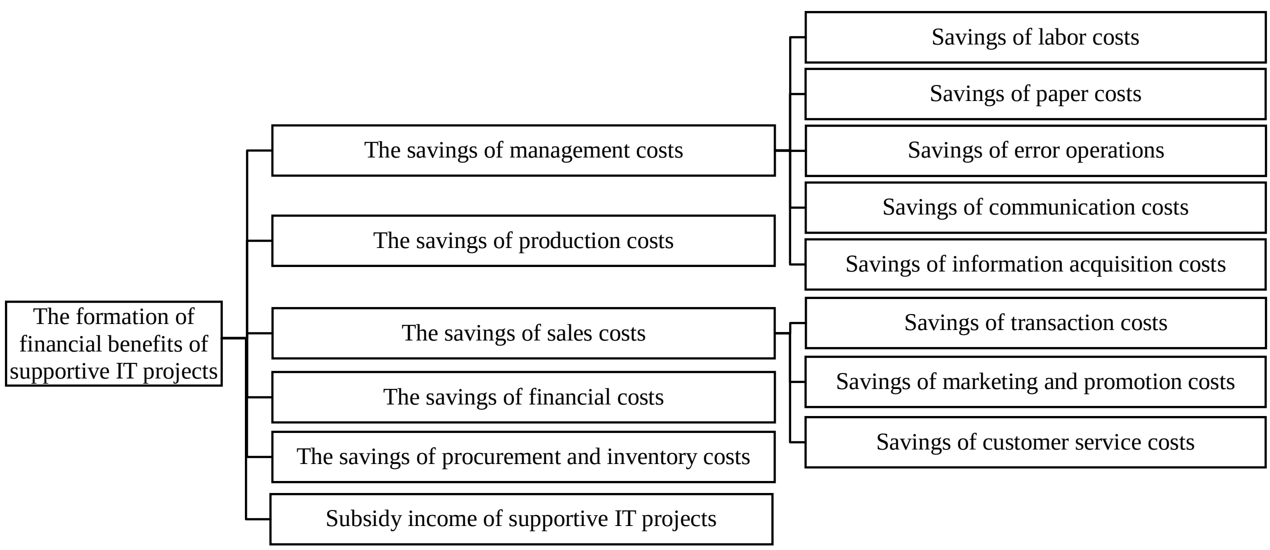


Figure 1. The formation of financial benefits of supportive IT projects.

accidents; the second is the costs reduced with decreased transport mileages which can be calculated by the odometers.

2) Calculation methods of the financial benefits of type II supportive IT projects

The financial benefits of type II supportive IT projects usually can not reveal soon. On one hand, the savings of project costs can not be represented in a short time; on the other hand, the improvement of management efficiency can not be simply measured by cost reduction. Therefore The financial benefits of type II supportive IT projects can not calculated by routine methods.

Take the establishment of schools' information office system as an example. Office information system generally has the following functions: firstly it can reduce the amount used of office supplies such as papers and pencils; secondly, it can help reduce the number of management employees that are not necessary for automatic offices; thirdly, it can improve the information transmission efficiency in the organization. The third is the key function of this project, while the improvement of efficiency can not calculated by routine methods. It is not easy to exactly determine its financial benefits.

One feasible calculation method of the financial benefits of type II projects is cased-based reasoning. In the calculation of financial benefits, the major idea of cased-based reasoning is to set up a case base according to the financial benefits of similar IT projects actually happening before, and then compare the differences in their characteristics such as scales, characters, time etc. between the new and old projects. After a series adjustments and modifications of those differences, reuse the related information in the case base to calculate the financial benefits of the present IT projects. When there is only comparability between the project to be settled and the old projects in the case base, an suited choice should be found. That is a process of optimal matching. Optimal matching is to select one or several cases best related to the present problem from a group of candidate cases obtained in the first matching process.

The similarity measurement plays a very important role in case searches. Similarity measure methods include fuzzy relational clustering, single linkage clustering and so on. Similarity computation formulas are as follows [15]:

$$Sim(V_i, V_j) = 1 - d(V_i, V_j) = 1 - d_{ij} \quad (2)$$

$$d_{ij} = \left| V_i - V_j \right| \quad (3)$$

In formula (2) and (3), $Sim(V_i, V_j)$is the similarity between V_i and V_j; $d(V_i, V_j)$represents the distance between V_i and V_j.

The greater the distance between cases, the smaller the similarity. Through calculating similarity and searching the case base, the cases should be ranked in accordance with the order of similarity decreasingly.

5. Conclusions

1) Financial evaluation is an important work to determine the financial feasibility of the project and also an important basis for investment decision. The identification of financial costs and benefits of IT projects is an important source to obtain the basis data for financial evaluation and that is what that influences the quality of IT projects decision-making.

2) IT projects can be divided into productive kind and supportive kind. Accordingly, the financial benefits of IT projects can be accordingly divided into two types, one is of productive IT projects; the other is supportive IT projects.

3) Financial benefits of productive IT projects are mainly revealed as operating income and subsidy income, which can generally be calculated through routine methods; financial benefits of supportive IT projects are mainly revealed as savings of organizational costs which should be identified according to the principle of before and after comparison.

4) According to the difficult degree of financial benefits calculating, we divide supportive IT projects into two types. Financial benefits of type Ⅰ projects can be determined by routine methods, while type Ⅱ can not. It is suggested that case-based reasoning be introduced to calculate the financial benefits.

6. References

[1] M. Li and N. Yang, "Research on the Evaluation System of IT Projects," *Proceedings of International Conference on Computational Intelligence and Software Engineering (CiSE)*, Wuhan, China, December 2009, pp. 1-4.

[2] B. C. Esty, "Why Study Large Projects? An Introduction to Research on Project Finance," *European Financial Management*, Vol. 2, No. 10, 2004, pp. 213-224.

[3] National Development and Reform Commission, Ministry of Housing and Urban-Rural Development of the People's Republic of China, "Economic assessment Method and Parameters for Construction Project," 3rd Version, China Planning Press, Beijing, 2006.

[4] H. Pesaran, D. Pettenuzzo and A. Timmermann, "Learning, Structural Instability, and Present Value Calculations," *Econometric Reviews*, Vol. 2-4, No. 26, 2007, pp. 253-288.

[5] B. H. Reich, A. Gemino and C. Sauer. "Modeling the Knowledge Perspective of IT Projects," *Project Management Journal*, Vol. 39, 2008, pp. S4-S14.

[6] J. C. Feng, M. Li, D. C. Huang and Y. Wang, "Theories and Methods of IT Projects Management," China Water Power Press, Beijing, 2009.

[7] V. Pareja and Ignacio, "The Correct Definition for the Cash Flows to Value a Firm (Free Cash Flow and Cash Flow to Equity)," 2005. http://ssrn.com/abstract=597681.

[8] Q. W. Wang, X. G. Cao, T. Zhong, Q. F. Jin and B. Wang, "China Telecom Operators: Applications Platform Overview," *Bell Labs Technical Journal*, Vol. 13, No. 2, 2008, pp. 223-235.

[9] G. Boyd, "Editorial: Quatts, Virtual Currency for Gaming and Bartering Education on the Web," *British Journal of Educational Technology*, Vol. 33, No. 4, 2002, pp. 361-363.

[10] L. J. L. Kaouthar and J. T. Mahoney, "Revisiting Agency and Transaction Costs Theory Predictions on Vertical Financial Ownership and Contracting: Electronic Integration as an Organizational form Choice," *Managerial and Decision Economics*, Vol. 27, No. 7, 2006, pp. 573- 586.

[11] S. Shenghui, "Costs and Benefits Analysis of Enterprise Information," *ESAS World*, No. 9, 2000, pp. 43-44.

[12] Y. Z. Liu, "Problems and Solutions in Enterprise Information Construction in China," *Master dissertation*, Taiyuan University of Technology, Taiyuan, 2006.

[13] N. P. Napier, M. Keil and F. B. Tan, "IT Projects Managers' Construction of Successful Project Management Practice: A Repertory Grid Investigation," *Information Systems Journal*, Vol. 19, No. 3, 2009, pp. 255-282.

[14] Y. Z. Xu and D. Q. Li, "How Do Enterprises Make Cost-Benefit Analysis with the Application of Information Technology," *Market and computer*, No. 7, 1999, pp. 19-22.

[15] J. C. Gower and G. J. S. Ross, "Minimum Spanning Trees and Single Linkage Cluster Analysis," *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, Vol. 18, No. 1, 1969, pp. 54-64.

Intelligent Information Management, 2010, 2, 559-562
doi:10.4236/iim.2010.29064 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Improvement of Chen-Zhang-Liu's IRPB Signature Scheme

Dezhi Gao
College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao, Shandong, China
E-mail: *dezhi_gao@yahoo.com.cn*
Received June 8, 2010; *revised July* 18, 2010; *accepted August* 21, 2010

Abstract

Restrictive partially blind signatures incorporate the advantages of restrictive blind signatures and partially blind signatures, which play an important role in electronic commerce. Recently, Chen-Zhang-Liu first proposed an ID-based restrictive partially blind (IRPB) signature from bilinear pairings. Later, Hu-Huang showed that the Chen-Zhang-Liu's scheme has a security weakness, and pointed out that their scheme does not satisfy the property of restrictiveness as they claimed. In this paper, we improve Chen-Zhang-Liu's scheme and propose a new signature scheme from bilinear pairings. The improved scheme can resist the Hu-Huang's attack.

Keywords: Cryptography, Bilinear Pairings, Restrictiveness, Partially Blind Signature, ID-Based Restrictive Partially Blind Signature

1. Introduction

Blind signature scheme were first introduced by Chaum [1] to protect the right of an individual privacy. A blind signature allows a user to acquire a signature without giving the signer any information about the actual message or the resulting signature. And blind signature techniques have been widely used in anonymous electronic cash(e-cash) and anonymous voting systems.

Restrictive blind signatures firstly introduced by Brands [2], which allows a user to receive a blind signature on a message not known to the signer but the choice of message is restricted and must conform to certain rules. Furthermore, he proposed a highly efficient electronic cash system, where the bank ensures that the user is restricted to embed his identity in the resulting blind signature.

A partially blind signature scheme allows a signer to produce a blind signature on a message for a user, where the signature explicitly includes common agreed information which remains clearly visible despite the blinding process. The concept was first introduced by Abe and Fujisaki[3].

In an electronic cash system, embedding user identity information in the blind signature enables the bank to learn the identity of double spenders. Maitland and Boyd [4] first proposed a provably secure restrictive partially blind signature scheme, which satisfies the partial blindness and restrictive blindness.

Recently, Chen-Zhang-Liu[5] proposed the first ID-based restrictive partially blind signature scheme(IRPB) by combining an ID-based partially blind signature scheme proposed by Chow *et al.*[6] with an ID-based restrictive blind signature scheme proposed by Chen-Zhang-Liu[7]. However, X. M. Hu. and S. T. Huang.[8] found that Chen-Zhang-Liu's scheme had an important weakness. Their scheme does not achieve the restrictiveness property as they claimed. They showed, in an electronic cash system constructed by Chen-Zhang-Liu, an account-holder cannot be caught when he performs double-spending. In this paper, we propose an improvement of Chen-Zhang-Liu's scheme, and the new scheme can resist the Hu-Huang *et al*.'s attack.

The rest of the paper is organized as follows: In Section 2, we briefly review Chen-Zhang-Liu's scheme. We propose an improvement of Chen-Zhang-Liu's scheme in Section 3. The completeness and security of improved scheme are discussed in Section 4. Finally, conclusions will be made in Section 5.

2. Review of Chen-Zhang-Liu's Scheme

2.1. Basic Concepts on Bilinear Pairings

Let G_1 be an additive cyclic group with prime order q, G_2 be a multiplicative cyclic group of same order and P be a generator of G_1. Let $e: G_1 \times G_1 \to G_2$ be a bilinear mapping with the following properties:

1) bilinear: $e(aP, bQ) = e(P,Q)^{ab}$ for all

$$P, Q \in G_1 \text{ and } a, b \in Z_q^*;$$

2) non-degenerate: there exists P and $Q \in G_1$ such that $e(P,Q) \neq 1$;

3) computable: there exists an efficient algorithm to compute $e(P,Q)$ for $P, Q \in G_1$.

The bilinear pairings can be derived from the Weil or Tate pairings.

2.2. Complexity Assumptions

Let G be a cyclic multiplicative group generated by g, whose order is a prime q, assume that the inversion and multiplication in G can be computed efficiently.

Discrete Logarithm Problem (DLP): Given two elements g and h, to find an integer $n \in Z_q^*$, such that $h = g^n$ whenever such an integer exists.

We assume that the discrete logarithm problem (DLP) in both G_1 and G_2 are hard.

2.3. Chen-Zhang-Liu's ID-based Restrictive Partially Blind Signature Scheme

Chen-Zhang-Liu's ID-based restrictive partially blind signature scheme (Chen *et al*.,2007) consists of four phases: *system parameters generation, key generation, signature generation and signature verification*. For simplicity, we will use the same notation as Chen-Zhang-Liu's scheme.

Define two cryptographic secure hash functions

$$H: \{0,1\}^* \to G_1 \text{ and } H_1: G_1^3 \times G_2^4 \to Z_q.$$

-*System parameters generation:* On input security parameter k, output the master key $s \in_R Z_q^*$ and the system parameter

$$\text{params} = \{G_1, G_2, e, q, P, P_{pub}, k, H, H_1\}.$$

-*Key generation:* Given *params* and the signer's identity information ID, output the signer's private key

$$S_{ID} = sQ_{ID} = sH(ID).$$

-*Signature generation*: Let Δ be the shared information and a message M be from the receiver.

Publish $g = e(P, Q_{ID})$ and $y = e(P_{pub}, Q_{ID})$.

- The signer randomly chooses $Q \in_R G_1, r \in_R Z_q^*$ and computes $z = e(M, S_{ID}), a = e(P,Q), \quad b = e(M,Q)$, $U = rP$ and $Y = rQ_{ID}$. He then sends (z,a,b,U,Y) to the receiver.
- The receiver randomly generates $\alpha, \beta, u, v, \lambda, \mu, \gamma \in_R Z_q^*$, and computes $M' = \alpha M + \beta P$, $z' = z^{\alpha} y^{\beta}$ $A = e(M', Q_{ID})$, $a' = a^u g^v$, $b' = a^{u\beta} b^{u\alpha} A^v$, $Y' = \lambda Y + \lambda\mu Q_{ID} - \gamma H(\Delta)$, $U' = \lambda U + \gamma P_{pub}$, $h = \lambda^{-1} H_1(M', Y', U', A, z', a', b') + \mu$ and $c' = hu$. He then sends h to the signer.
- The signer computes $S_1 = Q + hS_{ID}$, $S_2 = (r+h) S_{ID} + rH(\Delta)$ and sends (S_1, S_2) to the receiver.
- If the equations $e(P, S_1) = ay^h$ and $e(M, S_1) = bz^h$ hold, the receiver computes $s_1' = uS_1 + vQ_{ID}$ and $s_2' = \lambda S_2$. Thus, the tuple $(Y', U', z', c', s_1', s_2')$ is the signature for Δ and M'.

-*Signature verification*: Given a tuple $(Y', U', z', c', s_1', s_2')$ for Δ and M', the verifier computes $A = e(M', Q_{ID})$, $a' = e(P, s_1') y^{-c'}$ and $b' = e(M', s_1') z'^{-c'}$. The verifier accepts the signature if the following equation holds:

$$e(s_2', P) = e(Y' + H_1(M', Y', U', A, z', a'b') Q_{ID}, P_{pub}) \times e(H(\Delta), U').$$

Unfortunately, Hu-Huang [8] pointed out that above scheme is insecure and can not achieve the property of restrictiveness as they claimed. Any adversary receiver can obtain a valid signature for a message.

3. Improvement of Chen-Zhang-Liu's Scheme

In [8], X.M. Hu *et al*. found that above scheme had an important weakness. Any adversary receiver can obtain a valid signature for a message M' with any form. The main reason is that the Chen-Zhang-Liu's scheme had more variable parameters. In this section, we present an improvement of Chen-Zhang-Liu's scheme. The system initialization phase is the same as the one presented in Section 2. In the following, we only describe the *Signature generation* and *Signature verification*.

-*System parameters generation*: The parameters generation is just as before.

-*Key generation*: The key generation is just as before.

-*Signature generation*: Let Δ be the shared information and a message M be from the receiver.

Publish $g = e(P, Q_{ID})$ and $y = e(P_{pub}, Q_{ID})$.

- The signer randomly chooses $Q \in_R G_1, r \in_R Z_q^*$ and computes $z = e(M, S_{ID})$, $a = e(P,Q)$, $b = e(M,Q)$, $U = rP$ and $Y = rQ_{ID}$. He then sends (z,a,b,U,Y) to the receiver.
- The receiver randomly generates $\alpha, \beta, u, v \in_R Z_q^*$, and computes $M' = \alpha M + \beta P$, $A = e(M', Q_{ID})$, $z' = z^{\alpha} y^{\beta}$, $a' = a^u g^v$, $b' = a^{u\beta} b^{u\alpha} A^v$, $Y' = (\alpha+\beta)Y + (\alpha+\beta)(\beta+u)Q_{ID} - (u+v)H(\Delta)$, $U' = (\alpha+\beta)U + (u+v)P_{pub}$, $h = (\alpha+\beta)^{-1} H_1(M', Y', U', A, z', a', b')$

$+(\beta+u)$ and $c'=hu$. He then sends h to the signer.

- The signer computes $S_1 = Q + hS_{ID}$, $S_2 = (r+h)S_{ID} + rH(\Delta)$, and sends (S_1, S_2) to the receiver.
- If the equations $e(P,S_1) = ay^h$ and $e(M,S_1) = bz^h$ hold, the receiver computes $s_1' = uS_1 + vQ_{ID}$ and $s_2' = \lambda S_2$. Thus, the tuple (Y',U',z',c',s_1',s_2') is the signature for Δ and M'.

-*Signature verification*: Given a tuple (Y',U',z',c',s_1', s_2') for Δ and M', the verifier computes $A = e(M', Q_{ID})$, $a' = e(P,s_1')y^{-c'}$ and $b' = e(M',s_1')z'^{-c'}$. The verifier accepts the signature if the following equation holds:

$$e(s_2',P) = e(Y' + H_1(M',Y',U',A,z',a',b')Q_{ID}, P_{pub}) \times e(H(\Delta),U').$$

4. Discussion

In this section, we first discuss the completeness of our improved scheme, and then show the new scheme can resist against the proposed attack by X.M. Hu. *et al*.

Theorem 1. The improved scheme achieves the property of completeness.

Proof. Note that

$$e(P,s_1') = e(P,S_1)^u \cdot e(P,Q_{ID})^v = (ay^h)^u g^v = a' y^{c'}$$

$$e(M',s_1') = e(M',S_1)^u \cdot e(M',Q_{ID})^v$$
$$= e(\alpha M + \beta P, S_1)^u \cdot A^v = b' z'^{c'}$$

and

$$\begin{aligned}
e(s_2',P) &= e((\alpha+\beta)S_2, P) \\
&= e((\alpha+\beta)(r+h)S_{ID} + (\alpha+\beta)H(\Delta), P) \\
&= e((\alpha+\beta)r + H_1(M',Y',U',A,z',a',b') \\
&\quad + (\alpha+\beta)(\beta+u)Q_{ID}, P_{pub}) \cdot e(H(\Delta),(\alpha+\beta)rP) \\
&= e((\alpha+\beta)r + H_1(M',Y',U',A,z',a',b') \\
&\quad + (\alpha+\beta)(\beta+u)Q_{ID}, P_{pub}) \cdot e(H(\Delta), U' - \gamma P_{pub}) \\
&= e((\alpha+\beta)r + H_1(M',Y',U',A,z',a',b') \\
&\quad + (\alpha+\beta)(\beta+u)Q_{ID} - \gamma H(\Delta), P_{pub}) \cdot e(H(\Delta),U') \\
&= e((Y' + H_1(M',Y',U',A,z',a',b')Q_{ID}, P_{pub}) \\
&\quad \cdot e(H(\Delta),U')
\end{aligned}$$

Thus, the improved scheme achieves the property of completeness.

Theorem 2. The improved scheme is secure and can resist the attack proposed by Hu-Huang.

Proof. Hu-Huang pointed out that the Chen-Zhang-Liu's scheme is insecure, the key reason is that their scheme was constructed by simply assembling Chow *et al*.'s scheme (2005) and Chen *et al*.'s scheme (2005). In their scheme, the receiver randomly generated seven numbers, and these numbers have redundancy. Any adversary can constructs (z',a',b') without using the values of (z,a, b) in step in the Chen-Zhang-Liu's scheme. Comparison with the Chen-Zhang-Liu's scheme, our improved scheme only choose $\alpha+\beta$, $\beta+u$ and $u+v$, instead of choosing the parameters λ, μ and γ, respectively, and the verification equation is same as that of Chen-Zhang-Liu. Thus, we reduce the variable parameters and only need four parameters instead of seven parameters in Chen-Zhang-Liu's scheme. Since the parameters $\alpha+\beta$, $\beta+u$ and $u+v$ are depend on the parameters α, β, u and v in the improved scheme, the recipient obtains a signature on a message that can only be the form $M' = \alpha M + \beta P$ with α and β randomly chosen by the recipient. Similar to Chen-Zhang-Liu's analysis, our scheme achieves the property of restrictiveness and can resist the Hu-Huang's attack.

The other properties of the improved scheme are same as the Chen-Zhang-Liu's scheme, we omit them.

5. Conclusions

Chen-Zhang-Liu proposed the first ID-based restrictive partially blind signature scheme (IRPB) by combing the ID-based partially blind signature scheme with the ID-based restrictive blind signature scheme. Recently, X.M Hu *et al*. showed the Chen-Zhang-Liu's scheme did not satisfy the property of restrictiveness as they claimed. In this paper, we give an improved version of Chen-Zhang-Liu's scheme, and the improved scheme can resist the attack proposed by X. M. Hu *et al*.

6. Acknowledgements

The author wishes to thank the anonymous referees for their valuable comments and suggestions.

7. References

[1] D. Chaum, "Blind Signatures for Untraceable Payments," *Advances in Cryptology Crypto'*82, Springer-Verlag, Germany, 1982, pp. 199-203.

[2] S. Brands, "Untraceable off-Line Cash in Wallets with Observers," *Advances in Cryptology Crypto'*93, *LNCS 773*, Springer-Verlag, Germany, 1993, pp. 302-318.

[3] M. Abe and E. Fujisaki, "How to Date Blind Signatures," *Advances in Cryptology-Asiacrypt* 1996, *LNCS* 1163, Springer-Verlag, Germany, 1996, pp. 244-251.

[4] G. Maitland and C. Boyd, "A Provably Secure Restrictive Blind Signature Scheme," *PKC'02, LNCS* 2274, Springer-Verlag, Germany, 2002, pp. 99-114.

[5] X. F. Chen, F. G. Zhang and S. L. Liu, "ID-Based Restrictive Partially Blind Signatures and Applications," *The Journal of Systems and Software*, Vol. 80, No. 2,

2007, pp. 64-71.

[6] S. M. Chow, C. K. Hui and S. M. Yin, “Two Improved Partially Blind Signature Schemes from Bilinear Pairings,” *ACISP’05*, *LNCS* 3574, Springer-Verlag, Germany, 2005, pp. 316-328.

[7] X. F. Chen, F. G. Zhang and S. L. Liu, “ID-Based Restrictive Partially Blind Signatures and Applications,” 2007. http://eprint. iacr.org/2005/3/319/.

[8] X. M. Hu and S. T. Huang, “Analysis of ID-Based Restrictive Partially Blind Signatures and Applications,” *The Journal of Systems and Software*, Vol. 81, No. 11, 2008, pp. 1951-1954.

Intelligent Information Management, **2010, 2, 563-568**
doi:10.4236/iim.2010.29065 Published Online September 2010 (http://www.SciRP.org/journal/iim)

Simulation of Learners' Behaviors Based on the Modified Cellular Automata Model*

Zhenyan Liang[1], Haiyan Liu[1], Chaoying Zhang[1]*, Shangyuan Yang[2]
[1]*Computer Science and Information Engineering College, Guangxi Normal University, Guilin, P. R. China*
[2]*College of Teaching Chinese as a Second Language, Qingdao University, Qingdao, Shandong, P. R. China*
E-mail: *zhangcy@gxnu.edu.cn*
Received June 11, 2010; *revised July* 25, 2010; *accepted August* 29, 2010

Abstract

This study develops a computational model for simulation of behaviors of learners under the influence of motivation and engagement environment based on Cellular Automata (CA). It investigates the changing patterns of learners' behaviors when motivation and engagement environment are assigned with different values respectively. The simulation process indicates that the internal factor, which is the motivation in this paper, plays a key role in changing learners' behaviors under certain circumstance and the engagement environment also significantly influences learner's perception. The results obtained also show good agreement with the phenomenon generally being observed in practice.

Keywords: Motivation, Engagement, Cellular Automata, Simulation, Learner Behaviors

1. Introduction

Motivation and engagement environment are always focus points for educators and interesting topics for researchers. They are the essential elements for successful knowledge acquisition and learning efficiency. In the past decades, many researchers have been focusing their studies on how the learners' behaviors change when different impacts involve. Andrew J. Martin developed the Motivation and Engagement Wheel for the seminal motivation and engagement theory [1]. Researches indicate that students' behaviors are influenced by different factors such as: self-complete intervention [2], emotional withdraw and poor identification with the school [3], class participation[4], school culture and structure[5], the relationship the students have with their teachers[6], educational correlates (educational aspirations, class participation, enjoyment of subject)[7] etc. These studies made great contribution to the formation of motivation and engagement theory in educational area and have been widely accepted in practice. However, these theories involve complex dynamic interactions of internal and external factors. The mixture of different types of impacts imposed on learners' learning process makes quantitative analysis on behavior orientation a very difficult task. The main purpose of the present study is to apply micro-evolutionary mechanism of CA to simulate learners' behaviors in a quantitative method.

The present study develops an approach through which the behavior orientation of learners can be observed intuitively. Based on CA and basic thoughts proposed in the motivation and engagement theories in education, this paper proposes a model for the simulation of complicated behaviors of learners influenced by two impacts which are the motivation strength and engagement environment, to investigate the global behavior of a selected population of learners. Simulation results obtained show that the proposed model is a promising computational approach in the mimic of behavior orientation of learners and provides a way to intuitively observe the changing pattern of learners' attitude toward study under the complicated influence of impacts.

2. Method

2.1. CA

CA model was developed by Von Neumann and Ulam in their study on self-reproducing systems in 1951[8]. CA is a bi-dimensional continuum. Cells in CA interact in a codified space-time torus wrapped from a CA grid under

*Supported by the Education and Teaching Key Project for the Eleventh Five-year Plan for the New Century (Guang Xi Provence, P. R. China) No.2008A020.

behavioral rules[9]. They may change their status during the evolution, and each state represents the state of a specific cell at moment *t*. CA model may also be viewed as simulation of global results of local interactions among individual cells in a selected grid size[10,11]. The model can be considered as an evolving system in which individuals interacting with each other under certain stochastic transition rules [11]. The transferring of information among members in the system may result in the change of behaviors.

Most widely used neighborhood structures are Von Neumann's model and Moore's model as shown in **Figure 1** [12]. In Von Neumann's model, the central agent (cell) is surrounded by four neighboring cells, while in Moore's model the central one is surrounded by eight neighboring cells [13]. The state of the central cell at time *t* depends on the collective interaction of the neighboring cells at time *t*-1.

A classical definition for CA is shown in the following equation[14]:

$$A = (L_d, S, N, f) \tag{1}$$

where, *L* and d are the spatial extension of the automata and the dimension of the system respectively; *S* is the finite state set and *N* stands for the set of the cells; *f* is the transfer function of the mapped states.

2.2. Simulation Model

The simulation model presented in this paper consists of three major components which are CA, motivation and engagement environment respectively.

2.2.1. CA Component

The selected population of learners is treated as a complex system in which the individuals' decisions on the perception of learning efficiency are closely related to that of their neighbors and impacts. The individual cells (treated as learners) in a two-dimensional grid may be represented with the following matrix (see **Figure 2**):

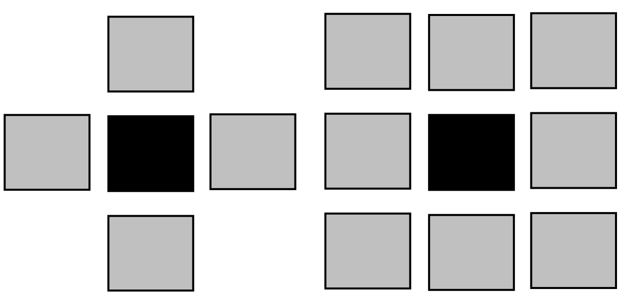


Figure1. Types of neighborhood of bi-dimensional CA.

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{matrix}$$

Figure2. CA matrix.

The subscripts (i, j) in the matrix indicate the positions of the cells in the automata.

A 30 × 30 CA automata grid is chosen for the simulation and Moore type of neighborhood structure is applied in this paper. Each cell in the CA grid is pre-assigned with a state and has three states to choose during the simulation. The state is represented as $S = \{-1, 0, 1\}$, where -1, 0 and 1 stand for the maladaptive intended state, natural intended state and adaptive intended state respectively.

The maladaptive intended state is for those who are negatively affected by neighbors in learning. They tend to have the perception of disengagement, self-handicapping, anxiety, failure avoidance and uncertain control.

The natural intended state represents the state for which the perception of learners on learning result is between the maladaptive and adaptive groups.

The adaptive intended state is suitable for those who are positively affected by neighbors in learning. They tend to have positive perception on self-efficacy, attributions, valuing, control, self-determination, goal orientation and so on.

The state of the central cell in CA component can be represented as the following:

$$S(t) = S(t-1) \tag{2}$$

S (*t*) represents the intended state of a cell at time t and *S*(*t* - 1) stands for the average influence from neighboring cells in the neighborhood at time *t* - 1. The value of *S*(*t*) is determined by the average value obtained from its all neighboring cells at time *t* - 1. The influences imposed on the central cell at time *t* - 1 are: the average influence created by the neighbors is $n_1(t-1)$ when neighboring cells are in the state of -1; the average influence from the neighbors is $n_2(t-1)$ when the neighboring cells are in the state of 0, and the average influence from the neighbors is $n_3(t-1)$ when they are in the state of 1. Therefore, the average influences imposed on the central cell from its neighboring cells are as shown in **Table 1**.

2.2.2 Motivation Component

M(*t* - 1) is assumed to be individual's intrinsic drive on learning at time *t* - 1 or the motivation on learning This paper assumes that *M*(*t* - 1) represents individual's perception, at time *t* - 1, on self-efficacy, attributions, valuing, control, self-determination, goal orientation and so on. the value of M is chosen from [r1, r2], $0 \leqslant r1 \leqslant 1$ and $0 \leqslant r2 \leqslant 1$. In order to investigate the influence of M , the values of r1 and r2 are stochastically taken from sub-ranges within [0, 1] during the simulation. The sub-ranges [0.3, 0.4], [0.4, 0.5], [0.5, 0.6] and [0.6, 0.7] are adopted to represent different perception (drive) levels while [0, 0.3] and [0.7, 1] are considered to be the extreme bad and extreme good perceptions on learning result. This arrangement of data-assignment typically represents the fact that motivation has certain level for certain group of learners. Through this way, we can assign different, but specific, levels (sub-ranges) of motivation on internal drive of individuals during the simulation.

2.2.3. Engagement Environment Component

$E(t\text{-}1)$ is taken as engagement environment at time t - 1, which is assumed to be the perception of an individual has on the engagement environment he is in. It is assumed that [0, e] is the range of perception that learners have on teaching method, school condition, interaction between teacher and students, relationship among students and so on and $0 \leqslant e \leqslant 1$. The higher the value e is, the better the perception of the learners have. Values of Et are obtained through stochastically assigning values from the range of [0, e] to $E(t-1)$ at time t-1during the simulation. This arrangement of data-assignment typically imitates the practice where the perception on engagement environment is widely distributed over individual learners in the range from the low to high due to different personal values.

2.3. Determination of Resulted State

The final state for each cell at each time step is determined by the total $S_{i,j}^t$ of $S(t)$, $M(t)$ and $E(t)$ as shown in Equation 3. The value for $S_{i,j}^t$ is ≧ 0 and may exceed 1 in some cases.

$$S_{i,j}^t = S(t) + M(t) + E(t) \tag{3}$$

With this special design, we can combine CA with some impacts to dynamically simulate state change of individual cells in a social system, thus to create a global orientation of a selected population of learners.

2.4. Transit Rules

According to the simulation model proposed above, following transit rules for the simulation are formulated:

1) The state of a cell may change stochastically from $S_{i,j}^{t-1}$ to $S_{i,j}^t$ during the simulation. It may change from its own state at time t-1 to other states at time t depending on the calculated $S_{i,j}^t$ in each time step. Here, critical values of 0.95 and 0.55 for $S_{i,j}^t$ exist. Learner's state changes in the direction of adaptive only when $S_{i,j}^t \geqslant 0.95$, while learner's state transfers to maladaptive only when $S_{i,j}^t \leqslant 0.55$.

2) The state change for each cell follows the path of maladaptive-natural-adaptive or adaptive-natural-maladaptive. Direct changing from maladaptive to adaptive or adaptive to maladaptive is prohibited.

Through numerous experiments, this paper proposes a set of suitable transit rules for the simulation process as shown in **Figure 3**.

3. Simulation and Discussion

Based on the transit rules described above, numerous simulations were carried out. Typical evolving results of the simulation are as shown in **Figure 4** where the star represents maladaptive and oval stands for adaptive. State change for learners is simulated under different levels of M (t) and E (t), which is discussed in following sections. The results shown in **Figure 5** and **Figure 6** are the average of ten simulations, which clearly indicate the changing trend of learners' perception under the influence of M and E.

Table 1. Central cell state under the average influence of neighboring cells.

Central cell state $S(t)$	Neighboring cell average $S(t\text{-}1)$
-1	$\overline{n_2(t-1)} + \overline{n_3(t-1)}$
0	$\overline{n_3(t-1)} - \overline{n_1(t-1)}$
1	$-\overline{n_1(t-1)} - \overline{n_2(t-1)}$

If

$S_{i,j}^{t-1}$ == -1

If

$S(t)$ >= 0.55

$S_{i,j}^t$ = 0

Else

$S_{i,j}^t$ = -1

If $S_{i,j}^{t-1}$ == 0

If $S(t)$ >= 0.95

$S_{i,j}^t$ = 1

If $S(t)$ <= 0.55

$S_{i,j}^t$ =-1

Else

$S_{i,j}^t$ =0

If

$S_{i,j}^{t-1}$ == 1

If

$S(t)$ <=0.55

$S_{i,j}^t$ = 0

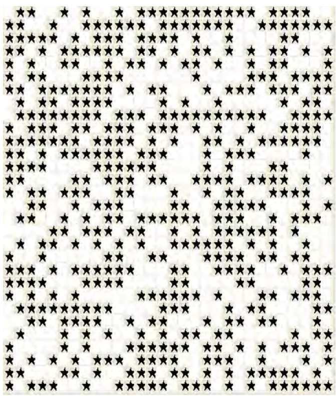
Else

$S_{i,j}^t$ = 1

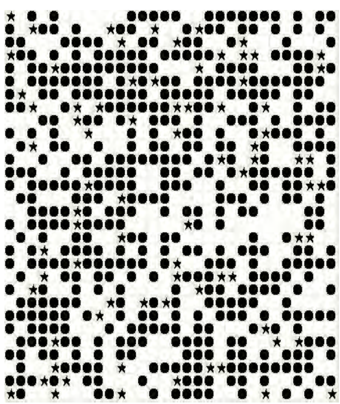
Figure3. Transit rules.

3.1. The Impact of Engagement Environment on Learner Behaviors

Learners' perceptions on learning results, as shown in **Figure 5 (a)**, **(b)** and **(c)**, are simulated under the influence of E when M is taken from [0.35, 0.45], [0.45, 0.55] and [0.55, 0.65] respectively. The results show the influence of engagement environment on learners' perceptions at different levels of motivation strength. Following interesting points are noticed: the change of E causes the numbers of adaptive and maladaptive learners increasing or decreasing at different levels of M; the numbers of adaptive and maladaptive learners increases and decreases respectively at the same level of M when E is improved; the numbers for both types of learners also increase and decrease at the same level of E when the strength of M is increased to a higher level. Another important implication is that the change of M's level brings more remarkable change on the numbers of both types of learners than the change of E. These result show good agreement with the fact that internal factor plays more important role in changing the perceptions of group of learners than the external factors.

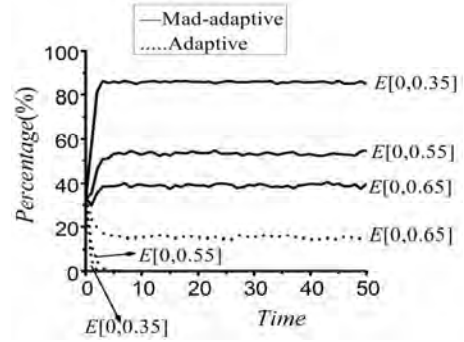


(a) E = 0.5, M = [0.35, 0.45]

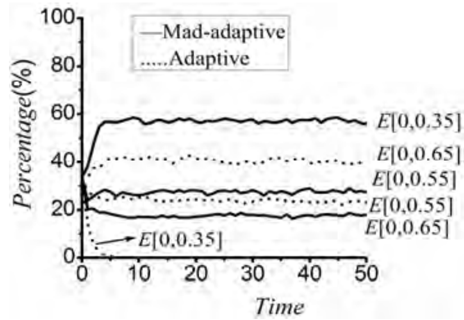


(b) E = 0.5, M = [0.55, 0.65]

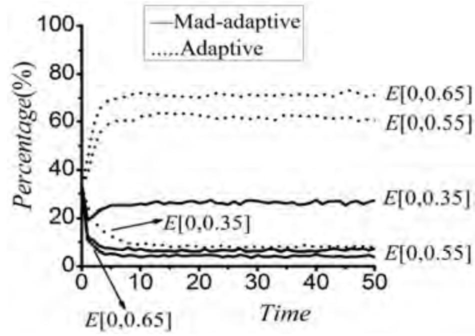
Figure 4. Typical evolving results of the simulation at the 50th time step.



(a)



(b)

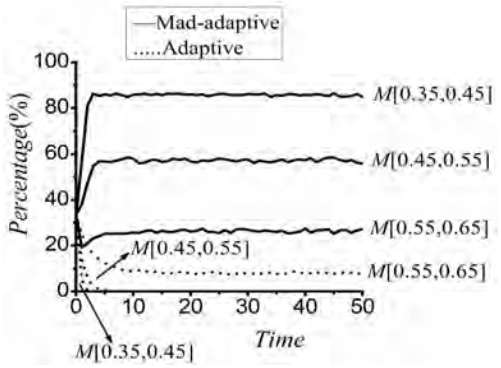


(c)

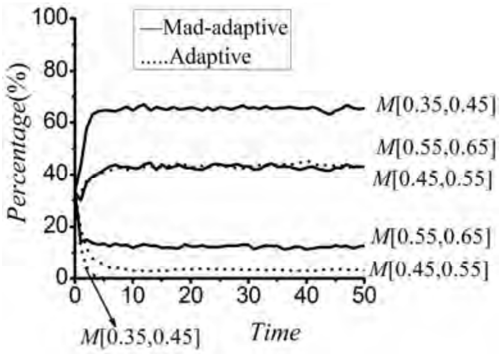
Figure 5. (a) Influence of E when M is taken from [0.35, 0.45]; (b) Influence of E when M is taken from [0.45, 0.55]; (c) Influence of E when M is taken from [0.55, 0.65].

3.2. The Impact of Motivation on Learners' Behaviors

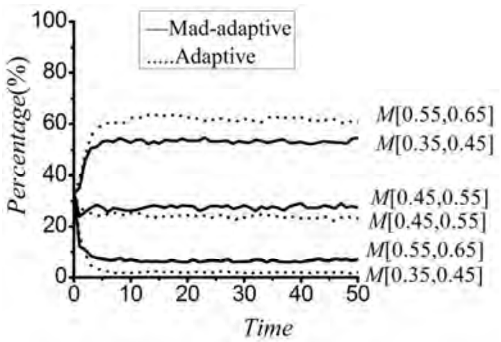
To further prove the findings in **Figure 5**, another group of simulations are also implemented. **Figure 6 (a)**, **(b)**, **(c)** and (d)show the influence power of M when E is taken from [0, 0.35], [0, 0.45], [0, 0.55] and [0, 0.65] respectively. These figures again clearly indicate the importance of M on learners' perceptions on learning results. At different levels of engagement environment, M imposes remarkable influence on learners' perception. The most significant impact on the number change of learners both in adaptive and maladaptive groups happens at the middle and better levels of engagement environments, which



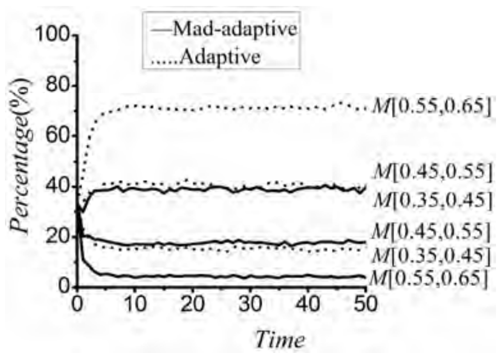
(a)



(b)



(c)



(d)

Figure 6. (a) Influence of *M* when *E* is taken from [0, 0.35]; (b) Influence of *M* when *E* is taken from [0, 0.45]; (c) Influence of *M* when *E* is taken from [0, 0.55]; (d) Influence of *M* when *E* is taken from [0, 0.65].

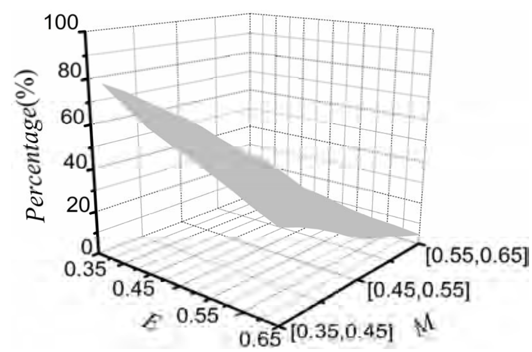
also has good agreement with the fact that basic external environment is necessary for better outcomes produced by the internal factors.

3.3. Comparison of Effectiveness of Impact Factors

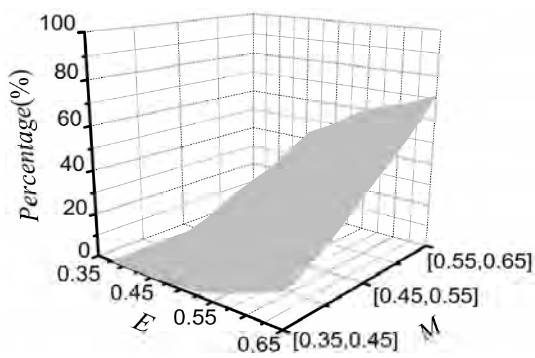
In order to separately show the changing pattern of adaptive and maladaptive learners' behaviors under changing M and E, three-dimensional figures are plotted based on the average of 100 simulations as shown in **Figure 7** where percentage represents the number of relevant group of learners. From these figures, intuitive changing trend for both groups of learners is observed when M and E both change to different values. Stronger influence power of M comparing with E can be seen clearly at different points of the axes.

4. Conclusions

The model proposed in this paper combines CA with two strictly defined impacts. Through using the simulation model and transit rules carefully designed, it can be applied to simulate the global behavior of learners. The simulation process indicates that the internal factor, which is the motivation in this paper, plays a key role in changing



(a)



(b)

Figure 7. (a) Changing pattern of maladaptive learners; (b) Changing pattern of adaptive learners.

learners' behaviors under certain circumstance and the engagement environment is also a very important force of influence on learners' perceptions. Results also show that the simulation model and transit rules proposed in this paper are adequately designed and be able to reveal the changing patterns of behaviors of learners in some extent in learning environment and the results are close to the phenomenon observed in practice. Simulation results obtained show that the proposed model is a promising computational approach in the assessment of behavior orientation of a selected population although the accuracy is not validated due to the lack of real data. The model proposed also provides a way to intuitively observe the changing pattern of learners' attitude toward learning under the influence of impacts based on the social environment provided by CA.

5. References

[1] A. J. Martin, "How Domain Specific is Motivation and Engagement across School, Sport, and Music? A Subst-Antive-Methodological Synergy Assessing Young Sportsp-Eople And Musicians," *Contemporary Eduction Psychology*, Vol. 33, 2008, pp. 785-813.

[2] A. J. Martin, "Enhancing Student Motivation and En-Gagement:The Effects of a Multidimensional Intervention," *Contemporary Eduction Psychology*, Vol. 33, No. 2, 2008, pp. 239-269.

[3] J. D. Finn, "Withdrawing from School," *Review of Educational Research*" Vol. 59, No. 2, pp. 117-142.

[4] F. D. Ritcher and D. Tjosvold, "Effects of Student Partciption in Classtoom Decision Making on Attitudes, Peer Interaction, Motivation and Learing," *Journal of Applied Phychology*, Vol. 65, 1980, pp. 74-80.

[5] E. A. Anderman and M. L. Maehr, "Motivation and Schooling in the Middle Grades," *Review of Educational Research*, Vol. 64, No. 2, pp. 287-310.

[6] J. A. Kelly and D. J. Hansen, (1987), "Social Interactions and Adjustment," In V. B. Can Hasselt and M. Hersen, Ed., *Handbook of adolescent psychology*, Pergamon Press: Springer, New York, pp. 131-146.

[7] J. Freen, A. J. Martin and H. W. Marsh, "Motivation and Engagement in English, Mathematics and Science High School Subjects: Towards an Understanding of Multidimensional Domain Specificity," *Learning and Individual Differences*, Vol. 17, No. 3, 2007, pp. 269-279.

[8] J. vonNeumann, "The General and Logical Theory of Automata," L. A. Jiffries, Ed., *Cerebral Mechanism in Behavior- the Hixon Symposium* [*C*], Wiley, New York, 1951.

[9] H. Y. Liu, Y. L. Li, C. Y. Zhang and Q. Wang, "Simulation of Learners' Behaviors Based on Cellular Automata," *IEEE 2009 International Conference on Computational Intelligence and Software Engineering*, China, 2009, pp. 1-4.

[10] S. Wolfram, "Statistical Mechanics of Cellular Automata," *Reviews of Modern Physics*, Vol. 55, No. 3, 1983, pp. 601-644.

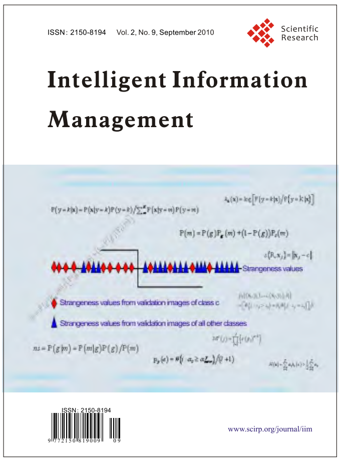
[11] N. H. Packad, S. W. Fram, "Two Dimensional Cellular Automata," *Journal of Statistical Physics*, Vol. 38, No. 5-6, 1985.

[12] J. Nemmann, "Theory of Self Reproducing Automata," University of Illionois, Urbana, 1966.

[13] Frishu, Hasslacherb, Y. Pomeau," Two Dimensional Cellular Automata," *Physical Review Letters*, 1986.

[14] S. W. Fram, "Theory and Applications of Cellular Automata," World Scientific, Singapore, 1986.

Call for Papers



Intelligent Information Management

ISSN 2150-8194 (print) ISSN 2150-8208 (online)
www.scirp.org/journal/iim

IIM is a peer reviewed international journal dedicated to the latest advancement of intelligent information management. The goal of this journal is to keep a record of the state-of-the-art research and promote the research work in these fast moving areas. The journal publishes the highest quality, original papers includ but are not limited to the fields:

★ Computational intelligence
★ Data mining
★ Database management
★ Distributed artificial intelligence
★ General systems theory and methodology
★ Information security
★ Intelligent information management systems
★ Knowledge discovery and management
★ Nonlinear system and control theory
★ Optimal algorithms
★ Other related areas and applications

We are also interested in short papers (letters) that clearly address a specific problem, and short survey or position papers that sketch the results or problems on a specific topic. Authors of selected short papers would be invited to write a regular paper on the same topic for future issues of the IIM.

Website and E-Mail

http://www.scirp.org/journal/iim **E-Mail:iim@scirp.org**

Intelligent Information Management, 2010, 2, 499-568.
Published Online September 2010 in SciRes (http://www.SciRP.org/journal/iim/).

TABLE OF CONTENTS

Volume 2 Number 9 September 2010